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MEASURES ON THE GEOMETRIC LIMIT SET IN HIGHER RANK SYMMETRIC SPACES

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Abstract

For a discrete isometry group of a higher rank symmetric space we present certain families of measures on its geometric limit set. We further introduce a notion of Hausdorff measure and give an estimate of the Hausdorff dimension of the radial limit set.

1. Introduction

Let X be a globally symmetric space of noncompact type and ∂X its geometric boundary endowed with the cone topology (see [Ba, chapter II]). We denote by $G = \mathrm{Isom}^o(X)$ the connected component of the identity, and let $\Gamma \subset G$ be a discrete subgroup. The geometric limit set of Γ is defined by $L_\Gamma := \overline{\Gamma \cdot o} \cap \partial X$ where $o \in X$ is an arbitrary point. In order to measure the size of the limit set of discrete isometry groups of real hyperbolic spaces, S. J. Patterson ([P]) and D. Sullivan ([S]) developed a theory of conformal densities. These densities allow to relate the Hausdorff dimension of the limit set to the critical exponent of Γ

$$\delta(\Gamma) := \inf\Big\{s > 0 \mid \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} < \infty\Big\}.$$

Part of the theory has been extended by P. Albuquerque ([Al]) to Zariski dense discrete isometry groups of arbitrary symmetric spaces $X = G/K$ of noncompact type. However, if the rank of X is greater than one, the support of a $\delta(\Gamma)$-dimensional conformal density is a proper Γ-invariant subset of the limit set. In order to obtain densities supported on every Γ-invariant subset of the geometric limit set, we recently constructed so-called $(b, \Gamma\cdot\xi)$-densities ([Li]). We remark that the projection of these densities to the Furstenberg boundary gives precisely the "(Γ, φ) Patterson measures" constructed independently by J. F. Quint ([Q]) using different methods. The measures on the Furstenberg

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boundary, however, do not allow to capture an essential piece of information concerning the geometry of Γ-orbits.

In this note we are going to describe the ideas of our construction of $(b, \Gamma\cdot\xi)$-densities and give some estimates on the Hausdorff dimension of the limit set. For a detailed description and more general results we refer the reader to [Li].

The paper is organized as follows: In section 2 we recall some basic facts about Riemannian symmetric spaces of noncompact type. We describe the G-orbit structure of the geometric boundary ∂X and introduce a family of (possibly nonsymmetric) G-invariant pseudo distances on X. In section 3 we give a definition and describe our construction of $(b, \Gamma\cdot\xi)$-densities. In section 4 we introduce an appropriate notion of Hausdorff measure and estimate the Hausdorff dimension of the radial limit set.

2. Preliminaries

In this section we recall some basic facts about symmetric spaces of noncompact type (see also [H], [BGS], [E]) and fix some notations for the sequel.

2.1. Polar coordinates

Let X be a simply connected symmetric space of noncompact type with base point $o \in X$, $G = \mathrm{Isom}^o(X)$, and K the isotropy subgroup of o in G. It is well-known that G is a semisimple Lie group with trivial center and no compact factors, and K a maximal compact subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K. Since G acts transitively on X we have the identification $X \cong G/K$. The geodesic symmetry in o induces a Cartan involution on $\mathfrak{g}$, hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{g}$ denotes its -1 eigenspace. The tangent space T_oX of X in o is identified with $\mathfrak{p}$, and the Riemannian exponential map at o is a diffeomorphism of $\mathfrak{p}$ onto X. The Killing form of $\mathfrak{g}$ restricted to $\mathfrak{p}$ induces an inner product $\langle\cdot,\cdot\rangle$ on T_oX and hence a G-invariant Riemannian metric on X with associated distance d. With respect to this metric, X has nonpositive sectional curvature, and up to rescaling in each factor, this metric is the original one.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Its dimension r is called the rank of X. The choice of an open Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ determines a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$, where $\overline{\mathfrak{a}^+}$ denotes the closure of $\mathfrak{a}^+$. The component of g in $\overline{\mathfrak{a}^+}$ is uniquely determined by g and will be called the Cartan projection $H(g)$. We will denote by $\mathfrak{a}_1$ the unit sphere in $\mathfrak{a}$.

Let Σ denote the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$, and $\Sigma^+ \subset \Sigma$ the set of positive roots determined by the Weyl chamber $\mathfrak{a}^+$. We fix a set of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of Σ^+. For $i \in \{1, 2, \ldots, r\}$ we call the unique vector $H_i \in \overline{\mathfrak{a}_1^+}$ with the property $\alpha_j(H_i) = 0$ for all $j \neq i$ the i-th maximal singular direction.

2.2. Compactification of X

The geometric boundary ∂X of X is the set of equivalence classes of asymptotic geodesic rays endowed with the cone topology. This boundary is homeomorphic to the unit tangent space of an arbitrary point in X, hence by the Cartan decomposition $\mathrm{Ad}(K)\overline{\mathfrak{a}_1^+} \cong \partial X$.

For $k \in K$ and $H \in \overline{\mathfrak{a}_1^+}$ we denote by (k, H) the unique class in ∂X which contains the geodesic ray $\sigma(t) = ke^{Ht}o$, $t > 0$. We will call H the Cartan projection of (k, H). Note that the writing is not unique, because $(k_1, H) = (k_2, H)$ if and only if $k_1^{-1}k_2$ belongs to the centralizer of H in K.

Put $\overline{X} := X \cup \partial X$. For $x \in X$ and $z \in \overline{X}$ we denote by $\sigma_{x,z}$ the unique unit speed geodesic emanating from x which contains z.

The isometry group of X has a natural action by homeomorphisms on the geometric boundary, and $G\cdot\xi = K\cdot\xi$ for any $\xi \in \partial X$. Furthermore, G acts transitively on ∂X if and only if $\mathrm{rank}(X) = 1$.

If $r = \mathrm{rank}(X) > 1$, we define the regular boundary $\partial X^{\mathrm{reg}}$ as the set of classes with Cartan projection $H \in \mathfrak{a}_1^+$, and the i-th maximal singular boundary component ∂X^i, $1 \leqslant i \leqslant r$, as the set of classes with Cartan projection $H_i \in \overline{\mathfrak{a}_1^+}$ as defined in the previous section. If $r = 1$, we use the convention $\partial X^{\mathrm{reg}} = \partial X$.

2.3. Directional distances

For $x, y \in X$, $\xi \in \partial X$ let $\mathscr{B}_\xi(x, y) := \lim_{s\to\infty}\big(d(x, \sigma_{x,\xi}(s)) - d(y, \sigma_{x,\xi}(s))\big)$ be the Busemann function centered at ξ (compare [Ba, chapter II]). Using these functions we will construct an important family of (possibly nonsymmetric) pseudo distances which will play a crucial role in the remainder of this note.

Definition 2.1. — *Let $\xi \in \partial X$. The* directional distance *of the ordered pair $(x, y) \in X \times X$ with respect to the subset $G\cdot\xi \subseteq \partial X$ is defined by*

$$\begin{array}{rccl} \mathscr{B}_{G\cdot\xi}: & X\times X & \to & \mathbb{R} \\ & (x,y) & \mapsto & \mathscr{B}_{G\cdot\xi}(x,y) := \sup\limits_{g\in G}\mathscr{B}_{g\cdot\xi}(x,y)\,. \end{array}$$

Note that in rank one symmetric spaces $G\cdot\xi = \partial X$, and $\mathscr{B}_{G\cdot\xi}$ equals the Riemannian distance d for any $\xi \in \partial X$. In general, the corresponding estimate for the Buseman functions implies $\mathscr{B}_{G\cdot\xi}(x, y) \leqslant d(x, y)$ for any $\xi \in \partial X$, $x, y \in X$.

Furthermore, $\mathscr{B}_{G\cdot\xi}$ is a (possibly nonsymmetric) G-invariant pseudo distance on X (for a proof see [L, Proposition 3.7]), and we have

$$\mathscr{B}_{G\cdot\xi}(x, y) = d(x, y)\cos\angle_x(y, G\cdot\xi)\,, \quad \text{where} \quad \angle_x(y, G\cdot\xi) := \inf_{g\in G}\angle_x(y, g\xi)\,.$$

In particular, if $H_\xi \in \overline{\mathfrak{a}_1^+}$ denotes the Cartan projection of ξ, we have

$$\mathscr{B}_{G\cdot\xi}(o, ke^H o) = \langle H_\xi, H\rangle \qquad \forall\, k \in K \quad \forall\, H \in \overline{\mathfrak{a}^+}.$$

If $\xi \in \partial X^i$, $1 \leqslant i \leqslant r$, we will write d_i instead of $\mathscr{B}_{G\cdot\xi}$.

3. Construction of $(b, \Gamma\cdot\xi)$-densities

We denote by π^B the projection
$$\begin{array}{rccc} \pi^B : & \partial X^{\mathrm{reg}} & \to & K/M \\ & (k, H) & \mapsto & kM. \end{array}$$

It is well-known (see for example [L, Theorem 5.15], [Be]) that in the higher rank case the regular geometric limit set splits as a product $K_\Gamma \times (P_\Gamma \cap \mathfrak{a}_1^+)$ where $K_\Gamma = \pi^B(L_\Gamma \cap \partial X^{\mathrm{reg}})$ and $P_\Gamma \subseteq \overline{\mathfrak{a}_1^+}$ is the set of Cartan projections of limit points. In particular, for any $H \in P_\Gamma \cap \mathfrak{a}_1^+$, $\xi = (\mathrm{id}, H) \in \partial X^{\mathrm{reg}}$, the set $L_\Gamma \cap G\cdot\xi$ is a Γ-invariant subset of the limit set isomorphic to $K_\Gamma \times \{H\}$.

In this section we will give an idea of how to construct the following kind of densities for each Γ-invariant subset of the regular limit set. Recall that r denotes the rank of X.

Definition 3.1. — *Let $\mathscr{M}^+(\partial X)$ denote the cone of positive finite Borel measures on ∂X, $\xi \in \partial X^{\mathrm{reg}}$ and $b = (b^1, b^2, \dots b^r) \in \mathbb{R}^r$. A* $(b, \Gamma\cdot\xi)$-density *is a continuous map*

$$\begin{array}{rccc} \mu : & X & \to & \mathscr{M}^+(\partial X) \\ & x & \mapsto & \mu_x \end{array}$$

with the properties

$$\begin{array}{ll} (i) & \mathrm{supp}(\mu_o) \subseteq L_\Gamma \cap G\cdot\xi, \\ (ii) & \gamma * \mu_x = \mu_{\gamma^{-1}x} \text{ for any } \gamma \in \Gamma,\ x \in X, \\ (iii) & \dfrac{d\mu_x}{d\mu_o}(\eta) = e^{\sum_{i=1}^r b^i \mathscr{B}_{\eta_i}(o,x)} \quad \text{for any } x \in X,\ \eta \in \mathrm{supp}(\mu_o). \end{array}$$

Here $\eta_i \in \partial X^i$ denotes the unique point in the i-th maximal singular boundary component which is contained in the closure of the Weyl chamber at infinity determined by $\eta \in \partial X^{\mathrm{reg}}$.

Remarks. — If $r = 1$ then for any $\xi \in \partial X$ we have $L_\Gamma \cap G\cdot\xi = L_\Gamma$ and the above definition for a $(b, \Gamma\cdot\xi)$-density is exactly the definition of a b-dimensional conformal density.

If $r > 1$ let H_ξ denote the Cartan projection of $\xi \in \partial X^{\mathrm{reg}}$ and $H_1, H_2, \dots, H_r$ the maximal singular directions defined in section 2.1. A $(b, \Gamma\cdot\xi)$-density is an α-dimensional conformal density with support in $G\cdot\xi$ as defined in [Al] if and only if $\sum_{i=1}^r b^i H_i = \alpha H_\xi$.

We briefly recall the Patterson-Sullivan construction of $\delta(\Gamma)$-dimensional conformal densities. Denote by D the unit Dirac measure and by

$$P^s(x,y) := \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}$$

the Poincaré series of Γ. Define on $\overline{X}$ a family of measures

$$\mu_x^s := \frac{1}{P^s(o,o)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma o)} D(\gamma o)\,,\ x \in X\,,\ s > \delta(\Gamma)$$

supported on the orbit $\Gamma\cdot o$. We remark that if $P^{\delta(\Gamma)}(o,o)$ converges, the definition of the measures is modified by adding a slowly incrasing function (see [P] for details). Any weak limit $\mu = (\mu_x)_{x\in X}$ as s tends to $\delta(\Gamma)$ of the family of measures $(\mu_x^s)_{x\in X}$ then yields a $\delta(\Gamma)$-dimensional conformal density.

3.1. Exponential growth rate in direction $G\cdot\xi$

If $\xi \in \partial X^{\mathrm{reg}}$ is given, a necessary condition for a sequence $(y_j) \subset X$ to converge to a point $\eta \in G\cdot\xi$ is

$$\angle_x(y_j, G\cdot\xi) := \inf_{g\in G} \angle_x(y_j, g\xi) \to 0 \qquad \text{for any point } x \in X\,.$$

Hence if the rank of X is greater than one, only the subset

$$\Gamma(x,y) := \{\gamma \in \Gamma \mid \gamma y \neq x\,,\ \angle_x(\gamma y, G\cdot\xi) < \varphi\}\,,\ x,y \in X\,,$$

for arbitrarily small $\varphi > 0$ contributes to the limit set in $G\cdot\xi$. Therefore, in order to obtain measures supported on $L_\Gamma \cap G\cdot\xi$, one should rather consider sums of the form

$$Q_{G\cdot\xi}^{s,\varphi}(x,y) = \sum_{\gamma\in\Gamma(x,y)} e^{-sd(x,\gamma y)}$$

instead of using the complete Poincaré series.

It is not possible, however, to obtain a family of Γ-equivariant measures directly from such a sum, because $\Gamma(x,y)$ is not Γ-invariant. Furthermore, the exponent of convergence $\delta_{G\cdot\xi}^{\varphi}(x,y)$ of $Q_{G\cdot\xi}^{s,\varphi}(x,y)$ depends on the points $x,y \in X$.

However, the number $\delta_{G\cdot\xi}(\Gamma) := \liminf_{\varphi\to 0} \delta_{G\cdot\xi}^{\varphi}(x,y)$ is independent of $x,y \in X$ (see [L, Lemma 6.2]) and will be called the exponent of growth of Γ in direction $G\cdot\xi$. Since

$$\begin{aligned} \delta_{G\cdot\xi}(\Gamma) &= \liminf_{\varphi\to 0}\left(\limsup_{R\to\infty} \frac{\log \Delta N_{G\cdot\xi}^{\varphi}(x,y;R)}{R}\right) \quad \text{with} \\ \Delta N_{G\cdot\xi}^{\varphi}(x,y;R) &= \#\{\gamma \in \Gamma \mid R-1 \leqslant d(x,\gamma y) < R\,,\ \angle_x(\gamma y, G\cdot\xi) < \varphi\}\,, \end{aligned}$$

this number can also be interpreted as an exponential growth rate of the number of orbit points close in direction to $G\cdot\xi$.

3.2. The modified Poincaré series

From here on we fix $\xi \in \partial X^{\mathrm{reg}}$ with $\delta_{G\cdot\xi}(\Gamma) > 0$. Recall that $\mathscr{B}_{G\cdot\xi}$ and $d_1, d_2, \dots, d_r$ are the directional distances introduced in section 2.3. In order to have hardly any contribution of elements far in direction from $G\cdot\xi$, i.e. those with $\angle_x(yy, G\cdot\xi)$ large, we add weights

$$e^{-\tau(d(x,yy)-\mathscr{B}_{G\cdot\xi}(x,yy))} = e^{-\tau d(x,yy)(1-\cos\angle_x(yy,G\cdot\xi))}$$

with $\tau > 0$ large to the terms in the Poincaré series. It turns out that we also have to introduce more degrees of freedom which is done by replacing the Riemannian distance d with a linear combination of $d_1, d_2, \dots d_r$.

For $\tau > 0$ we denote by $\mathscr{R}^{\tau}_{G\cdot\xi} \subset \mathbb{R}^r$ the set of r-tuples $b = (b^1, b^2, \dots, b^r) \in \mathbb{R}^r$ for which the series

$$P^{s,b,\tau}_{G\cdot\xi}(x,y) = \sum_{\gamma\in\Gamma} e^{-s\left(\sum_{i=1}^r b^i d_i(x,yy)+\tau\left(d(x,yy)-\mathscr{B}_{G\cdot\xi}(x,yy)\right)\right)}$$

has exponent of convergence equal to one. Notice that $\mathscr{R}^{\tau}_{G\cdot\xi}$ is independent of $x, y \in X$ by the triangle inequalities for d, $\mathscr{B}_{G\cdot\xi}$ and $d_1, d_2, \dots d_r$.

It is shown in detail in [Li, section 3.3], that for any $b \in \mathscr{R}^{\tau}_{G\cdot\xi}$ the Patterson-Sullivan construction yields a family of Γ-equivariant measures $\mu = \mu^{b,\tau}_{G\cdot\xi}$ supported on the limit set. However, these measures are in general not absolutely continuous with respect to each other.

Recall that $H_1, H_2, \dots, H_r$ are the maximal singular directions defined in section 2.1. Suppose there exists $b \in \mathbb{R}^r$ and $\varphi_0 \in (0, \pi/4)$ such that

$$(*) \qquad \sum_{i=1}^r b^i\langle H_i, H_\xi\rangle = \delta_{G\cdot\xi}(\Gamma)\ , \ \text{and} \quad \sum_{i=1}^r b^i\langle H_i, H_\eta\rangle \geqslant \delta_{G\cdot\eta}(\Gamma)$$

for any $\eta \in \partial X$ with Cartan projection $H_\eta \in \overline{\mathfrak{a}_1^+}$ and $\angle_o(\eta, G\cdot\xi) < \varphi_0$. This condition on the behavior of the exponent of growth of Γ in the neighboring directions of $G\cdot\xi$ is satisfied for Zariski dense discrete groups Γ by a result of J. F. Quint (see [Q]) and [Li, Proposition 3.12]). Then there exists $\tau_0 = \tau_0(b, \varphi_0) \geqslant 0$ such that for all $\tau \geqslant \tau_0$ the family of measures $\mu = \mu^{b,\tau}_{G\cdot\xi}$ is supported on $L_\Gamma \cap G\cdot\xi$ and satisfies

$$\frac{d\mu_x}{d\mu_o}(\eta) = e^{\sum_{i=1}^r b^i \mathscr{B}_{\eta_i}(o,x)} \qquad \text{for any } x \in X\ ,\ \eta \in \operatorname{supp}(\mu_o)\ .$$

Hence μ is a $(b;\Gamma\cdot\xi)$-density.

3.3. The case of lattices

In this section we are going to precise the parameters of $(b,\Gamma\cdot\xi)$-densities for lattices $\Gamma \subset G$. The calculation in [A] shows that in this case the exponent of growth $\delta_{G\cdot\xi}(\Gamma)$ in a direction $G\cdot\xi$ with Cartan projection H_ξ is equal to $\rho(H_\xi)$, where ρ denotes the sum

of the positive roots counted with multiplicities. Furthermore, the critical exponent of Γ is equal to $\|\rho\|$.

Since ρ is a linear functional on $\mathfrak{a}$ and the maximal singular directions $H_1, \dots, H_r$ form a basis of $\mathfrak{a}$, there exist parameters $b^1, b^2, \dots, b^r \in \mathbb{R}$ with the property $\sum_{i=1}^r b^i \langle H_i, H\rangle = \rho(H)$ for all $H \in \mathfrak{a}$. This implies that for any $\xi \in \partial X$ the conditions $(*)$ above are satisfied for arbitrary $\varphi_0 > 0$ with the same tuple $b = (b^1, b^2, \dots, b^r)$. Using τ sufficiently large, we are able to construct a $(b, \Gamma\cdot\xi)$-density for every $\xi \in \partial X^{\mathrm{reg}}$.

According to the second remark after Definition 3.1, the direction H_* which supports the conformal density constructed by P. Albuquerque is given by the condition $\sum_{i=1}^r b^i H_i = \alpha H_*$ for some $\alpha > 0$. By choice of our parameters b we know that $\sum_{i=1}^r b^i H_i$ equals the dual vector of ρ in $\mathfrak{a}^+$. Hence $H_* \in \mathfrak{a}_1^+$ is the normalized dual vector of ρ and $\alpha = \rho(H_*) = \|\rho\| = \delta(\Gamma)$. This shows that any $\delta(\Gamma)$-dimensional conformal density is supported in $G\cdot\xi_*$ for $\xi_* = (\mathrm{id}, H_*) \in \partial X^{\mathrm{reg}}$.

4. Hausdorff measure

In this section, we deal with an important subset of the limit set, the radial limit set. We introduce an appropriate notion of Hausdorff measure and Hausdorff dimension on the geometric boundary ∂X and estimate the size of the radial limit set in each G-invariant subset $G\cdot\xi \subseteq \partial X$. For so-called radially cocompact groups we obtain a sharp estimate for the Hausdorff dimension of the radial limit set in any given subset $G\cdot\xi \subseteq \partial X^{\mathrm{reg}}$.

4.1. The upper bound on the Hausdorff dimension

We will use the following definition of Hausdorff measure on the geometric boundary which was introduced by G. Knieper in [Kn, §4]. For $\xi \in \partial X$, $c > 0$ and $0 < r < e^{-c}$ we call the set

$$B_r^c(\xi) := \{\eta \in \partial X \mid d(\sigma_{o,\eta}(-\log r), \sigma_{o,\xi}(-\log r)) < c\}$$

a c-ball of radius r centered at ξ. With this notion of c-balls we define as in the case of metric spaces Hausdorff measure and Hausdorff dimension on the geometric boundary.

Definition 4.1. — *Let E be a Borel subset of ∂X,*

$$\mathrm{Hd}_\varepsilon^\alpha(E) = \inf\Big\{\sum r_i^\alpha \mid E \subseteq \bigcup B_{\xi_i}^c(r_i)\,,\ r_i < \varepsilon\Big\}\,.$$

$\mathrm{Hd}^\alpha(E) := \lim_{\varepsilon\to 0} \mathrm{Hd}_\varepsilon^\alpha(E)$ *is called the α-dimensional* Hausdorff measure *of E,*
$\dim_{Hd}(E) := \inf\{\alpha \geqslant 0 \mid \mathrm{Hd}^\alpha(E) < \infty\}$ *the* Hausdorff dimension *of E.*

In this note we are going to use the following definition of radial limit points in order to simplify the estimates concerning the upper bound on the Hausdorff dimension. For a proof of the more general result we refer the interested reader to [Li, section 6].

DEFINITION 4.2. — *A point $\xi \in L_\Gamma$ is called a* radial limit point *for the action of Γ if there exists a sequence $(\gamma_j) \subset \Gamma$ which remains at bounded distance of a geodesic ray with extremity ξ. The set of radial limit points in ∂X is denoted by L_Γ^{rad}.*

For $c > 0$, $x, y \in X$ with $d(x,y) > c$ we further put

$$\mathrm{sh}_x(B_y(c)) := \{\eta \in \partial X \mid d(y, \sigma_{x,\eta}) < c\}.$$

THEOREM 4.3. — *If $\Gamma \subset G$ is a discrete group and $\xi \in \partial X^{\mathrm{reg}}$, then the Hausdorff dimension of the radial limit set in $G\cdot\xi$ is bounded above by $\delta_{G\cdot\xi}(\Gamma)$.*

Proof. — Fix $\xi \in \partial X^{\mathrm{reg}}$ with Cartan projection $H_\xi \in \mathfrak{a}_1^+$, and $c > 0$ sufficiently large. By definition of the radial limit set,

$$L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi \subseteq \bigcap_{R>0} \bigcup_{\substack{\gamma\in\Gamma \\ d(o,\gamma o)>R \\ d(\gamma o, K\cdot\sigma_{o,\xi})<c/6}} \mathrm{sh}_o(B_{\gamma o}(c/2)).$$

Let $\varepsilon > 0$ be arbitrarily small. For $\gamma \in \Gamma$, put $\xi_\gamma := \sigma_{o,\gamma o}(+\infty)$, $r_\gamma := e^{-d(o,\gamma o)}$ and let $\Gamma' := \{\gamma \in \Gamma \mid r_\gamma < \varepsilon,\ d(\gamma o, K\cdot\sigma_{o,\xi}) < c/6\}$. Since $\mathrm{sh}_o(B_{\gamma o}(c/2)) \subseteq B^c_{r_\gamma}(\xi_\gamma)$ we have

$$L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi \subseteq \bigcup_{\gamma\in\Gamma'} B^c_{r_\gamma}(\xi_\gamma).$$

Using the definition of $\mathrm{Hd}^\alpha_\varepsilon$ we estimate

$$\mathrm{Hd}^\alpha_\varepsilon(L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi) \leqslant \sum_{\gamma\in\Gamma'} r_\gamma^\alpha = \sum_{\gamma\in\Gamma'} e^{-\alpha d(o,\gamma o)} \leqslant Q^{\alpha,\varphi}_{G\cdot\xi}(o,o) \quad \text{if } \varphi < \frac{c}{-6\log\varepsilon}.$$

Hence $\mathrm{Hd}^\alpha_\varepsilon(L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi)$ is finite for $\alpha > \delta^\varphi_{G\cdot\xi}(o,o)$. Taking the limit as $\varphi \searrow 0$ we conclude that the same is true for $\alpha > \delta_{G\cdot\xi}(\Gamma)$. Letting $\varepsilon \searrow 0$, we obtain $\mathrm{Hd}^\alpha(L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi) < \infty$ if $\alpha > \delta_{G\cdot\xi}(\Gamma)$, hence $\dim_{Hd}(L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi) \leqslant \delta_{G\cdot\xi}(\Gamma)$. □

4.2. Radially cocompact groups

For convex cocompact and geometrically finite discrete groups of real hyperbolic spaces D. Sullivan proved that the Hausdorff dimension of the radial limit set is equal to the critical exponent ([S, Theorem 25]). In 1990, K. Corlette ([C]) extended this result to all rank one symmetric spaces of noncompact type. In order to give a sharp estimate for the Hausdorff dimension of the radial limit set in higher rank symmetric spaces, we use the following definition.

DEFINITION 4.4. — *A discrete group $\Gamma \subset G$ is called* radially cocompact *if there exists a constant $c_\Gamma > 0$ such that for any $\eta \in L_\Gamma^{\mathrm{rad}}$ and for all $t > 0$ there exists $\gamma \in \Gamma$ with $d(\gamma o, \sigma_{o,\eta}(t)) < c_\Gamma$.*

Examples of radially cocompact groups are convex cocompact isometry groups of rank one symmetric spaces, uniform lattices in higher rank symmetric spaces and products of convex cocompact groups acting on the Riemannian product of rank one symmetric spaces.

For radially cocompact discrete groups $\Gamma \subset G$, the existence of a $(b, \Gamma\cdot\xi)$-density μ together with the following theorem (see [Li, section 4.3] for a proof) allows to also obtain a lower bound for the Hausdorff dimension of the radial limit set.

THEOREM 4.5 (Shadow lemma). — *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup, $\xi \in \partial X^{\mathrm{reg}}$, and μ a $(b, \Gamma\cdot\xi)$-density. Then there exists a constant $c_0 > 0$ such that for any $c > c_0$ there exists a constant $D(c) > 1$ with the property*

$$\frac{1}{D(c)} e^{-\sum_{i=1}^{r} b^i d_i(o,\gamma o)} \leqslant \mu_o(sh_o(B_{\gamma o}(c))) \leqslant D(c) e^{-\sum_{i=1}^{r} b^i d_i(o,\gamma o)}$$

for all $\gamma \in \Gamma$ such that $d(o, \gamma o) > c$ and $d(\gamma o, K\cdot\sigma_{o,\xi}) < c/3$.

From here on, we fix $c > 2\max\{c_\Gamma, c_0\}$ with c_Γ as in Definition 4.4 and c_0 as in Theorem 4.5.

THEOREM 4.6. — *Let $\Gamma \subset G$ be a radially cocompact Zariski dense discrete group, $\xi \in \partial X^{\mathrm{reg}}$ with Cartan projection $H_\xi \in \mathfrak{a}_1^+$, and μ a $(b, \Gamma\cdot\xi)$-density. Then there exists a constant $C_0 > 0$ such that for any Borel subset $E \subseteq L_\Gamma^{\mathrm{rad}}$*

$$\mathrm{Hd}^\alpha(E) \geqslant C_0 \cdot \mu_o(E), \quad \alpha = \sum_{i=1}^{r} b^i \langle H_i, H_\xi\rangle .$$

Proof. — Set $\alpha := \sum_{i=1}^{r} b^i\langle H_i, H_\xi\rangle$. Since $\mathrm{Hd}^\alpha(E) \geqslant \mathrm{Hd}^\alpha(E \cap G\cdot\xi)$ and $\mu_o(E) = \mu_o(E \cap G\cdot\xi)$, it suffices to prove the assertion for $E \subseteq L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi$. Let $\varepsilon > 0$, $s > 0$ arbitrary, and choose a cover of E by balls $B^c_{r_j}(\eta_j)$, $r_j < \varepsilon$, such that

$$\mathrm{Hd}^\alpha_\varepsilon(E) \geqslant \sum_{j\in\mathbb{N}} r_j^\alpha - s .$$

If $B^c_{r_j}(\eta_j) \cap L_\Gamma^{\mathrm{rad}} = \varnothing$, we do not need $B^c_{r_j}(\eta_j)$ to cover $E \subseteq L_\Gamma^{\mathrm{rad}} \cap G\cdot\xi$, otherwise we choose $\xi_j \in B^c_{r_j}(\eta_j) \cap E$. Since Γ is radially cocompact, there exists $\gamma_j \in \Gamma$ such that $d(\gamma_j o, \sigma_{o,\eta_j}(-\log r_j)) \leqslant c$. This implies $d_i(o, \gamma_j o) \geqslant d_i(o, \sigma_{o,\eta_j}(-\log r_j)) - c$, hence

$$-\sum_{i=1}^{r} b^i d_i(o, \gamma_j o) \leqslant \sum_{i=1}^{r} b^i(\langle H_i, H_\xi\rangle \log r_j + c) \leqslant \alpha \log r_j + c\|b\|_1 .$$

Furthermore, we have $B^c_{r_j}(\eta_j) \subseteq \mathrm{sh}_o(B_{y_j o}(2c))$, hence $E \subseteq \bigcup_{j\in\mathbb{N}} \mathrm{sh}_o(B_{y_j o}(2c))$. We conclude

$$\begin{aligned}
\mu_o(E) &\leqslant \mu_o(\bigcup_{j\in\mathbb{N}} \mathrm{sh}_o(B_{y_j o}(2c)) \leqslant \sum_{j\in\mathbb{N}} \mu_o(\mathrm{sh}_o(B_{y_j o}(2c)) \\
&\leqslant D(2c) \sum_{j\in\mathbb{N}} e^{-\sum_{i=1}^r b^i d_i(o,y_j o)} \leqslant D(2c) \sum_{j\in\mathbb{N}} e^{\alpha \log r_j + c\|b\|_1} \\
&\leqslant D(2c) e^{c\|b\|_1} \sum_{j\in\mathbb{N}} r_j^\alpha \leqslant D(2c) e^{c\|b\|_1} (\mathrm{Hd}^\alpha_\varepsilon(E) + s) .
\end{aligned}$$

The claim now follows as $s \searrow 0$ and $\varepsilon \searrow 0$. □

THEOREM 4.7. — *Let $\Gamma \subset G$ be a radially cocompact Zariski dense discrete group, $\xi \in \partial X^{\mathrm{reg}}$ and μ a $(b, \Gamma\cdot\xi)$-density constructed as in section 3.2. Then*

$$\dim_{Hd}(L^{\mathrm{rad}}_\Gamma \cap G\cdot\xi) = \delta_{G\cdot\xi}(\Gamma) .$$

Proof. — Let $\xi \in \partial X^{\mathrm{reg}}$ with Cartan projection $H_\xi \in \mathfrak{a}_1^+$, and μ a $(b, \Gamma\cdot\xi)$-density constructed as in section 3.2. From the previous theorem we deduce that for $\alpha = \sum_{i=1}^r b^i \langle H_i, H_\xi\rangle$

$$\mathrm{Hd}^\alpha(L^{\mathrm{rad}}_\Gamma \cap G\cdot\xi) \geqslant C_0 \mu_o(L^{\mathrm{rad}}_\Gamma) \geqslant 0 ,$$

hence $\dim_{Hd}(L^{\mathrm{rad}}_\Gamma \cap G\cdot\xi) \geqslant \alpha = \sum_{i=1}^r b^i \langle H_i, H_\xi\rangle \geqslant \delta_{G\cdot\xi}(\Gamma)$ by condition (*) of section 3.2. The assertion now follows directly from Theorem 4.3. □

Using the results of section 3.3, we deduce the following

COROLLARY 4.8. — *Let X be a globally symmetric space of noncompact type, and $\Gamma \subset Isom^o(X)$ a cocompact lattice. Then for any $\xi \in \partial X^{\mathrm{reg}}$ with Cartan projection $H_\xi \in \mathfrak{a}_1^+$ we have*

$$\dim_{Hd}(L^{\mathrm{rad}}_\Gamma \cap G\cdot\xi) = \rho(H_\xi) .$$

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