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SPECTRAL ASYMPTOTICS FOR SCHRÖDINGER OPERATORS WITH A DEGENERATE POTENTIAL

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Abstract

This paper is devoted to the asymptotic growth of the number of eigenvalues less than an energy λ of a Schrödinger operator $H_h = -h^2\Delta + V$ on $L^2(\mathbf{R}^m)$, in the case when the potential V does not fulfill the non degeneracy condition: $V(x) \to +\infty$ as $|x| \to +\infty$. For such a model, the point is that the set defined in the phase space by: $H_h \leqslant \lambda$ may no longer have an a finite volume, so that the Weyl formula has no sense.

We present a survey of various results in this area. in the classical context ($h = 1$ and $\lambda \to +\infty$), as well as in the semi-classical one ($h \to 0$) and comment the different methods. The last result presented deals with a general potential of the form: $V(x) = f(y)g(z)$, $f \in C(\mathbf{R}^n; \mathbf{R}_+^*)$, $g \in C(\mathbf{R}^p; \mathbf{R}_+)$, g homogeneous and $f \to +\infty$ as $|y| \to +\infty$.

1. Introduction

Let V be a nonnegative, real and continuous potential on $\mathbf{R}^m$, and h a parameter in $]0,1]$. The spectral asymptotics of the operator $H_h = -h^2\Delta + V$ on $L^2(\mathbf{R}^m)$ have been intensively studied. More precisely it is well known [7] that H_h is essentially selfadjoint with compact resolvent when $V(x) \to +\infty$ as $|x| \to +\infty$ (we shall say that V is non degenerate). Moreover, denoting by $N(\lambda, H_h)$ the number of eigenvalues less than a fixed energy λ, the following semiclassical asymptotics hold, as $h \to 0$:

$$N(\lambda, H_h) \sim h^{-m}(2\pi)^{-m} v_m \int_{\mathbf{R}^m} (\lambda - V(x))_+^{m/2} dx\,. \tag{1}$$

In this formula, v_m denotes the volume of the unit ball in $\mathbf{R}^m$, and the notation W_+ means the positive value of W.

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Let us note that the classical asymptotics are also given by the formula (1), provided we let $h = 1$ and $\lambda \to +\infty$.

In both cases, the result points out the asymptotic correspondance between the number of eigenstates with energy less than λ and the volume in phase space of the set $\{(x,\xi), f(x,\xi) \leqslant \lambda\}$, where $f(x,\xi) = \xi^2 + V(x)$ is the principal symbol of H_h.

In this paper we propose a review of results concerning the degenerate case: the potential V does not tend to infinity with $|x|$, so that the volume in phase space of the previous set may be infinite.

2. The Tauberian approach

Let us explain how the problem of the degenerate case came from the non degenerate one.

In 1950 De Wet and Mandl [3] proved the formula (1) in its classical version, provided $V(x) \geqslant 1$ and two more conditions on V:

1) a smoothness condition: V differentiable and $|\nabla V|/V \to 0$ as $x \to +\infty$

2) a Tauberian type condition: let $\Phi(V,\lambda) = \int_{\mathbf{R}^m} (\lambda - V(x))_+^{m/2} dx$; it is assumed that there exists strictly positive constants c and c' such that:

$$c\Phi(V,\lambda) \leqslant \lambda\Phi'(V,\lambda) \leqslant c'\Phi(V,\lambda)\,, \text{for all } \lambda \in \mathbf{R}_+.$$

The first condition is local and the second is global. This last condition was needed to use a Tauberian technique, which consists on studying the asymptotic behavior of the Green's function of the operator H_1 and applying a Tauberian theorem.

Refinements were done by Titchmarsh, Levitan and Kostjucenko, [12], [4], [5] and then Rosenbljum [9] proved that the formula (1) holds with "maximal" weakening conditions on V:

1) the smoothness condition is replaced by a condition on the "L^1-modulus of continuity" on unit cubes and by the following assumption: $V(y) \leqslant C'V(x)$ if $|x - y| \leqslant 1$.

2) the Tauberian type condition becomes: $\sigma(2\lambda,V) \leqslant C\sigma(\lambda,V)$ (for large λ), where $\sigma(\lambda,V)$ denotes the volume of the set $\{x \in \mathbf{R}^m; V(x) < \lambda\}$.

Solomyak [11] makes the following remark:

LEMMA 2.1. —*Let V be a positive a-homogeneous potential, so that for a given $a > 0$:*

$$V(x) \geqslant 0\,;\ V(tx) = t^a V(x) \ \textit{for any}\ t \geqslant 0\,.$$

If moreover $V(x)$ is strictly positive ($V(x) \neq 0$ if $x \neq 0$) the spectrum of H_1 is discrete and the formula (1) *takes the form:*

$$N(\lambda,H_1) \sim \gamma_{m,a}\lambda^{\frac{2m+am}{2a}} \int_{S^{m-1}} (V(x))^{-m/a} dx\,,\ \textit{as}\ \lambda \to +\infty$$

($\gamma_{m,a}$ is a constant depending only on the parameters m and a.)

From that lemma comes out naturally the idea of investigating the spectrum whithout the condition of strict positivity (and thus in a case of degeneracy of the potential); the two main results are [11]:

THEOREM 2.2. — *The formula of Lemma 2.1 still holds for a positive a-homogeneous potential such that* $J(V) = \int_{S^{m-1}} (V(x))^{-m/a} dx$ *is finite.*

The second result deals with a case where $J(V)$ is infinite:

Let $V(x) = F(y,z)$, $y \in \mathbf{R}^n$, $z \in \mathbf{R}^p$, $n + p = m$, $m \geqslant 2$, such that $F(sy,tz) = s^b t^{a-b} F(y,z)$ (with $0 < b < a$) and $F(y,z) > 0$ for $|z||y| \neq 0$. Denote by $\lambda_j(y)$ the eigenvalues of the operator $-\Delta_z + F(y,z)$ in $L^2(\mathbf{R}^p)$ and let $s = \frac{2b}{2+a-b}$, then:

THEOREM 2.3. — *If n/b, m/a are related as follows, then as* $\lambda \to \infty$

$n/b > m/a \quad N(\lambda,H_1) \sim \gamma_{n,s} \lambda^{n(2+a)/2b} \int_{S^{m-1}} \Sigma(\lambda_j(y))^{-n/s} dx;$

$n/b = m/a \quad N(\lambda,H_1) \sim \frac{a(a+2)}{2b(a-b)} \gamma_{m,a} \lambda^{m(2+a)/2a} \ln \lambda \int_{S^{n-1}S^{p-1}} F(y,z)^{-m/a} dx.$

The proof is based on variational techniques and spectral estimates proved in [9]. But on a heuristic level the result can be understood in the framework of the theory of Schrödinger operators with operator potential.

This last approach can be found in [8] where D.Robert extended the theory of pseudodifferential operators in the form developped by Hörmander to pseudodifferential operators with operator symbols. It was thus possible to study cases where the operator has a compact resolvent but the condition $\lim_\infty V(x) = +\infty$ is not fulfilled. As an example it gives the asymptotics of $N(\lambda,H_1)$ for the 2-dimensional potential $V(y,z) = y^{2k}(1 + z^2)^l$, where k et l are strictly positive. The asymptotics are the following:

THEOREM 2.4. — *If k,l are related as follows, then as* $\lambda \to +\infty$

$k > l \quad N(\lambda,H_1) \sim \gamma_1 \lambda^{(l+k+1)/2l};$

$k = l \quad N(\lambda,H_1) \sim \gamma_2 \lambda^{(2k+1)/2k} \ln \lambda;$

$k < l \quad N(\lambda,H_1) \sim \gamma_3 \lambda^{(2k+1)/2k}.$

The constants γ_i depend only on k and l, but the first one γ_1 takes in account the trace of the operator $(-\Delta_z + z^{2k})^{-(k+1)/2l}$ in $L^2(\mathbf{R})$.

In the 2-dimensional case let us mention the results of B. Simon [10]. He first recalls Weyl's famous result: let H be the Dirichlet Laplacian in a bounded region Ω in $\mathbf{R}^2$, then the following asymptotics hold:

$$N(\lambda,H) \sim \frac{1}{2}\lambda|\Omega|, \text{ as } \lambda \to \infty$$

and then he considers special regions Ω for which the volume (denoted by $|\Omega|$) is infinite but the spectrum of the Laplacian is still discrete. These regions are of the type: $\Omega_\mu = \{(y,z); |y||z|^\mu \leqslant 1\}$.

Actually the problem can be derived from the study of the asymptotics of Schrödinger operators with the homogeneous potential: $V(y,z) = |y|^\alpha |z|^\beta$.

In order to get these "non-Weyl" asymptotics, Simon uses the Feynman-Kac formula and the Karamata-Tauberian theorem, but the main tool is what he calls "sliced bread inequalities", which can be seen as a kind of Born-Oppenheimer approximation. More precisely let $H = -\Delta + V(y,z)$ be defined on $\mathbf{R}^{n+p}$, and denote by $\lambda_j(y)$ the eigenvalues of the operator $-\Delta_z + V(y,z)$ in $L^2(\mathbf{R}^p)$. (If the z's are electron coordinates and the y's are nuclear coordinates, the $\lambda_j(y)$ are the Born- Oppenheimer curves). Simon proves the following result:

$$\mathrm{Tr} e^{-tH} \leqslant \sum_j e^{-t(-\Delta_y + \lambda_j(y))}$$

(when the second term exists).

Thus he gets the two following coupled results:

THEOREM 2.5. — *If* $H = -\Delta + |y|^\alpha |z|^\beta$ *and* $\alpha < \beta$, *then as* $\lambda \to +\infty$

$$N(\lambda,H) \sim c_\nu \lambda^{(2\nu+1)/2} \qquad (\nu = (\beta+2)/2\alpha).$$

Corollary: *if* $H = -\Delta_{\Omega_\mu}$ $(\mu > 1)$, *then* $N(\lambda,H) \sim c_\mu \lambda^{1/(2\mu+1)}$ *as* $\lambda \to +\infty$.

THEOREM 2.6. — *If* $H = -\Delta + |y|^\alpha |z|^\alpha$, *then* $N(\lambda,H) \sim \frac{1}{\pi}\lambda^{1+\frac{1}{\alpha}} \ln \lambda$

Corollary: *if* $H = -\Delta_{\Omega_\mu}$ $(\mu = 1)$, *then* $N(\lambda,H) \sim \frac{1}{\pi}\lambda \ln \lambda$.

The constant c_μ depends only on μ, and the constant c_μ takes in account the trace of the operator $(-\Delta_z + |z|^\beta)^{-\nu}$ in $L^2(\mathbf{R})$.

3. The min-max approach

The result presented in this section is based on the method of Courant and Hilbert, the min-max variational principle. It turns out that this method can be applied to operators in $L^2(\mathbf{R}^m)$ with principal symbols which can degenerate on some non bounded manifold of $T^*(\mathbf{R}^m)$. It is the case for the Schrödinger operator with a magnetic field $H = (D_x - A(x))^2$, which degenerates on $\{(x,\xi) \in T^*(\mathbf{R}^m); \xi = A(x)\}$. If the magnetic field $B = dA$ fulfills the so-called magnetic bottle conditions (mainly: $lim_\infty \|B(x)\| = \infty$) the spectrum is discrete [1] and the classical asymptotics were established by Colin de Verdière [2] using the min-max method. The semiclassical version of the result is given in [13].

In [6], the min-max method is used to get semiclassical asymptotics for a large class of non degenerate potentials, namely potentials of the following form: $x = (y,z) \in \mathbf{R}^n \times \mathbf{R}^p$, $n + p = m$, $m \geqslant 2$

$$V(x) = f(y)g(z),\ f \in C(\mathbf{R}^n; \mathbf{R}_+^*),\ g \in C(\mathbf{R}^p; \mathbf{R}_+),$$

such that for any $t > 0$, $g(tz) = t^a g(z)$ (with $0 < a$) and $g(z) > 0$ for $z \neq 0$.

The spectrum of the operator $-\Delta_z + g(z)$ in $L^2(\mathbf{R}^p)$ is discrete and positive. Let us denote by μ_j its eigenvalues. It is straightforward to make the following remark:

Remark. — If $f(y) \to +\infty$ as $|y| \to +\infty$ then $H_h = -h^2\Delta + V$ has a compact resolvent.

Of course if f was supposed to be homogeneous, the asymptotics would be given by Theorem 2.3. Here the assumption on f is only a locally uniform regularity:

$$\exists\ b,c > 0 \text{ s.t. } c^{-1} \leqslant f(y) \text{ and } |f(y) - f(y')| \leqslant c f(y)|y - y'|^b,$$

for any y, y' verifying $|y - y'| \leqslant 1$.

THEOREM 3.1. — *Assume the previous conditions on f and g. Then there exists $\sigma,\tau \in]0,1[$ such that, for any $\lambda > 0$, one can find $h_0 \in]0,1[$, $C_1, C_2 > 0$ in order to have*

$$(1 - h^\sigma C_1) n_{h,f}(\lambda - h^\tau C_2) \leqslant N(\lambda; H_h) \leqslant (1 + h^\sigma C_1) n_{h,f}(\lambda + h^\tau C_2)\ ,\quad \forall h \in]0,h_0[$$

where $n_{h,f}(\lambda) = h^{-n}(2\pi)^{-n} v_n \displaystyle\int_{\mathbf{R}^n} \sum_{j \in \mathbf{N}} \left[\lambda - h^{2a/(2+a)} f^{2/(2+a)}(y)\mu_j\right]_+^{n/2} dy\ .$

If additional conditions are assumed to hold on f, the previous result can be refined as follows:

THEOREM 3.2. — *If moreover one can find a constant C_3 such that, for any $\mu > 1$:*

$$\int_{\{y,\, f(y) < 2\mu\}} f^{-p/a}(y)dy \leqslant C_3 \int_{\{y,\, f(y) < \mu\}} f^{-p/a}(y)dy\ ,$$

then one can take $C_2 = 0$ in Theorem 3.1:

$$(1 - h^\sigma C_1) n_{h,f}(\lambda) \leqslant N(\lambda; H_h) \leqslant (1 + h^\sigma C_1) n_{h,f}(\lambda) \quad \forall h \in]0,h_0[$$

Remark. — If moreover $f^{-p/a} \in L^1(\mathbf{R}^n)$ and $g \in C^1(\mathbf{R}^p \setminus \{0\})$, then the formula (1) holds.

The proof of Theorem 3.1 uses a suitable covering of $\mathbf{R}^n$, so that the min-max variational principle applies to the Dirichlet and Neumann problems in cylinders for the restrained operator (with a fixed y). The proof of Theorem 3.2 is based on an asymptotic

formula of the moment of eigenvalues of $-h^2\Delta_z + g(z)$, which is again obtained using the min-max principle.

As a conclusion, let us notice that if there is some information on the growth of f, then the asymptotics can be computed in terms of power of h:

Remark. — Suppose there exists $k > 0$ and $C > 0$ such that $\frac{1}{C}|y|^k \leqslant f(y) \leqslant C|y|^k$ for $|y| > 1$. If k, a are related as follows, then as $h \to 0$:

$k > a \quad N(\lambda, H_h) \approx h^{-m}$;

$k = a \quad N(\lambda, H_h) \approx h^{-m} \ln 1/h$;

$k < a \quad N(\lambda, H_h) \approx h^{-n-pa/k}$.

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