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TWO LECTURES ON SPECTRAL INVARIANTS FOR THE SCHRÖDINGER OPERATOR

Mikhail V. NOVITSKII

Abstract

An introduction into spectral invariants for the Schrödinger operators with periodic and almost periodic potentials is given. The following problems are considered: a description for the fundamental series of the spectral invariants, a completeness problem for these collections, spectral invariants for the Hill operator as motion integrals for the KdV equation, a connection of the spectrum of the periodic multidimensional Schrödinger operator with the spectrum of a collection of the Hill operators obtained by averaging of the potential over a family of closed geodesic on a torus, the direct and inverse problems. Some open problems are formulated.

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Introduction

Spectral invariants as functionals on the collection of the Schrödinger operator with periodic potential appear in a natural way in investigations of the direct and inverse problems for these operators. Most interesting are functionals defined in terms of polynomials of a potential and finite number of its derivatives. We will call them *polynomial* spectral invariants. The most known collection of such functionals are the Minakshisundaram-Pleijel coefficients as a collection of the coefficients of the complete asymptotic expansion of the trace of the fundamental solution for the parabolic equation associated with the Schrödinger operator. In the same way, the Minakshisundaram-Pleijel coefficients can be introduced for the Schrödinger operator on smooth compact manifolds [30].

It turns out that even if a manifold is a circle, the theory of spectral invariants for the 1-D Schrödinger operator with periodic potential (the Hill operator) is not trivial. First of all, this is because the Minakshisundaram-Pleijel coefficients are the motion integrals of the Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx}$$

(see [11], [34]) and, as a result, have some additional algebraic properties. In particular, they are solutions of some moment problems on the real axis. The KdV equation presents the *first* nontrivial case of continuous isospectral deformations for the Hill operator. Other isospectral deformations for the Hill operator are related to the high order KdV equations and their linear combinations. If a potential is an almost periodic function, then the theory of spectral invariants becomes much more complicated, however some results can be proven in this case, too.

Spectral invariants for the multi-dimensional Schrödinger operator with periodic potential is a subject of Lecture 2. Some new effects like spectral rigidity appear, but as compared to the one-dimensional case the general situation is far from the end. Essential part of results are stated for analytic potentials only. No classification is known yet for potentials which permit continuous isospectral deformations. One of the interesting results is a connection of the spectrum of the periodic multi-dimensional Schrödinger operator with the spectrum of a collection of the Hill operators whose potentials are obtained from the potential by averaging over a family of closed geodesic on a torus. Some open problems are formulated.

1. Spectral invariants for 1-D Schrödinger operators with periodic and almost periodic potentials

1.1. Periodic case - the Hill operator

1.1.1. Spectral theory. The first step.

Let $u(x)$ be an infinitely differentiable function with the period one, i.e., $u(x+1) = u(x)$. The Hill operator is the Schrödinger operator with the periodic potential $u(x)$:

$$Hy = -\frac{d^2y}{dx^2} + u(x)y.$$

In the space $L^2(-\infty,\infty)$, this operator is selfajoint and bounded from below, its spectrum is absolutely continuous, has multiplicity two and is the union of intervals

$$\sigma(H) = \bigcup_{k=0}^{\infty} [\lambda_{2k},\lambda_{2k+1}]$$

divided by the gaps $(\lambda_{2k+1},\lambda_{2k+2})$, $k = 0,1,2,\dots$ For some potentials, the number of the gaps is finite. For example, if $u = \frac{n(n+1)}{2}P(x)$ with $P(x)$ the Weierstrass function, then the spectrum of this operator has exactly n gaps. For the potential $u = \cos 2\pi x$, "all" the gaps are open.

In the space $L^2[0,2]$, consider another operator H_{per} defined by $Hy = -\frac{d^2y}{dx^2} + u(x)y$ with the periodic boundary conditions $y(0) = y(2)$, $y'(0) = y'(2)$. What is a relation between $\sigma(H)$ and $\sigma(H_{per})$? The answer to this question is

LEMMA 1.1.

$$\sigma(H_{per}) = \{\lambda_k\}_{k=0}^{\infty} \tag{1}$$

Sketch of the proof. — The Hill discriminant can be defined by the equation

$$\Delta(\lambda,u) = \frac{1}{2}[c(\lambda,1) + s'(\lambda,1)],$$

where $s(\lambda,x),c(\lambda,x)$ is the fundamental system of solutions of the equation $Hy = \lambda y$ with the initial conditions $s(\lambda,x) = c'(\lambda,0) = 0$, $s'(\lambda,0) = c(\lambda,0) = 1$. The central point of the proof is the following description of the spectrum $\sigma(H_{per})$ and $\sigma(H)$ in the terms of $\Delta(\lambda,u)$:

$$\sigma(H) = \{\lambda : |\Delta(\lambda,u)| \leqslant 1\} \tag{2}$$

and

$$\sigma(H_{per}) = \{\lambda : |\Delta^2(\lambda,u)| - 1 = 0\}. \quad (3)$$

The function $\Delta(\lambda,u)$ is an entire function of order $\frac{1}{2}$ and the function $\Delta^2(\lambda,u) - 1$ may be expressed as a canonical product formed by the roots $\lambda_k, k = 1,2,\ldots$:

$$\Delta^2(\lambda,u) - 1 = c\prod_{k=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_k}\right), \quad (4)$$

where the constant c is known. The proof follows from (1),(2),(3) and (4).

The meaning of this Lemma is very simple: if we know the spectrum H we can find H_{per} and vice versa. For details of the proof and other information on the spectral theory of the Hill operator see [25], [26], [27], [28], [45].

Remark. — In the space $L^2[0,1]$, consider the operator H_θ defined by $Hy = -\frac{d^2y}{dx^2} + u(x)y$ with the *Floquet boundary conditions* $y(0) = e^{i\theta}y(1), y'(0) = e^{i\theta}y'(1)$. The eigenvalues of this problem are the roots of the equation

$$\Delta(\lambda,u) - \cos\theta = 0. \quad (5)$$

The union of the eigenvalues of the periodic ($\theta = 0$) and antiperiodic ($\theta = \pi$) spectral problems is exactly the collection $\{\lambda_k, k = 1,2,\ldots\}$. If the spectrum $\sigma(H_{\theta_1})$ is known, then one can find the discriminant $\Delta(\lambda,u)$ and, as a result, any spectrum $\sigma(H_\theta), \theta \in (0,\pi]$.

1.1.2. Spectral invariants for the Hill operator.

Let us introduce

DEFINITION 1.2.

*a) Two Hill's operators H_1 and H_2 are called **isospectral Hill's operator** if* $\lambda_i(H_1) = \lambda_i(H_2), i = 0,1,2,\ldots$.

*b) The functional $Q(u)$ is called a **spectral invariant** of H if $Q(u)$ has the following property: if the spectra of the operators $H_i = -\frac{d^2}{dx^2} + u_i(x), i = 1,2$, coincide, then $Q(u_1) = Q(u_2)$.*

DEFINITION 1.3. — *We call a system of functionals $\{Q_\alpha\}, \alpha \in \Omega$, **a complete system of spectral invariants** for the Hill operator $H = -\frac{d^2}{dx^2} + u(x)$, if the equality $Q_\alpha(u_1) = Q_\alpha(u_2)$ for all values of $\alpha \in \Omega$ implies that the spectra of the operators $H_i = -\frac{d^2}{dx^2} + u_i(x), i = 1,2$, coincide, and conversely.*

Examples.

i) the eigenvalues $\lambda_k(H), k = 0,1,2,\ldots$ and the functionals $Q(\lambda_0,\ldots,\lambda_N)$, where Q is an arbitrary function of $N+1$ variables, are spectral invariants of the Hill operator;

ii) the collection of functionals

$$Q(\lambda,u) = \Delta(\lambda,u), \quad \lambda \in C$$

is a *complete system.* It follows from (4).

iii) the Hill discriminant $\Delta(z,u)$ has the representation

$$\Delta(z^2,u) = \cos(\psi(z,u)), \tag{6}$$

where the function $\psi(z,u)$ is a conformal mapping of the upper half-plane $C_+ = \{z : \operatorname{Im} z > 0\}$ onto a region of the form

$$C_+ \setminus \bigcup_{k=-\infty}^{\infty} \{[k\pi, k\pi + ih_k]\}$$

where $h_k = h_{-k}$, $h_0 = 0$(see [26]). The function $\psi(z)$ is normalized by the conditions $\psi(0) = 0$ and $\lim_{y\to\infty} \Theta(iy)/iy = 1$ and uniquely determined by the collection $\{h_k\}$. *The collection of functionals* $\{\psi(z,u), \quad \operatorname{Im} z > 0\}$ *and the collection* $\{h_k\}_{k=-\infty}^{\infty}$ *are complete systems.*

iv) the trace of the fundamental solution of the parabolic equation associated with the Hill operator is the collection of the functionals

$$Q(t,u) = \Theta(t,u) = \sum_{i=0}^{\infty} e^{-\lambda_i t}, \quad t \in [0,\infty].$$

The collection $\{\Theta(t,u), t > 0\}$ *is a complete system.*

v) the *Minakshisundaram-Pleijel coefficients* is a collection of the coefficients of the asymptotic expansion of $\Theta(t,u)$:

$$\Theta(t,u) \sim \frac{1}{\sqrt{4\pi t}}\left(1 + \sum_{k=1}^{\infty} c_k I_k(u) t^k\right), \qquad t \downarrow 0.$$

Here $c_k = (-1)^{k+1}2^k/(2k-1)!!$, $k = 1,2,..$ The functionals $\{I_n\}_{n=0}^{\infty}$ are spectral invariants for the Hill operator H. Generally, this collection is not complete.

DEFINITION 1.4. — *A potential* $u_1(x)$ *is called* ***resonance*** *if there exists another potential* $u_2(x)$ *such that* $I_n(u_1) = I_n(u_2)$ *for all* $n = 1,2,..$ *but the spectra of the operators* $H_i = -\frac{d^2}{dx^2} + u_i(x)$, $i = 1,2$, *are different.*

Analytic potentials are not resonance. About that and about a criterion for the collection $\{I_n\}_{n=0}^{\infty}$ to be complete, see Theorem 1.9.

A description of the fundamental series of spectral invariants for the Hill operator.

The counting number for discrete series of spectral invariants for the Hill operator can be introduced using the generalized Jacobi formula

$$\Theta(t,u) = \sum_{n=-\infty}^{\infty} \Theta_n(t,u).$$

Here $\Theta_n(t,u)$ has the form

$$\Theta_n(t,u) = \int_0^1 e(t,x,x+n)\,dx = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{n^2}{4t}\right) F_n(t,u). \tag{7}$$

The function $e(t,x,y)$ is a fundamental solution of the equation $\partial e/\partial t = He$ on the whole line. According to the Kac-Feynman formula, the function $F_n(t)$ admits the representation

$$F_n(t,u) = \int_0^1 \mathrm{M}\left[\exp(-t\int_0^1 u(x+n\tau+\sqrt{t}w(\tau))\,d\tau)\right]dx. \tag{8}$$

Here M is the mathematical expectation of a random variable, $w(\tau)$ is the "Wiener bridge", which is defined as a one-dimensional conditional Wiener process determined by the following conditions:

i) $w(0) = w(1) = 0$;

ii) $w(\tau)$ is a Gaussian process with the zero mean and a correlation function $B(s,t) = (s \wedge t)(1 - s \vee t)$.

LEMMA 1.5 ([44], [26]). — *For each n the set of functionals $F_n(t,u)$, $n \in Z$, $t \geqslant 0$, constitutes a complete system of spectral invariants.*

Proof follows from the formula

$$\int_0^\infty e^{\lambda t}\Theta_n(t,u)\,dt = \frac{-\Delta'(\lambda)}{\sqrt{\Delta^2(\lambda)-1}}\left[\frac{\Delta(\lambda)-\sqrt{\Delta^2(\lambda)-1}}{2}\right]^{|n|}.$$

According to (6), we can rewrite the previous equality as

$$\int_0^\infty e^{\lambda t}\Theta_n(t,u)\,dt = \frac{1}{|n|2^{|n|}}\frac{d}{d\lambda}e^{i|n|\psi(\lambda)}, \quad n \neq 0.$$

Hence $F_n(t,u)$ defines $\psi(\lambda)$ and the Hill discriminant $\Delta(\lambda,u)$.

DEFINITION 1.6. — *By an n-series of the spectral invariants for the Hill operator we will mean the set of values $\{I_k^n\}_1^\infty$ given by the asymptotic expansion*

$$F_n(t,u) \sim 1 + \sum_{k=1}^{\infty} I_k^n(u)t^k, \qquad t \downarrow 0.$$

It follows from (8) that

$$I_k^n = \int_0^1 P_{k,n}(u,u',\dots)\,dx,$$

where $P_{k,n}$ is a polynomial in u and its derivatives. The polynomial spectral invariants I_k^n are our main subject of consideration.

Explicit formulas for calculating the coefficients of n - series from the set $\{I_k^n\}_1^\infty$ can be presented.

Theorem 1.7 ([38], [39]). — *The coefficients of an n- series $\{I_k^n\}_{k=1}^\infty$ can be expressed in terms of the coefficients of the 0-series by the relation*

$$I_k^n = \sum_{m=1}^{k} \frac{1}{n^{2(k-m)}} I_k^0, \qquad k = 1,2,\dots$$

This implies that the collections $\{I_k^n\}_0^\infty$ with different n are equivalent and it is sufficient to investigate the collection $\{I_k^0\}$ only. Define a new collection $\{I_n\}$ by the formula

$$I_n = \frac{(-1)^{n+1}(2n-1)!!}{2^n} I_{n+1}^0, \qquad n = 0,1,2,\dots$$

The collection I_n has the following properties:

i) I_n is given by the formula

$$I_n = \int_0^1 \sigma_{2n+1}(u,u',\dots)\,dx,$$

where $\sigma_n(u,u',..)\,dx$ are universal polynomials of degree n that have no constant terms and are defined by the recurrence relation

$$\sigma_{k+1} = -\sigma_k' - \sum_{j=1}^{k-1} \sigma_{k-j}\sigma_j, \quad k = 1,2,\dots$$

with the condition $\sigma_1 = u(x)$. Using this recurrence relation we get

$$I_0 = \int_0^2 u(x)\,dx, \quad I_1 = \int_0^2 u^2(x)\,dx, \quad I_2 = \int_0^2 (u^3 + (u')^2/2)\,dx, \ \dots$$

and so on.

ii) Let $\psi(z,u)$ be the "comb-like" comformal mapping from (6). The function $\psi(z,u)$ has the representation

$$\psi(z,u) = z + \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z}, \tag{9}$$

where the measure $d\sigma(t)$ has the form $d\sigma(t) = \frac{1}{\pi}\,\mathrm{Im}\,\psi(t)\,dt$ (see [27]). The sequence $\{I_n\}_{n=1}^{\infty}$ coincides with the coefficients of the asymptotic expansion

$$\psi(z) \sim z - \sum_{n=0}^{\infty} \frac{I_n}{z^{2n+1}}, \quad z = iy, \quad y \to +\infty. \tag{10}$$

By the Hamburger-Nevanlinna theorem (see [1]) and (10), the coefficients I_n are a solution to the moment problem on the whole line:

$$I_n = \int_{-\infty}^{\infty} t^{2n}\, d\sigma(t). \tag{11}$$

A criterion for the collection $\{I_n\}_{n=0}^{\infty}$ to be a complete system.

Finite gap potentials are not resonance: the periodic spectrum can be recovered from the collection $\{I_n\}_{n=0}^{\infty}$ in a unique way. The same statement is true for real analytic potentials. Let $\{m_n\}_{n=1}^{\infty}$ be a fixed sequence of positive numbers. We assume with no loss of generality that $\{m_n\}_{n=1}^{\infty}$ grows faster than any power of n and the sequence $\{\ln m_n\}_{n=1}^{\infty}$ is convex with respect to n.

DEFINITION 1.8. — *The Carleman class $C(m_n)$ is the class of all C^{∞} one -periodic functions u such that*

$$||u^{(n)}||_{L^2} \leqslant C^n(u)m_n, \quad n = 0,1,2,\ldots,$$

where the constant C depends on u.

The class $C(m_n)$ is called *quasianalytic* if it possesses the following property: each function $u \in C(m_n)$ satisfying $u^{(k)}(x_0) = 0$ for all $k = 0,1,2,\ldots$ at a fixed point x_0 is identically zero. Any function from a quasianalytic class can be recovered by its Taylor coefficient at any fixed point.

THEOREM 1.9 ([40]).

1. Let $C(m_n)$ be a quasianalytic class. Then for every two potentials $u_1 \in C(m_n)$ and $u_2 \in C(m_n)$ the equalities $I_k(u_1) = I_k(u_2)$, $k = 0,1,2,\ldots$, imply $\lambda_k(u_1) = \lambda_k(u_2)$, $k = 0,1,2,\ldots$.

2. Let $C(m_n)$ be a nonquasianalytic class. Then there exist two potentials $u_1 \in C(m_n)$ and $u_2 \in C(m_n)$ such that $I_k(u_1) = I_k(u_2)$, $k = 0,1,2,\ldots$, but the spectra of the periodic boundary value problems in $L^2[0,2]$ for the Hill operators with these potentials are different.

Remarks.

1. It follows from this theorem that resonance potentials exist in nonquasianalytic classes only.

2. According to Theorem 1.7, all n-series of spectral invariants of the Hill operator are equivalent to each other. Hence Theorem 1.9 is true if we change $\{I_n\}$ into each of the n- series $\{I_k^n\}_{k=0}^{\infty}$ (n being fixed).

1.1.3. Isospectral set.

For any periodic potential u, we denote by $\mathrm{ISO}(u)$ the set of potentials having the same spectrum as u. What is the structure of an isospectral set $\mathrm{ISO}(u)$?

THEOREM 1.10 ([46]). — *Isospectral set* $\mathrm{ISO}(u)$ *of the Hill operator* H *with potential* u *is topologically equivalent to the torus* T^N, *where* N *is the number of the gaps in the spectrum* H :

$$\mathrm{ISO}(u) \approx T^N. \tag{12}$$

Sketch of the proof. — Consider the collection of operators

$$H_t y(x) = \Big[-\frac{d^2}{dx^2} + u(x+t)\Big] y(x), \qquad t \in [0,1]$$

in the space $L^2[0,1]$. Let $\{\mu_k(t)\}_{k=1}^{\infty}$, $t \in [0,1]$ be the auxiliary spectrum of the operator H_t generated by the equation $H_t y = \mu y$ and the boundary conditions $y(0) = y(1) = 0$. Then the first trace formula for the Hill operator is

$$u(t) = \lambda_0 + \sum_{k=1}^{+\infty} [\lambda_{2k} + \lambda_{2k-1} - 2\mu_k(t)]. \tag{13}$$

It follows from (13) that

$$u(x) \iff \{\{\lambda_k\}_{k=0}^{\infty}, \{\mu_k(t)\}_{k=0}^{\infty}, \quad t \in [0,1]\}. \tag{14}$$

The collection (14) is overdetermined: to recover a function $u(x)$ we have to know a countable collection of the functions $\{\mu_k(t)\}_{k=0}^{\infty}$. The functions $\{\mu_k(t)\}_{k=1}^{\infty}$ are not arbitrary, they are solutions of some system of ordinary differential equations. They are known as the Dubrovin-Trubowitz equations. These equations can be deduced in the following way (see [46], [3]). Consider the Green function $g(s,t,\lambda)$ of the operator $H = -\frac{d^2}{dx^2} + u(x)$ in $L^2(-\infty,\infty)$. The diagonal function $g(t,t,\lambda)$ is well defined if λ belongs to a gap $(\lambda_{2k+1}, \lambda_{2k+2}), k = 0,1,2,\ldots$ of the operator H. The resolvent equation for $g(t,t,\lambda)$ implies

$$\frac{d}{d\lambda} g(t,t,\lambda) = \int_{-\infty}^{\infty} g^2(t,t+s,\lambda)\,ds > 0$$

and, as a result, this function grows from $-\infty$ to ∞ on each gap. The value $\mu_k(t)$ is exactly the intersection point of this function with the real line, i.e.,

$$g(t,t,\mu_k(t)) = 0, \quad t \in [0,1].$$

Differentiation of this equation with respect t implies

$$\frac{\partial g}{\partial t}(t,t,\mu_k(t)) + \frac{d\mu_k(t)}{dt}\frac{\partial g}{\partial \mu}(t,t,\mu_k(t)) = 0, \quad t \in [0,1]$$

and consequently

$$\frac{d\mu_k(t)}{dt} = -\frac{\frac{\partial g}{\partial t}(t,t,\mu_k(t))}{\frac{\partial g}{\partial \mu}(t,t,\mu_k(t))}.$$

A more detailed analysis of $g(t,t,\lambda)$ shows that

$$\frac{\partial g}{\partial t}(t,t,\mu_k(t)) = \pm 1 = -\sigma_k(t),$$

where ± 1 corresponds to the left and right half-line Dirichlet eigenvalue. Finally, the system for $\mu_k(t)$ is

$$\frac{d\mu_k(t)}{dt} = \frac{\sigma_k(t)}{\frac{\partial g}{\partial \mu}(t,t,\mu_k(t))}.$$

This system is autonomous, and the existence and uniqueness theorems can be proven. Furthermore, if we fix some point $t_0 \in [0,1]$, then

$$u(x) \Longleftrightarrow \left\{\{\lambda_k\}_{k=0}^{\infty}, \{\mu_k(t_0)\}_{k=1}^{\infty}, \{\sigma_k(t_0)\}_{k=1}^{\infty}\right\}.$$

Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_N), 1 \leqslant N \leqslant \infty$, be the coordinates of the torus T. Define the constants

$$\zeta_k = (\lambda_{2k} - \lambda_{2k-1})/2, \quad \xi_k = (\lambda_{2k} + \lambda_{2k-1})/2$$

The point $\varphi \in T$ is defined by

$$\zeta_k \cos\varphi_k = \mu_k(t_0) - \xi_k, \quad \lambda_{2k-1} \leqslant \mu_k \leqslant \lambda_{2k},$$

$$0 < \varphi_k < \pi, \quad \text{for} \quad \sigma_k = 1, \quad \pi \leqslant \varphi_k < 2\pi, \quad \text{for} \quad \sigma_k = -1$$

There is no ambiguity in φ_k when $\mu_k = \lambda_{2k-1}$ or λ_{2k}. Hence,

$$u(x) \Longleftrightarrow \varphi, \quad \varphi \in T^N.$$

Remark. — The global geometry of the isospectral torus. Global geometry properties of isospectral sets M corresponding to infinite gap potentials are considered in [29]. By *the tangent space T_u at the point u* we mean the closed span in $L^2[0,1]$ of the system of the functions

$$\frac{d}{dx}\frac{\delta\Delta(\lambda,u)}{\delta u}, \qquad \lambda \in R.$$

CONJECTURE 1 (H.P. McKean, E. Trubowitz [29]). — *The motion integrals $\{I_n(u)\}_{n=0}^{\infty}$ determine the isospectral set M_u if and only if the local tangent vectors*

$$V_n(u) = \frac{d}{dx}\frac{\delta I_n}{\delta u}, \quad n = 0,1,2,\ldots,$$

span the tangent space T_u.

The following assertion holds:

THEOREM 1.11 ([40]). — *The two properties are equivalent:*

1) for every isospectral set M in $C(m_n)$ and for every $u \in M$, the span of the set of the local tangent vectors $\{V_n(u)\}_{n=0}^{\infty}$ coincides with the whole T_u;

2) the class $C(m_n)$ is quasianalytic.

It follows from Theorem 1.9 and 1.11 that *the conjecture is true for every quasianalytic class $C(m_n)$.*

1.1.4. The KdV equation and isospectral deformations.

One of the problems of the spectral geometry related to the Schrödinger operator on a manifold is an analysis of the set of operators having the same spectrum. Even in the case if the manifold is a circle, we have a nontrivial problem. We are interested in a construction of a continuous collection of operators $H(t) = -\frac{d^2}{dx^2} + u(x,t), t \in [0,\infty]$ with the same spectrum: $\sigma(H_{t_1}) = \sigma(H_{t_2})$, where t_1 and t_2 are from $[0,\infty]$. A trivial example of an isospectral deformation is the collection of operators

$$H(t) = -\frac{d^2}{dx^2} + u(x+t), \quad t \in R.$$

Nontrivial cases appear from

THEOREM 1.12 ([11]). — *Consider the initial value problem for the KdV equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} = 6uu_x - u_{xxx}, \quad t \in [0,\infty),\ x \in (-\infty,\infty) \\ u(x,0) = u_0(x), \end{cases} \tag{15}$$

where the function $u_0(x)$ is periodic with period 1. Then:

1) there exists a unique solution of equation (15) in the class of periodic function with period 1;

2) the spectra of the operators

$$H(t) = -\frac{d^2}{dx^2} + u(x,t),$$

where $u(x,t)$ is a solution of the KdV equation (15), do not depend upon t:

$$\sigma(H(t)) = \sigma(H(0)).$$

To prove this theorem, we use the Lax representation of the KdV equation and formulate the following general lemma.

LEMMA 1.13 ([22],[23]). — *Let $H(t)$ and $A(t)$ be collections of selfajoint and skew-selfajoint operators in a Hilbert space respectively:*

$$H(t) = H^*(t), \quad A(t) = -A^*(t).$$

Suppose that they satisfy the equation

$$\frac{d}{dt}H(t) = [A(t),H(t)] = A(t)H(t) - H(t)A(t). \tag{16}$$

Then there exists a unique collection of unitary operators $U(t)$, $t \in (0,\infty)$, which present $H(t)$ in the form

$$H(t) = U^*(t)H(0)U(t) \tag{17}$$

and consequently $H(t)$ is an isospectral deformation of the operator $H(0)$.

To prove the theorem, we set

$$H(t) = -\frac{d^2}{dx^2} + u(x,t), \quad A(t) = -4\frac{d^3}{dx^3} + 6u\frac{d}{dx} + 3u_x. \tag{18}$$

Direct calculation shows that in this case $[A(t),H(t)]$ is an operator of order zero. More exactly, it is the operator of multiplication by the function $6uu_x - u_{xxx}$ and the KdV equation is equivalent to equation (16).

Remark. — Representation of the KdV equation in the form $\frac{d}{dt}H(t) = [A(t),H(t)]$, where $A(t)$ and $H(t)$ are of the form (18) is known as the *Lax representation of the KdV equation.*

Theorem 1.12 has the following consequences:

i) the eigenvalues of the Hill operator are the motion integrals of the KdV equation:

$$\lambda_k(u(x,t)) = \lambda_k(u(x,0)), \quad k = 0,1,2,\ldots; \tag{19}$$

ii) the spectral invariants $I_k, k = 0,1,2,\ldots$ are the motion integrals of the KdV equation.

The KdV equation can be rewritten in the form

$$u_t = \frac{\partial}{\partial x}\frac{\delta I_2}{\delta u}. \tag{20}$$

Here $I_2(u) = \int_0^2 (u^3 + \frac{(u')^2}{2})dx$ and, as a result, $\frac{\delta I_2}{\delta u} = 3u^2 - u_{xx}$.

One can consider higher order KdV equations (see [34]),

$$u_t = \frac{\partial}{\partial x}\frac{\delta I_n}{\delta u}, \quad n = 3,4,\ldots \tag{21}$$

All these higher order KdV equations also admit Lax representation with a common operator $H(t) = -\frac{d^2}{dx^2} + u(x,t)$ and an appropriate skew-adjoint operator $A_n(t)$:

$$\frac{d}{dt}H(t) = [A_n(t),H(t)] \Leftrightarrow u_t = Q(u,\dots,u^{(2n+1)}). \tag{22}$$

A more general case

$$u_t = \sum_{k=2}^{N} c_k \frac{\partial}{\partial x}\frac{\delta I_k}{\delta u}, \quad N = 2,3,4,\dots, \tag{23}$$

where $c_k, k = 2,3,\dots,N$ are arbitrary constant, can also be considered. As a result, a rich collection of isospectral deformations of the Hill operator can be constructed using the spectral invariants $I_k, k = 2,3,4,\dots$ The potentials of these isospectral operators are solutions of some partial differential equations (formulas (22) and (23)).

1.2. Almost periodic case

1.2.1. The rotation number and other functionals.

i) *Definition of almost periodic function.*

Definition 1.14. — *Let u be a bounded and continuous function on R. We call the function $u(x), t \in R$, Bohr – almost periodic (a.p.) if the set of its translates $\{u_s(x) = u(x+s))\}$ is relatively compact, i.e., for any sequence of points $s_k \in R, k = 1,2,\dots$ the sequence of functions $\{u_{s_k}\}$ has a subsequence which converges uniformly in R.*

Trigonometric polynomials $u(x) = \sum_{k=1}^{N} c_k e^{\xi_k x}$, where $\xi_k \in R, k = 1,2,\dots,N$ are almost periodic. A function is almost periodic if and only if it is the uniform limit of a sequence of trigonometric polynomials. Obviously, every continuous periodic function is almost periodic. The functional

$$\mathbf{E}_x(u) = \lim_{N\to\infty} \frac{1}{2N}\int_{-N}^{N} u(x)\,dx$$

is defined on the class of almost periodic functions and called the *mean value* of u. There exists only a countable set $\{\xi_k, k = 1,2,\dots,N, \quad N \leqslant \infty\}$ of numbers ξ for which $E_x(u(x)e^{\xi_k x}) \neq 0$. The *frequency module* is the set

$$M(u) = \Big\{\sum_{k=1}^{N} n_k\xi_k, n_k \in Z\Big\}$$

of finite integer combinations of these frequencies.

ii) *Rotation number* [15]. Let $u(x)$ be an a.p. function of the class $C^\infty(R)$, let

$$H = -\frac{d^2}{dx^2} + u(x)$$

be the one - dimensional Schrödinger operator in $L^2(R)$. The nature of the spectrum of the Schrödinger operator with almost periodic potential is very complicated and in the decomposition

$$\sigma(H) = \sigma(H)_{pp} \cup \sigma(H)_{sc} \cup \sigma(H)_{ac}$$

"almost all" combinations of the spectral components $\sigma(H)_{pp}, \sigma(H)_{sc}, \sigma(H)_{ac}$ can appear (see [10]).

DEFINITION 1.15. — *The functional $Q(u)$ is called a spectral invariant on the class of the Schrödinger operators with almost periodic potentials H if $Q(u)$ has the following property: if the operators $H_i = -\frac{d^2}{dx^2} + u_i(x)$, $i = 1,2$, are unitary equivalent, then $Q(u_1) = Q(u_2)$.*

Trivial examples of the spectral invariants are the points of the spectrum and the multiplicity of the spectrum. Another collection of spectral invariants can be constructed in the following way.

Let $G(x,y,z)$ be the Green function of the operator H, where z belongs to the resolvent set of the operator $zI - H$. The function $G(x,x,z) \neq 0$ is an a.p. function (see [15]). Thus, consider *the function of the rotation number (or the rotation number)*

$$w(z,u) = \mathbf{E}_x \left(\frac{-1}{2G(x,x,z)} \right).$$

The rotation number has some interesting analytic properties. We will say that a function $f(z)$ holomorphic on $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ belongs to the *Nevanlinna class N* if it maps $\mathbb{C}_+$ into itself. The function $w(z,u)$ belongs to the class of holomorphic functions on $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ and satisfies *the 3-time Nevanlinna property*

$$w(z,u) \in N, \quad -iw(z,u) \in N, \quad w'(z,u) \in N.$$

For the periodic Schrödinger operator (the Hill operator), the function of rotation number $w(z,u)$ is connected to a "comb-like" mapping $\psi(z,u)$ by the relation $w(z,u) = i\psi(\sqrt{z},u)$, $\operatorname{Im} z > 0$.

The function $w(z,u)$ can be defined in the different way. Let $y(x,\lambda)$ be any non-zero solution of $H(u)y = \lambda y$, then $y'(x,\lambda) + iy(x,\lambda)$ does not vanish for any x. For any real λ we define

$$\alpha(\lambda,u) = \lim_{x\to\infty} \frac{1}{x} \arg (y'(x,\lambda) + iy(x,\lambda)). \tag{24}$$

This limit exists and is independent of the particular solution chosen. The functional $\alpha(\lambda,u)$ measures the average increase of the angle in the (y',y)-plane. $\alpha(\lambda,u)$, as function of λ, can be extended to some harmonic function in the upper-half plane. It turns out that this harmonic function is the imaginary part of the rotation number $w(z,u)$, $\operatorname{Im} z > 0$. Hence

$$\alpha(\lambda,u) = \lim_{\varepsilon\to 0} \operatorname{Im} w(\lambda + \varepsilon,u), \quad \lambda \in R.$$

iii) *Integrated density of states and the Lyapunov exponent.*

Operator $H = -d^2/d^2x + u(x)$ with a.p. potential $u(x)$ has the spectral decomposition

$$H = \int_{\lambda_0}^{\infty} \lambda dE_\lambda, \quad \lambda_0 = \inf_{\lambda\in\sigma(H)} \{\lambda\},$$

where the spectral projector E_λ is an integral operator with a jointly continuous and uniformly bounded kernel $e(x,y,\lambda)$. The *integrated density of states* is defined as

$$N(\lambda) = \mathbf{E}_x(e(x,x,\lambda)). \tag{25}$$

$N(\lambda)$ can be find by means of the following procedure. Consider the spectral problem

$$\begin{cases} -y'' + u(x)y = \lambda y & x \in [-T,T], \\ y(-T) = y(T) = 0 & . \end{cases} \tag{26}$$

Let $\lambda_k(T), k = 1,2,\dots$ be the spectrum of (26). Then it can be proven that

$$N(\lambda) = \lim_{T\to\infty} \frac{\#\{\lambda_k(T) \leqslant \lambda\}}{2T} \tag{27}$$

and

$$N(\lambda) = \frac{\alpha(\lambda)}{2\pi}. \tag{28}$$

For large values of x the envelope of $y(x,\lambda)$, where $y(x,\lambda)$ is a solution of equation $Hy = \lambda y$, behaves as $e^{\pm\gamma(\lambda)|x|(1+o(1))}$. The nonnegative quantity $\gamma(\lambda)$ is called *the Lyapunov exponent.* In terms of the rotation number:

$$\gamma(\lambda) = -\operatorname{Re} w(\lambda + i0). \tag{29}$$

It is known (see, for example,[10], [21])that:

a) the set of the points of growth of the integrated density of states $N(\lambda)$ coincides with the spectrum $\sigma(H)$ of the operator H;

b) the absolutely continuous spectrum $\sigma_{ac}(H)$ coincides with the essential closure with respect to the Lebesgue measure of the points of R where Lyapunov exponent $\gamma(\lambda)$ equals zero:

$$\sigma_{ac}(H) = ess\overline{\{\lambda : \gamma(\lambda) = 0\}}.$$

1.2.2. Spectral invariants.

i) *The rotation number, the integrated density of states $N(\lambda)$ and Lyapunov exponent $\gamma(\lambda)$ are spectral invariants of the Schrödinger operator.*

The mean of the Green functions is related to the rotation number by

$$w'(z,u) = \mathbf{E}_x G(x,x,z), \tag{30}$$

so $w(z,u)$ determines the collection of the spectral invariants of the Schrödinger operator H : two unitary equivalent Schrödinger operators with almost periodic potentials u_1 and u_2 have the same rotation number $w(z,u_1) = w(z,u_2)$. From (24) and (29) it follows that $N(\lambda)$ and $\gamma(\lambda)$ are spectral invariants of the Schrödinger operator as well.

Let $u(x)$ and all its derivatives $u^{(k)}(x)$ be almost periodic functions. Then the function $w(z,u)$ admits an asymptotic expansion of the form

$$w(z,u) \sim \sqrt{-z}\left(1 - \sum_{n=0}^{\infty} \frac{I_n(u)}{z^{n+1}}\right), \qquad z \to -\infty,$$

and defines the collection $\{I_n\}_0^\infty$ of spectral invariants of the Schröodinger operator.

The function $-iw(z^2,u)$ belongs to the Nevanlinna class N, i.e., it is a holomorphic function on $C_+ = z : \operatorname{Im} z > 0$ and maps the half-plane $\operatorname{Im} z > 0$ into itself. By the Hamburger- Nevanlinna theorem [1], the coefficients I_n are a solution of the moment problem on the whole line:

$$I_n = \int_{-\infty}^{\infty} t^{2n} d\,\sigma(t), \quad n = 0,1,2,\dots$$

where $d\,\sigma(t) = \gamma(t)d\,t$. Hence *the problem of unique recovery of the rotation number $w(z,u)$ from the set $\{I_n\}_0^\infty$ is equivalent to the problem of unique solvability of this moment problem in the class of absolutely continuous measures with the Lyapunov exponents as the densities.*

ii) *The problem of unique recovery of the rotation number $w(z,u)$ from the collection $\{I_n\}_0^\infty$.*

Let us formulate a criterion for recovery the rotation number in terms of the growth of the sequence $\{I_n\}_0^\infty$.

Let $\{m_n\}_{n=1}^\infty$ be a fixed sequence of positive numbers as in the periodic case, i.e., we assume that $\{m_n\}_{n=1}^\infty$ grows faster than any power of n and the sequence $\{\ln m_n\}_{n=1}^\infty$ is convex with respect to n. Define the standard object of theory of quasianalytic classes, the function

$$T(r,m_n) = \sup_{n\geqslant 1} \frac{r^n}{m_n}.$$

In the set of a.p. functions of the class $C^\infty(R)$ we distinguish the family of functions

$$\Omega(\{m_n\}) = \{u \in C^\infty(R), u \text{ is an a.p.f. } |I_n(u)| \leqslant C(u)m_n\}$$

and its subset $\Omega(\{m_n\},M)$ consisting of those a.p.f. whose Fourier exponents are contained in the module M of the Fourier exponents of some fixed a.p.f.

THEOREM 1.16 ([35]). — *In order that in the class $\Omega(\{m_n\})$ implication*

$$\forall u_1,u_2 \in \Omega(\{m_n\}), I_n(u_1) = I_2(u_2), n = 0,1,2,\dots \Rightarrow w(z,u_1) = w(z,u_2)$$

hold, it is necessary and sufficient that

$$\int_0^{\infty} \frac{\ln T(r,m_n)}{1+r^{\frac{3}{2}}}\, dr = +\infty. \tag{31}$$

Similar assertion holds also for the subset $\Omega(\{m_n\},M)$.

Hence, *in the general case the spectral invariants* $\{I_n\}_{n=0}^{\infty}$ *do not form a complete collection of spectral invariants of the Schrödinger operator* H.

Corollary 1.17. — *On the basis of the collection* $\{I_n\}_0^{\infty}$ *in the classes* $\Omega(\{m_n\})$ *and* $\Omega(\{m_n\},M)$, *the spectrum* $\sigma(H)$ *and the absolutely continuous spectrum* $\sigma_{ac}(H)$ *of the operator* H *as a closed subset of* R *can be uniquely recovered if and only if the integral from (31) is divergent.*

Sketch of the proof. — Let u_1 and u_2 be two potentials which have the same collection of the functionals $\{I_n\}_{n=1}^{\infty}$ ($I_n(u_1) = I_n(u_2), n = 0,1,2,\dots$). We get that for $\operatorname{Im} z \geqslant \delta > 0$ the function $\varphi(z) = -i[w(z^2,u_1) - w(z^2,u_2)]$ satisfies the system of inequalities

$$|z^{2n}\varphi(z)| \leqslant Cm_n, n = 0,1,2,\dots$$

and the problem can be reduced to the classical Carleman theorem. The divergence of the integral in (31) implies $\varphi(z) = 0$. Hence $w(z,u_1) = w(z,u_2)$.

In the proof of the second part of Theorem 1.16 it suffices to restrict attention to the classes $\Omega(\{m_n\})$ of the Hill operators. Consider the periodic potentials of the class $C^{\infty}(R)$ with period 1 such that the lengths of the gaps $\lambda_{2k} - \lambda_{2k-1}, k = 1,2,\dots$ in the spectrum of the operator $H = -d^2/dx^2 + u$ in $L^2(R)$ are open and have the asymptotic

$$\lambda_{2k} - \lambda_{2k-1} \sim [T(k^2)]^{-1/2}, k = 1,2,\dots$$

We prove that if

$$\int_0^{\infty} \frac{\ln T(r)}{1+r^{3/2}}\, d\,r < \infty,$$

then every potential u_1 from this class is resonance, i.e., there exists $u_2 \in \Omega(\{m_n\})$ such that $I_n(u_1) = I_n(u_2), n = 0,1,2,\dots$, but $w(z,u_1) \neq w(z,u_2)$. Here we use the Marchenko - Ostrovskii theorem [27] on characterization of the Hill discriminant in the class of entire functions.

1.2.3. Open problems.

1) To decide whether two almost periodic potentials u_1 and u_1 for which

$$w(z,u_1) = w(z,u_2), \operatorname{Im} z > 0$$

give rise to unitary equivalent operators

$$H_1 = -d^2/dx^2 + u_1 \text{ and } H_2 = -d^2/dx^2 + u_2.$$

In other words, is the collection of functionals $w(z,u), z \in C_+$, a complete system of spectral invariants in the class of the Schrödinger operators with almost periodic potentials?

2) To construct the Schrödinger operator with an almost periodic potential and pure point spectrum. To study its spectral invariants and the isospectral deformations.

2. Spectral invariants for the Schrödinger operator with periodic potential: multi-dimensional case

2.1. Spectral theory. The Bette-Sommerfeld conjecture

Consider an n-dimensional lattice L in R^n and an infinitely differentiable potential $V(x)$ on R^n satisfying the periodicity condition

$$V(x+d) = V(x), \quad \forall d \in L.$$

For such potentials one has the eigenvalue problems parameterized by $k \in R^n$,

$$\begin{cases} -\Delta y(x) + V y(x) = \lambda y(x) \\ y(x+d) = e^{2\pi i (k,d)} y(x) \qquad \forall x \in R^n, \forall d \in L. \end{cases} \tag{32}$$

The eigenvalues of (32), with multiplicities, are denoted by $\lambda_i(k), i = 1,2,\ldots$, and the spectrum by $\sigma(H_k)$.

DEFINITION 2.1. — *The Floquet spectrum $\sigma_F(H)$ of the Schrödinger operator $H = -\Delta + V$ is the collection of the functions $\lambda_i(k), i = 1,2,\ldots$ for all $k \in R^n$.*

In the physics literature, $\sigma_F(H)$ and the collection $\lambda_i(k), i = 1,2,\ldots$ are known as the *Bloch spectrum* and *the band functions*. The Floquet spectrum $\sigma_F(H)$ is an overdetermined system. The collection $\{\lambda_i(0), i = 1,2,\ldots\}$ $(k = 0)$ corresponds to the *periodic spectrum*, $\sigma(H_0)$ is the spectrum of the operator H on the torus R^n/L. We denote this spectrum by $\sigma_0(H)$.

Let $\sigma(H)$ be the spectrum of the Schrödinger operator $H = -\Delta + V$ in $L^2(R)$. It is known (see [42]) that H is the direct sum of the operators H_k and, as a result,

$$\sigma(H) = \bigcup_{k \in R^n/L^*} \sigma(H_k), \tag{33}$$

where L^* is the lattice dual to L. Using (33) it can be proven (see, for example, [42]) that the spectrum $\sigma(H)$ is absolutely continuous and, as in the one-dimensional case, is the union of intervals divided by gaps.

Bette and Sommerfeld conjectured that in the multi-dimensional case the spectrum $\sigma(H)$ contains only *finite number of gaps*, i.e.

$$\sigma(H) = \bigcup_{i=1}^{N-1} [a_i,b_i] \cup [a_N,\infty).$$

This conjecture was proven in [43] with some restrictions on L and in [47] for the general case (see also [20]).

2.2. Spectral invariants

2.2.1. Definitions and problems.

DEFINITION 2.2.

a) A functional $Q(u)$ *is called* ***the Floquet spectral invariant of*** H *if* $Q(u)$ *has the following property:*

$$\sigma_F(H_1) = \sigma_F(H_2) \quad \Longrightarrow \quad Q(u_1) = Q(u_2);$$

b) A functional $Q(u)$ *is called* ***a spectral invariant of periodic problem*** *if* $Q(u)$ *has the following property:*

$$\sigma_0(H_1) = \sigma_0(H_2) \quad \Longrightarrow \quad Q(u_1) = Q(u_2).$$

The following problems are interesting for us:

- Let us suppose that we know the Floquet spectrum $\sigma_F(H)$. What is known about complete collections of the Floquet spectral invariants? Does the Floquet or periodic spectrum determine the potential $u(x)$ up to isometries of R^n/L?
- Does the periodic spectrum $\sigma_0(H)$ determine the Floquet spectrum $\sigma_F(H)$, i.e., are the eigenvalues $\lambda_i(k), i = 1,2,\dots$ the spectral invariant of periodic problem for $\forall k \in R^n$?

2.2.2. Floquet spectral invariants.

1) *Complete collections of the Floquet spectral invariants.*

In this subsection the main results are proven without restriction on the lattice L. Let $e(t,x,y)$ be a fundamental solution of the heat equation $\partial e/\partial t = He$ on R^n. Denote

$$\Theta_d(t) = \int_{R^n/L} e(t,x,x+d)\,dx.$$

Using the trace formula for the heat equation we obtain

$$\sum_{m=0}^{\infty} e^{-\lambda_m(k)t} = \sum_{d\in L} e^{-2\pi i(k,d)}\Theta_d(t). \tag{34}$$

The right-hand side of (34) is the Fourier expansion of the function with respect to the variable k. Hence, if we know the whole collection $\lambda_m(k), m = 1,2,\dots, k \in R^n$ then the coefficients $\Theta_d(t)$ of the expansion (34) are known as well.

Hence, we have

LEMMA 2.3 ([4]). — *The collection*

$$\Theta_d(t) = \int_{R^n/L} e(t,x,x+d)dx, \quad t > 0, \quad d \in L$$

is a complete collection of the Floquet spectral invariants.

The Kac-Feynman formula for a fundamental solution of the parabolic equation associated with Schrödinger operator implies that $\Theta_d(t)$ has the representation

$$\Theta_d(t) = \frac{e^{\frac{-|d|^2}{4t}}}{(4\pi t)^2} \int_{R^n/L} \mathrm{M}\left[\exp\left(-t\int_0^1 u(x+d\tau+\sqrt{t}w(\tau))d\tau\right)\right] dx. \tag{35}$$

Then $\Theta_d(t)$ admits for $t \downarrow 0$ the asymptotic expansion

$$\Theta_d(t) \sim \frac{e^{\frac{-|d|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}\left(\sum_{k=0}^{\infty} I_k^d t^k\right), \qquad t \to 0. \tag{36}$$

DEFINITION 2.4. — *We call the coefficients I_k^d of the expansion (36) the generalized Minakshisundaram-Pleijel coefficients for the operator $H = -\Delta + V$.*

From Lemma 2.3 and (36) it follows that $\{I_k^d, k = 1,2,\dots, d \in L\}$ is a collection of the Floquet spectral invariants.

THEOREM 2.5 ([33], [32]). — *The generalized Minakshisundaram-Pleijel coefficients $\{I_n^d\}_0^\infty, d \in L, n = 0,1,2,\dots,$ form a complete system of the Floquet spectral invariants in the class of real-analytic potentials on R^n/L.*

The proof of this theorem is based on the transformation formula

$$\Theta_d(t) = \frac{2}{\sqrt{4\pi t}} \int_0^\infty \exp\left\{-\frac{s^2}{4t}\right\} E_d(s)ds,$$

where

$$E_d(s) = \int_{R^n/L} E(s,x,x+d)dx, \tag{37}$$

and $E(s,x,y)$ is the fundamental solution of the hyperbolic problem

$$\frac{\partial^2 E}{\partial s^2} = HE = -\Delta E + VE,$$

$$E|_{s=0} = \delta_x(y), \frac{\partial E}{\partial s}|_{s=0} = 0, \quad s \geqslant 0, \quad x,y \in R^n.$$

The properties of the function $E_d(s)$ for the real analytic potential V are described in the following lemma.

LEMMA 2.6. — *The function $E_d(s)$ admits the representation*

$$E_d(s) = \chi_{(|d|,\infty)}(\Phi_d(s) + \varphi_d(s))$$

Here $\Phi_d(s)$ is a distribution of the form

$$\Phi_d(s) = \alpha_0^d \frac{d}{ds}\Gamma_{-\frac{1}{2}}(s^2 - |d|^2) + \alpha_1^d I_1^d \frac{s}{\sqrt{s^2 - |d|^2}},$$

$\varphi_d(s)$ is an analytic function for $s \geqslant |d|$. The function $\varphi_d(s)$ has at most an exponential growth for $s \to \infty$ and has an expansion

$$\varphi_d(s) = \sum_{n=2}^{\infty} \alpha_n^d I_n^d (s^2 - |d|^2)^n, \qquad s \downarrow |d|.$$

The distribution $\Gamma_\lambda(s)$ is defined as an analytic extension of the function $s_+^\lambda/\Gamma(\lambda+1)$ for $\mathrm{Re}\,\lambda > 0$. *The sequence $\{\alpha_n^d\}_{n=1}^{\infty}$ is a sequence of universal constants which do not depend upon V.*

Remark. — The structure of the coefficients $\{I_k^d\}_0^\infty$ has been studied sufficiently well in the one-dimensional case (see Lecture 1). Here the main series $\{I_k^0\}_0^\infty$ coincides with the polynomial series of the first integrals I_n of the KdV equation while the remaining coefficients $\{I_k^d\}_0^\infty, d \neq 0$, are linear combinations of the main series $\{I_k^0\}_0^\infty$ (Theorem 1.7). Whether the higher coefficients $\{I_k^d\}_0^\infty$ for $n \geqslant 2$ reduce to the coefficients of the main series $\{I_k^0\}_0^\infty$ is unknown. We suppose that they do not.

2.2.3. Explicit expressions for the polynomial Floquet spectral invariants.

Using (35) we prove

LEMMA 2.7 ([32]). — *The following computational formulas hold:*

$$I_n^d = \int_{R^n/L} P_n^d(V, DV, \ldots)\,dx.$$

Here $P_n^d, d \in L$ are polynomials in the function V and its derivatives:

$$P_n^d(V, DV, \ldots) = \sum_{2r_2+3r_3+\cdots+2nr_{2n}=2n} \mathrm{M} \prod_{l=2}^{2n} \frac{\xi_{l-2,d}^{r_l}}{r_l!},$$

where $\{r_i\}$ *are all possible collections of nonnegative integers satisfying*

$$2r_2 + 3r_3 + \cdots + 2nr_{2n} = 2n,$$

and $p = (p_1, p_2, \ldots, p_n)$,

$$\xi_{l,d}(x) = -\sum_{p_1+p_2+\cdots+p_n=l} \int_0^1 \frac{\partial^{p_1+p_2+\cdots+p_n} V}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}} (x+\tau d) \omega_1^{p_1}(\tau) \omega_2^{p_2}(\tau) \cdots \omega_n^{p_n}(\tau) \, d\tau.$$

It is convenient to compute the coefficients $\{I_k^d\}_0^\infty$ in terms of the Fourier transform of the potential V. We present explicit expressions for the first four coefficients $I_n^d, n = 0,1,2,3$, in each d -series, $d \in L$.

LEMMA 2.8 ([6] [32]). — *Let* L^* *be the lattice dual to L, and let*

$$V(x) = \sum_{y \in L^*} c_y \exp\{2\pi i (y,x)\}.$$

Suppose the functions $\{V_d(x)\}, d \in L$ *are defined by*

$$V_d(x) = \sum_{(y,d)=0} c_y \exp\{2\pi i (y,x)\}.$$

Then

$$I_0^d = Vol R^n/L, \qquad I_1^d = -\int_{R^n/L} V(x)\,dx = -c_0,$$

$$I_2^d = \frac{1}{2} \sum_{(y,d)=0} c_y^2 = \frac{1}{2} \int_{R^n/L} V_d^2(x)\,dx, \tag{38}$$

$$I_3^d = \frac{1}{6} \int_{R^n/L} V_d^3(x)\,dx - \frac{1}{24} \int_{T^2} |DV_d|^2(x)\,dx + \frac{1}{2} \sum_{(y,d)\neq 0} \frac{c_y^2 |y|^2}{(d,y)^2}.$$

Proof of this Lemma follows from direct calculations and uses some exact formulas for the Wiener bridge moments.

COROLLARY 2.9. — *The collection*

$$\{d : d \in L, \quad V_d \equiv 0\} \tag{39}$$

is a collection of the Floquet spectral invariants.

COROLLARY 2.10. — *Let L be a rectangular lattice and potential V be separable:*

$$V(x) = \sum_{k=0}^{n} V_k(x_k).$$

Then all potentials Floquet isospectral to V are also separable.

Proof follows from the formula for the coefficient I_2^d.

2.2.4. Does the Floquet spectrum determine the potential up to isometries of torus?.

The central result here is an assertion regarding the connection of the spectrum of H with the class of the Hill operators whose potentials are obtained from V by averaging over a family of closed geodesic on T^2. To each element d of the lattice L we assign the family of geodesic $y = x + \tau d, 0 \leqslant \tau \leqslant 1, y \in R^n/L$. Introduce the functions

$$V_d(x) = \int_0^1 V(x+\tau d)d\tau = \sum_{(\gamma,d)=0} c_\gamma \exp\{2\pi i(\gamma,x)\}.$$

We shall call $V_d(x), d \in L$ the *reduced* potentials.

Theorem 2.11 ([6], [31]). — *The Floquet spectrum $\sigma_F(-\Delta+V)$ determines for arbitrary $d \in L$ the Floquet spectrum $\sigma_F(-\Delta+V_d)$ for reduced potential V_d.*

Proof (see [32]) is based on the following formula:

$$\lim_{k\to+\infty} \frac{\Theta_{\tilde{d}+kd}(t,V)}{\Theta_{\tilde{d}+kd}(t,0)} = \frac{\Theta_{\tilde{d}}(t,V_d)}{\Theta_{\tilde{d}}(t,0)}.$$

To prove this formula we note that according to (35) for any $k \geqslant 0$ we have

$$\begin{aligned}\frac{\Theta_{\tilde{d}+kd}(t,V)}{\Theta_{\tilde{d}+kd}(t,0)} &= \Theta_{\tilde{d}+kd}(t,V)\exp\left(\frac{l^2_{\tilde{d}+kd}}{4t}\right) \qquad (40)\\ &= \int_{R^n/L} d\,x\mathrm{M}\exp\left\{-t\int_0^1 u(x+(\tilde{d}+kd)\tau+\sqrt{t}w(\tau)))d\tau\right\}\\ &= \int_{R^n/L} d\,x\mathrm{M}\exp\left\{-t\sum_{(\gamma,d)=0} c_\gamma\int_0^1 \exp\{2\pi i(\gamma,x+\tilde{d}\tau+\sqrt{t}w(\tau))\}d\tau\right\}\\ &\quad\times\exp\left\{-t\sum_{(\gamma,d)\neq0} c_\gamma\int_0^1 \exp\{2\pi i(\gamma,x+\tilde{d}\tau+kd\tau+\sqrt{t}w(\tau))\}d\tau\right\}.\end{aligned}$$

Because of the rapid oscillation of the factor $\exp\{2\pi ik(\gamma,d)\tau\}$ for $(\gamma,d) \neq 0$ and the continuity of $w(\tau)$, we have

$$\sum_{(\gamma,d)\neq0} c_\gamma\int_0^1 \exp\{2\pi i(\gamma,x+\tilde{d}\tau+kd\tau+\sqrt{t}w(\tau))\}d\tau \to 0$$

as $k\to\infty$. The second factor in (40) tends to 1 as $k\to\infty$ and the desired limit coincides with the function $\Theta_{\tilde{d}}(t,V_d)/\Theta_{\tilde{d}}(t,0)$. As a result, the collection $\Theta_{\tilde{d}}(t,V_d)$ for arbitrary $\tilde{d} \in L$ is known. The Floquet spectrum $\sigma_F(-\Delta+V_d)$ for the reduced potential V_d can be find using (34).

Definition 2.12. — *A potential $Q \in L^2(R^n/L)$ is said to be a one-dimensional potential associated to q in the direction $\gamma \in L^*$ if there exists $\gamma \in L^*$ and a function q on R*

such that

$$Q = q(\gamma \cdot x) \tag{41}$$

for all $x \in R^n$.

Let

$$\Phi = \{\gamma \in L^* : \exists d_0 \in L, \quad \gamma \cdot d_0 = 1\}.$$

Φ is the set of indices parameterizing all possible lines in L^* passing through the point $\gamma = 0$. Given the condition $c_0 = 0$, the potential V can be expressed in terms of the collection of the one-dimensional potentials $\{V_\gamma\}$ by the formula

$$V(x) = \frac{1}{2} \sum_{g \in \Phi} V_\gamma(x),$$

where

$$V_\gamma(x) = \sum_{n=-\infty}^{+\infty} c_{n\gamma} e^{2\pi i \gamma \cdot x}.$$

We assign to the function V_γ, $\gamma \in \Phi$, the function $v_\gamma(t)$ defined on the circle of the length 1 and having the same Fourier coefficients as $V_\gamma(x)$:

$$v_\gamma(t) = \sum_{n=-\infty}^{+\infty} c_{n\gamma} e^{2\pi i n t}.$$

The function $V_\gamma(x)$ is the one-dimensional potential associated to $v_\gamma(t)$ in the direction $\gamma \in L^*$, i.e. $V_\gamma(x) = v_\gamma(\gamma \cdot x)$. The operator

$$h_\gamma = -|\gamma|^2 \frac{d^2}{d t^2} + v_\gamma(t)$$

is the Hill operator.

THEOREM 2.13 ([31], [6] (on quasireduction)). — *The Floquet spectrum* $\sigma_F(-\triangle + V)$ *determines:*

1) the Floquet spectrum $\sigma_F(-\triangle + V_\gamma)$ *of the one-dimensional potentials* $V_\gamma(x)$, $\gamma \in \Phi$;

2) the Floquet spectrum $\sigma_F(h_\gamma)$ *of the Hill operators* h_γ $\gamma \in \Phi$.

Proof.

1) For arbitrary $\gamma \in \Phi$ one can find linearly independent $d_1, d_2, \ldots, d_{n-1} \in L$ such that $\gamma \cdot d_k = 0, k = 1, 2, \ldots, n-1$. Reducing the potential V in directions $d_1, d_2, \ldots, d_{n-1}$ we obtain from Theorem 2.11 the first statement of our theorem.

2) The Floquet spectrum $\sigma_F(h_\gamma)$ of the Hill operators h_γ can be exactly calculated in terms of the Floquet spectrum $\sigma_F(-\triangle + V_\gamma)$ of the one-dimensional potentials $V_\gamma(x)$ and visa verse. ☐

In the class of separable potentials the Floquet spectrum of the Hill operators h_γ, $\gamma \in \Phi$ determines the Floquet spectrum $\sigma_F(-\Delta + V)$. In general case it is not true. It follows from

THEOREM 2.14 ([7]). — *There is a set M of potentials, dense in $C^\infty(R^2/L)$ and such that for $V \in M$ the Floquet isospectral set $\mathrm{ISO}^F(V)$ is finite up to isometries of R^2/L.*

COROLLARY 2.15. — *The potentials $V \in M$ do not admit any continuous isospectral deformations.*

2.2.5. Does the periodic spectrum determine the Floquet spectrum?.

There are trivial cases when the periodic spectrum $\sigma_0(H)$ does not determine the Floquet spectrum $\sigma_F(H)$. For example, if a lattice L is preserved under an orthogonal transformation $U \neq \pm I$ of R^n then the periodic potentials $V(x)$ and $V(Ux)$ have the same periodic spectrum but the Floquet spectrum are different.

Denote

$$[d] = \{d' \in L : |d'| = |d|\}, \quad \Theta_{[d]} = \sum_{d' \in [d]} \Theta_{d'}.$$

LEMMA 2.16. — *Let V be a real - analytic function on R^n/L. Then each term of the sum*

$$\Theta(t) = \sum_{[d]} \Theta_{[d]}(t)$$

can be recovered from $\Theta(t)$.

Proof. — We have

$$\Theta_d(t) = \frac{2}{\sqrt{4\pi t}} \int_0^\infty \exp\left\{-\frac{s^2}{4t}\right\} E_d(s) d\, s. \tag{42}$$

Here the function $E_d(s)$ is defined by formula (37). It is sufficient to prove that each term of the sum

$$E(t) = \sum_{[d]} E_{[d]}(t)$$

can be recovered from $E(t)$. The detailed analysis of the analytic wave front set of the fundamental solution $E(s,x,y)$ of a hyperbolic problem shows that $E(s,x,y)$ is a real – analytic function on the complement of $\{(t,x,y) : |x-y| = |t|\}$ and, as a consequence, the function $E_d(t)$ vanishes for $|t| < |d|$ and is a real analytic function for $|t| > |d|$. Then the distribution

$$E_d(t) = \int_{R^n/L} E(s,x+d,x) d\, x$$

vanishes for $|t| < |d|$ and is a real analytic function of the variable t for $|t| > |d|$. We can label the vectors of L as

$$0 = |[0]| < |[d_1]| < \ldots |[d_k]| < \ldots.$$

Denote $t_k = |[d_k]|$. On the interval $(0,t_1)$ we have $E(t) = E_{[0]}(t)$. The function $E_{[0]}(t)$ is analytic on $(0,\infty)$ and, hence, can be uniquely extended from the interval $(0,t_1)$ to $(0,\infty)$. On the interval (t_1,t_2) we have $E(t) = E_{[0]}(t) + E_{[d_1]}(t)$. The function $E_{[d_1]}(t)$ is analytic on (t_1,∞) and hence can be uniquely extended from the interval (t_1,t_2) to (t_1,∞). Using similar arguments we can find all the distributions $E_{[d]}(t)$.

The lemma is proved. □

THEOREM 2.17 ([6]). — *Let u_1 and u_2 be real analytic functions and a lattice L has the property*

$$|d| = |d'| \Rightarrow d = \pm d'. \tag{43}$$

Then the periodic spectrum $\sigma_0(H)$ determines the Floquet spectrum $\sigma_F(H)$.

For the *proof* we note that if the lattice L satisfies (43) then $[d] = \{d, -d\}$ and $E_d = E_{-d}$. Then $E_{[d]}(t) = 2E_d$ and each $E_d, d \in L$ can be recovered from $E(t)$. Using (42) we get that each $\Theta_d, d \in L$ can be recovered from $\Theta(t)$.

Theorems 2.11, 2.13 and 2.17 imply

THEOREM 2.18 ([31], [7]). — *Let u be the analytic and a lattice L have the property (43). Then the periodic spectrum $\sigma_0(-\triangle + V)$ determines:*

1) the periodic spectrum $\sigma_0(-\triangle + V_d)$ of the reduced potentials $V_d(x)$, $d \in L$;

2) the periodic spectrum $\sigma_0(-\triangle + V_\gamma)$ of the one-dimensional potentials $V_\gamma(x)$, $\gamma \in \Phi$;

3) the periodic spectrum $\sigma_0(h_\gamma)$ of the Hill operators h_γ, $\gamma \in \Phi$.

Denote by $\mathrm{ISO}_0(V)$ the set of potentials with the same periodic spectrum as the potential V. For $n = 1$ the following statement can be proven [40]: if a set $\mathrm{ISO}_0(V)$ contains a potential from a Carleman class $C(m_n)$, then all potentials from $\mathrm{ISO}_0(V)$ belong to the same class $C(m_n)$. In particular, if V is analytic then $\mathrm{ISO}_0(V)$ contains analytic functions only. In the multi-dimensional case $n \geqslant 2$ it is still an open problem: to prove that if a set $\mathrm{ISO}_0(V)$ contains at least one analytic potential then all potentials from $\mathrm{ISO}_0(V)$ are analytic.

DEFINITION 2.19. — *An analytically rigid potential V is a periodic analytic potential with the property: the intersection of $\mathrm{ISO}_0(V)$ with the class of analytic functions is exactly the collection $V(\pm x + a)$, $a \in R^n$.*

It is known

THEOREM 2.20 ([5]). — *Let u be real analytic and a lattice L have the property (43). Then the set of analytically rigid potentials is dense in the set of smooth potentials on R^2/L in C^∞ topology.*

The condition (43) coincides in dimension two with a generally less restrictive condition that the only isometries of R^n/L are the compositions of translations with $\pm I$. Nevertheless, it is surprising that

THEOREM 2.21 ([14], [13]). — *There exist a lattice $L_1 \subset R^4$ such that the only isometries of R^n/L_1 are the compositions of translations with $\pm I$ and real analytic potentials u_1 and u_2 on R^4/L_1 such that*

$$\sigma_0(H_1) = \sigma_0(H_1) \quad but \quad \sigma_F(H_1) \neq \sigma_F(H_1).$$

The tori R^4/L_1 were first constructed by J.H. Conway and N.J.A. Sloane [2] in the construction of flat tori in dimension four which are isospectral but not isometric.

2.3. Spectral rigidity theorem

Consider spectral invariants of the Schrödinger operator on the torus with coupling constant by potential. The operator

$$H(\beta) = -\Delta + \beta V(x)$$

is defined on the space $L^2([-a_1,a_1] \times [-a_2,a_2])$. The function $V(x_1,x_2)$ is a periodic function with respect to the variable x_1 with the period $2a_1$ and to the variable x_2 with the period $2a_2$; $\{\lambda_k(\beta)\}_{k=1}^{\infty}$ is the periodic spectrum of the operator $H(\beta)$. We consider *the problem of unique recovery of the potential $V(x_1,x_2)$ from the set $\{\lambda_k(\beta)\}_{k=1}^{\infty}, \beta \geqslant 0$,* and solve this problem in the class of even functions with respect to x_1 and x_2. The set $\{\lambda_k(\beta)\}_{k=1}^{\infty}, \beta \geqslant 0$, is overdetermined. Hence, instead of this set we consider the expansion

$$sp[R_\lambda(\beta) - R_\lambda(0)] = \beta I_1(\lambda) + \beta^2 I_2(\lambda) + \beta^3 I_3(\lambda) + \ldots, \beta \downarrow 0. \tag{44}$$

The constants $\{I_k(\beta)\}, k = 1,2,3,\ldots$ are spectral invariants of the operator $H(\beta), \beta \geqslant 0$. Let us suppose that we know the first three coefficients of this expansion. What partial information about the potential can we get? We prove

THEOREM 2.22 ([37] (the spectral rigidity theorem)). — *Let H^0 be the Hilbert space of even square integrable functions on the rectangle $L^2([-a_1,a_1] \times [-a_2,a_2])$ where a_1 and a_2 are incommensurable positive numbers. Let $V(x_1,x_2)$ be a function of H^0. Let us consider in the space H^0 the operator $H(\beta) = -\Delta + \beta V(x)$. Suppose the three coefficients $I_1(\lambda), I_2(\lambda), I_3(\lambda)$ in the expansion (44) are known. Then all other potentials with the same three coefficients are the superposition of this potential and the reflections:*

$$1)\ U_1 : x \to x,\quad y \to y; \qquad 2)\ U_2 : x \to x,\quad y \to a_2 - y$$
$$3)\ U_3 : x \to a_1 - x,\quad y \to y; \qquad 4)\ U_4 : x \to a_1 - x,\quad y \to a_2 - y.$$

The sets $\{\lambda_k(\beta)\}_{k=1}^{\infty}$, $\beta \geqslant 0$, and $sp[R_\lambda(\beta) - R_\lambda(0)]$ are equivalent. Then we have

COROLLARY 2.23. — *Let $V(x_1,x_2)$ be an even function of x_1 and x_2 on the rectangle $[0,a_1] \times [0,a_2]$. Then the set $\{\lambda_k(\beta)\}_{k=1}^{\infty}, \beta \geqslant 0$, recovers the potential $V(x_1,x_2)$ uniquely.*

2.4. Open problems

1. *Spectral rigidity.*

Let a lattice L have the property

$$|d| = |d'| \Rightarrow d = \pm d'. \tag{45}$$

To prove that for any periodic C^∞ potentials different from a directional potential $V_\delta(\delta \cdot x)$ the only Floquet isospectral potential have the form $V(\pm x + a)$, where $a \in R^n$.

2. *Complete systems.*

To give a criterion for the collection of $\{I_n^d, n = 1,2,\dots,d \in L\}$ to be a complete system of Floquet spectral invariants. We suppose that a criterion similar to the one-dimensional case can be proven: the collection $\{I_n^d, n = 1,2,\dots,d \in L\}$ can be a complete system of Floquet spectral invariants in quasianalytic classes only.

3. Let $E(t,x,y)$ be a fundamental solution for the hyperbolic equation associated with $-\Delta + V(x)$. Let V belong to the Carleman class $C(m_n, R^n/L)$. Is $E(t,x,y)$ from the same Carleman class $C(m_n, R^n/L)$ on the complement of $\{(t,x,y) : |x-y| = t\}$? We have a conjecture that it is true in the quasianalytic classes. Then the arguments of Theorem 2.17 could be used for quasianalytic potentials, i.e., the periodic spectrum $\sigma_0(H)$ will determine the Floquet spectrum $\sigma_F(H)$.

4. *The moment problem.*

Let L be an orthogonal lattice in R^2 and V be a separable periodic potential $V(x_1,x_1) = V_1(x_1) + V_2(x_2)$. Then the coefficients I_n^0 have the representation

$$I_n^0 = \int_{R^2} P_n(t,s)\, d\sigma(t,s), \tag{46}$$

where $P_n(t,s) = \sum_{k=0}^n t^k s^{n-k}$ and $d\sigma(t,s) = \gamma_1(t)\gamma_2(s)\,dt\,ds$. Define the moments $m_{n_1,n_2} = \int t^k s^{n-k} d\sigma(t,s)$. Then

$$I_n^0 = \sum_{n_1+n_2=n} m_{n_1,n_2} \tag{47}$$

and the function

$$w(z) = \int\int \frac{d\sigma(t,s)}{(z-t)(z-s)} \tag{48}$$

is the generating function for the collection I_n^0. Formulas (46) and (48) are similar to (11) and (9) for the Hill operator.

It would be interesting to find a representation similar to (46) and its generating function (48) for general periodic potentials.

5. *A set* $\mathrm{ISO}_0(V)$.

a) To describe $\mathrm{ISO}_0(V)$ for potentials of the form $V(x) = Q_1(\delta_1 \cdot x) + Q_2(\delta_2 \cdot x)$ or $V(x) = Q_1(\delta_1 \cdot x) + Q_2(\delta_2 \cdot x) + Q_3(\delta_3 \cdot x)$;

b) is a set $\mathrm{ISO}_0(V)$ compact in $C^\infty(R^n/L)$?

c) in the case $n \geqslant 2$, to prove that if a set $\mathrm{ISO}_0(V)$ contains at least one analytic potential then all potentials from $\mathrm{ISO}_0(V)$ are analytic.

Concluding remarks.

1. [31] is the first publications (unfortunately, in Russian) where Theorems 2.13 and 2.18 of the Lecture 2 where announced.

2. At present, some results formulated in these lectures are extended to the case of the Schrödinger operator with periodic vector potential [9] and for the Schrödinger operator with periodic magnetic and electric potentials [8]. Isospectral potentials on a discrete lattice are considered in [17]-[19].

3. On the trace formulas for multi-dimensional Schrödinger operator with periodic potential see [24],[12].

4. For detailed proofs of the results of these lectures, see [41].

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