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HYPERBOLIC DYNAMICS OF EULER-LAGRANGE FLOWS ON PRESCRIBED ENERGY LEVELS

Gabriel P. PATERNAIN

Abstract

The aim of this paper is to describe some recent results concerning the dynamics of Euler-Lagrange flows on prescribed energy levels. We show that if an Anosov energy level has a splitting of class C^1 then it must contain minimizing measures with non-zero homology.

1. Introduction

The aim of this paper is to describe some recent results concerning the dynamics of Euler-Lagrange flows on prescribed energy levels. These results have been obtained using variational methods. Throughout this paper the Euler-Lagrange flows the we shall consider are generated by *convex* and *superlinear* Lagrangians on closed connected manifolds M.

A very interesting aspect of the dynamics of the Euler-Lagrange flows is given by those orbits or invariant measures that satisfy some global variational properties, instead of the local ones that every orbit satisfy. Research on these special orbits goes back to M. Morse [39] and Hedlund [25] and has reappeared in recent years in the work of V. Bangert [2], M.J. Dias Carneiro [12], A. Fathi [14], [15], R. Mañé [35], [36] and J. Mather [37], [38]. For autonomous systems, like the ones we are considering, these distinguished orbits and measures have the remarkable property of living on certain energy levels related to minimal values of the action. This link was discovered by M.J. Dias Carneiro [12] and later exploited and enhanced by Mañé in his unfinished manuscript [35]. The proofs of the theorems stated in [35] have been given by Gonzalo Contreras, Jorge Delgado and Renato

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Iturriaga in [9], [10]. A key element in Mañé's work is what he called the *critical value* of a Lagrangian, whose meaning and properties shall be explained in Section 2 as well as its relation with Mather's theory. By considering the lift of the Lagrangian to a covering of the manifold it is possible to obtain many *different* critical values. Among them, there are two which are particularly relevant: the critical value $c_u(L)$ associated with the universal covering and the critical value $c_a(L)$ associated with the abelian covering. It is quite easy to see that $c_u(L) \leqslant c_a(L)$, but in general they may be different.

Using the machinery developed in Section 2, we shall be able to give various partial answers to the following general question:

Question. — *What type of dynamics can arise on a prescribed regular energy level?*

The "types" of dynamics considered here are those of hyperbolic nature. In Section 3 we shall study regular energy levels on which the Euler-Lagrange flow is Anosov and we shall explain the results obtained in [11], [40], [44], [45]. We find that the energy of these levels has to be strictly bigger than $c_u(L)$ and the levels are free of conjugate points. In Sections 4 and 5 we continue the study of Anosov energy levels, but we look for those which have regularity properties of the Anosov splitting. We shall say that the Anosov splitting is of class C^k if the strong stable and strong unstable bundles are both of class C^k. We find that Anosov energy levels with splitting of class C^1 do not exist for an important class of Lagrangians, namely that given by a Riemannain metric and a non-trivial magnetic field [46]. We prove in Section 5 a new result that links the existence of Anosov energy levels with C^1-splitting and the critical value $c_a(L)$. More precisely we show:

THEOREM 1.1. — *Suppose that M admits a Riemannian metric of negative curvature and suppose that the energy level k of the Lagrangian L is Anosov. If the Anosov splitting of the energy level k is of class C^1, then $k > c_a(L)$. In particular, the energy level has minimizing measures with non-zero homology.*

We should mention that there are examples of Anosov energy levels with energy $k < c_a(L)$ on surfaces of genus $\geqslant 2$ (cf. Section 2 and [44]). These levels do not support minimizing measures and by the theorem, their splitting is never of class C^1.

Finally in Section 6 we consider regular energy levels on which the Euler-Lagrange flow is *expansive* and $\dim M = 2$. We classify these levels up to topological equivalence, and we find that they are also free of conjugate points. The results in this section generalize and complete the results in [41].

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2. Critical values of Lagrangians

Let M^n be a closed manifold and let $L : TM \to \mathbb{R}$ be a C^∞ Lagrangian satisfying the following hypotheses:

- *Convexity.* For all $x \in M$, the restriction of L to T_xM has everywhere positive definite Hessian.
- *Superlinear growth.* Let $\| \ \|$ denote a Riemannian metric on M. Then

  $$\lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = +\infty,$$

  uniformly on $x \in M$. This condition is clearly independent of the choice of Riemannian metric, since M is compact.

Since M is compact, the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(x,\dot{x})\right) - \frac{\partial L}{\partial x}(x,\dot{x}) = 0$$

generates a smooth complete flow $\phi_t : TM \to TM$ which is defined as follows. Given $(x,v) \in TM$, consider the unique solution $x : \mathbb{R} \to M$ of the Euler-Lagrange equation with initial conditions

$$x(0) = x, \qquad \dot{x}(0) = v.$$

Now define $\phi_t : TM \to TM$ by

$$\phi_t(x,v) = (x(t), \dot{x}(t)).$$

Recall that the *energy* $E : TM \to \mathbb{R}$ is defined by

$$E(x,v) = \frac{\partial L}{\partial v}(x,v).v - L(x,v).$$

Since L is autonomous, E is a first integral of the flow ϕ_t. Observe that for all $x \in M$, E restricted to T_xM is a function that achives its minimum at $(x,0)$. Let us set

$$e = \max_{x\in M} E(x,0) = -\min_{x\in M} L(x,0).$$

Note that the energy level $E^{-1}(k)$ projects *onto* the manifold M if and only $k \geqslant e$ and for any $k > e$, the energy level $E^{-1}(k)$ is a smooth closed connected hypersurface of TM that intersects each tangent space T_xM in a sphere containing the origin in its interior.

We shall denote by $\mathcal{L} : TM \to T^*M$ the Legendre transform which is defined by $(x,v) \mapsto \frac{\partial L}{\partial v}(x,v)$. Our hypotheses on L assure that $\mathcal{L}$ is a diffeomorphism. If Θ denotes the canonical 1-form on T^*M, then the Euler-Lagrange flow of L can also be obtained as the Hamiltonian flow of E with respect to the symplectic form on TM given by $-\mathscr{L}^* d\Theta$. In other words, if X denotes the vector field associated with the Euler-Lagrange flow then $i_X \mathscr{L}^* d\Theta = -dE$.

Recall that the action of the Lagrangian L on an absolutely continuous curve $u : [a, b] \to M$ is defined by

$$A_L(u) = \int_a^b L(u(t), \dot{u}(t))\, dt.$$

Given two points, x_1 and x_2 in M, denote by $\mathscr{C}(x_1, x_2)$ the set of absolutely continuous curves $u : [0, T] \to M$, with $u(0) = x_1$ and $u(T) = x_2$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k : M \times M \to \mathbb{R}$ by

$$\Phi_k(x_1, x_2) = \inf\{A_{L+k}(u) : \ u \in \mathscr{C}(x_1, x_2)\}.$$

Mañé showed [35], [9] that there exists $c(L) \in \mathbb{R}$ such that

- if $k < c(L)$, then $\Phi_k(x_1, x_2) = -\infty$, for all x_1 and x_2;
- if $k \geqslant c(L)$, then $\Phi_k(x_1, x_2) > -\infty$ for all x_1 and x_2 and Φ_k is a Lipschitz function;
- if $k \geqslant c(L)$, then

$$\Phi_k(x_1, x_3) \leqslant \Phi_k(x_1, x_2) + \Phi_k(x_2, x_3),$$

  for all x_1, x_2 and x_3 and

$$\Phi_k(x_1, x_2) + \Phi_k(x_2, x_1) \geqslant 0,$$

  for all x_1 and x_2;
- if $k > c(L)$, then for $x_1 \neq x_2$ we have

$$\Phi_k(x_1, x_2) + \Phi_k(x_2, x_1) > 0.$$

Observe that in general the action potential Φ_k is **not** symmetric, however defining $d_k : M \times M \to \mathbb{R}$ by

$$d_k(x_1, x_2) = \Phi_k(x_1, x_2) + \Phi_k(x_2, x_1),$$

the properties above say that d_k is a metric for $k > c(L)$ and a pseudometric for $k = c(L)$. The number $c(L)$ is called the *critical value* of L.

It is important for our purposes to indicate that the results above also hold for coverings of M, i.e. suppose $\widehat{M}$ is a covering of M with covering projection p. Take the lift of the Lagrangian L to $\widehat{M}$ which is given by

$$\hat{L}(\hat{x}, \hat{v}) = L(p(\hat{x}), dp(\hat{v})).$$

Then we define for each $k \in \mathbb{R}$ the action potential $\hat{\Phi}_k$ just as above and the results hold for $\hat{L}$. Thus we have a critical value for $\hat{L}$.

Moreover, if M_1 and M_2 are coverings of M such that M_1 covers M_2, then

$$c(L_1) \leqslant c(L_2), \tag{2.1}$$

where L_1 and L_2 denote the lifts of the Lagrangian L to M_1 and M_2 respectively. Also note that if M_1 is a *finite* covering of M_2 then

$$c(L_1) = c(L_2). \tag{2.2}$$

Among all possible coverings of M there are two distinguished ones; the universal covering which we shall denote by $\widetilde{M}$, and the abelian covering which we shall denote by $\overline{M}$. The latter is defined as the covering of M whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \to H_1(M, \mathbb{R})$. When $\pi_1(M)$ is abelian, $\widetilde{M}$ is a finite covering of $\overline{M}$.

The universal covering of M gives rise to the critical value

$$c_u(L) \stackrel{\text{def}}{=} c(\widetilde{L}),$$

and the abelian covering of M gives rise to the critical value

$$c_a(L) \stackrel{\text{def}}{=} c(\overline{L}).$$

From inequality (2.1) it follows that

$$c_u(L) \leqslant c_a(L),$$

which naturally raises the following,

Question. — Is it true that $c_u(L) = c_a(L)$?

The answer to this question is negative. In [44] we gave an example of a Lagrangian on a closed surface of genus two for which the inequality becomes strict.

We shall explain now the relationship between the critical values and Mather's theory. We begin by recalling the main concepts introduced by Mather in [37].

Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra of TM that have compact support and are invariant under the flow ϕ_t. Let $H_1(M, \mathbb{R})$ be the first real homology group of M. Given a closed one-form ω on M and $\rho \in H_1(M, \mathbb{R})$, let $\langle \omega, \rho \rangle$ denote the integral of ω on any closed curve in the homology class ρ. If $\mu \in \mathcal{M}(L)$, its *homology* is defined as the unique $\rho(\mu) \in H_1(M, \mathbb{R})$ such that

$$\langle \omega, \rho(\mu) \rangle = \int \omega \, d\mu,$$

for all closed one-forms on M. The integral on the right-hand side is with respect to μ with ω considered as a function $\omega : TM \to \mathbb{R}$. The function $\rho : \mathcal{M}(L) \to H_1(M, \mathbb{R})$ is surjective [37]. The homology of an invariant measure is the projection of Schwartzmann's asymptotic cycle [51].

The *action* of $\mu \in \mathcal{M}(L)$ is defined by

$$A_L(\mu) = \int L \, d\mu.$$

Finally we define the function $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$ by

$$\beta(\gamma) = \inf\{A_L(\mu) : \ \rho(\mu) = \gamma\}.$$

The function β is *convex* and *superlinear* and the infimum can be shown to be a *minimum* [37] and the measures at which the minimum is attained are called *minimizing measures.* In other words, $\mu \in \mathcal{M}(L)$ is a minimizing measure iff

$$\beta(\rho(\mu)) = A_L(\mu).$$

Let us recall how the convex dual $\alpha : H^1(M, \mathbb{R}) \to \mathbb{R}$ of β is defined. Since β is convex and superlinear we can set

$$\alpha([\omega]) = \max\{\langle \omega, \gamma \rangle - \beta(\gamma) : \ \gamma \in H_1(M, \mathbb{R})\},$$

where ω is any closed one-form whose cohomology class is $[\omega]$. The function α is also convex and superlinear. It is not hard to see that [37]:

$$\alpha([\omega]) = -\min\left\{\int (L - \omega) \, d\mu : \ \mu \in \mathcal{M}(L)\right\}. \tag{2.3}$$

M.J. Dias Carneiro proved [12] that if μ is a minimizing measure with homology γ, then its support is contained in a fixed energy level k and $k = \alpha([\omega])$, where $[\omega]$ is the slope of a supporting hyperplane through $(\gamma, \beta(\gamma))$.

Mañé [35], [9] established a connection between the critical values of a Lagrangian and α, the convex dual of Mather's β function. He showed that

$$c(L) = -\min\left\{\int L \, d\mu : \ \mu \in \mathcal{M}(L)\right\}, \tag{2.4}$$

and therefore combining (2.3) and (2.4) we obtain the remarkable equality

$$c(L - \omega) = \alpha([\omega]), \tag{2.5}$$

for any closed one-form ω whose cohomology class is $[\omega]$.

Much more can be said about the support of the minimizing measures, but first we give a few definitions. Note that for every absolutely continuous curve $u : [a, b] \to M$ and all $k \geqslant c(L)$ we have

$$A_{L+k}(u) \geqslant \Phi_k(u(a), u(b)) \geqslant -\Phi_k(u(b), u(a)). \tag{2.6}$$

We say that an absolutely continuous curve $u : [a, b] \to M$ is *semistatic* if

$$A_{L+c(L)}(u|_{[t_0,t_1]}) = \Phi_{c(L)}(u(t_0), u(t_1)),$$

for all $a < t_0 \leqslant t_1 < b$; and that is *static* if

$$A_{L+c(L)}(u|_{[t_0,t_1]}) = -\Phi_{c(L)}(u(t_1), u(t_0)),$$

for all $a < t_0 \leqslant t_1 < b$. Clearly, by (2.6) a static curve is semistatic. Semistatic curves are solutions of the Euler-Lagrange equations because of their minimizing properties. Also it is not hard to check that semistatic curves have energy precisely $c(L)$. Mañé shows that a measure is minimizing with homology γ if and only if its support is contained in the set of all static curves of the Lagrangian $L - \omega$, where $[\omega]$ is the slope of a supporting hyperplane through $(\gamma, \beta(\gamma))$. He also shows that the set of statics curves of a Lagrangian is a Lipschitz graph, thus recovering and generalizing the celebrated Lipschitz Graph Theorem of Mather [37]. Mañé describes in [35] several recurrence properties between static and semistatic curves as well as coboundary properties and other graph properties. Using a "weak KAM theorem", A. Fathi [14], [15] gives new proofs of several of these properties and he obtains new ones. Fathi's weak KAM theorem is obtained studying a semigroup of nonlinear-operators $(T_t^-)_{t\geqslant 0}$ defined as follows: for $v \in C^0(M, \mathbb{R})$, set

$$T_t^- v(x) = \inf\{v(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds \mid \gamma : [0, t] \to M \ C^1 \text{ and } \gamma(t) = x\}.$$

Let $\mathcal{M}^\omega(L)$ denote the set of all minimizing measures μ such that $[\omega]$ is the slope of a supporting hyperplane through $(\rho(\mu), \beta(\rho(\mu)))$. Concerning the structure of the set $\mathcal{M}^\omega(L)$ we would like to mention the following important *genericity* result (cf. [35], [36], [10]): there exists a generic set $\mathcal{O}(\omega) \subset C^\infty(M, \mathbb{R})$ such that for all $\psi \in \mathcal{O}(\omega)$ the Lagrangian $L + \psi$ has a unique minimizing measure in $\mathcal{M}^\omega(L + \psi)$ and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic. It is conjectured in [35] that there exists a generic set $\mathcal{O}(\omega) \subset C^\infty(M, \mathbb{R})$ such that such that for all $\psi \in \mathcal{O}(\omega)$ the unique minimizing measure in $\mathcal{M}^\omega(L + \psi)$ is always supported on a periodic orbit.

Finally, Mañé defined the *strict critical value* of L as

$$c_0(L) \stackrel{\text{def}}{=} \min\{c(L - \omega) : [\omega] \in H^1(M, \mathbb{R})\} = -\beta(0).$$

We showed in [44] that the strict critical value of L equals the critical value of the lift of L to the abelian covering of M, that is, $c_a(L) = c_0(L)$.

In [35] Mañé describes an example of the form kinetic energy plus a magnetic field and a potential for which $e < c_0(L)$. The following theorem from [44] gives a large class of Lagrangians verifying the sharper inequality: $e < c_u(L)$ (it is quite easy to check that the inequality $e \leqslant c_u(L)$ always holds):

THEOREM 2.1. — *Let θ be the 1-form on M given by*

$$\theta_x(v) = \frac{\partial L}{\partial v}(x, 0)(v).$$

If θ is closed then

$$e = c_0(L).$$

Suppose in addition that $L(x,0) = 0$ for all $x \in M$. If $e = c_u(L)$, then θ is closed.

Let M be a closed manifold endowed with a Riemannian metric and let θ denote a smooth 1-form on M. Consider a Lagrangian of the type kinetic energy plus a magnetic field, i.e.,

$$L(x,v) = \frac{1}{2}\langle v,v\rangle_x + \theta_x(v).$$

The energy function associated with L is

$$E(x,v) = \frac{1}{2}\langle v,v\rangle_x,$$

therefore in this case $e = \max_{x\in M} E(x,0) = 0$. If θ is not closed, Theorem 2.1 immediately implies that $c_u(L) > 0$.

Finally, let us describe a new geometric way of obtaining the critical values. In [11] we showed:

THEOREM 2.2. — *For any covering $\widehat{M}$ we have*

$$c(\hat{L}) = \inf\{k \in \mathbb{R} : \widehat{H}^{-1}(-\infty,k) \text{ contains an exact Lagrangian graph}\},$$

where $\widehat{H}$ is the Hamiltonian associated with $\hat{L}$.

An exact Lagrangian graph is a set of the form $(x, d_x f)$, where f is a smooth function on $\widehat{M}$. Albert Fathi has also obtained independently a proof of the last theorem based on his weak KAM solutions.

3. Anosov energy levels

One of our aims will be the study of *Anosov energy levels*, that is, regular energy levels with a connected component on which the flow ϕ_t is an Anosov flow. Let $E^0 \oplus E^s \oplus E^u$ denote the Anosov splitting of such energy level, where E^0 denotes the one dimensional subbundle generated by the flow direction and $E^{s,u}$ denote the strong stable and strong unstable bundles respectively. It is well known that the bundles $E^0 \oplus E^s$ and $E^0 \oplus E^u$ are *Lagrangian* subbundles.

Let $\pi : TM \to M$ denote the canonical projection and, if $(x,v) \in TM$, let $V(x,v)$ denote the vertical fibre at (x,v) defined as usual as the kernel of $d\pi_{(x,v)} : T_{(x,v)}TM \to T_xM$.

We showed in [40] the following key result:

THEOREM 3.1. — *For every point in an Anosov energy level, the bundles $E^0 \oplus E^s$ and $E^0 \oplus E^u$ are transverse to the vertical bundle V.*

It is not hard to check that this property implies that the energy level has to project onto the manifold and therefore if k denotes the energy of the level, we must have $k > e$. Moreover, it can be proved that the transversality property stated in the theorem implies that there are no conjugate points in the energy level [40, Proposition 3]. Conjugate points, means, as usual, pair of points $(x_1, v_1) \neq (x_2, v_2) = \phi_t(x_1, v_1)$ such that $d\phi_t(V(x_1, v_1))$ intersects $V(x_2, v_2)$ non-trivially. In this way the theorem generalizes a well known result of Klingenberg [31] for geodesic flows (cf. also [33]). The results that we obtained in [40] are much more general than Theorem 3.1. We showed that if on the given energy level there exists a continuous invariant Lagrangian subbundle, then it must be transverse to the vertical bundle; from this result the theorem clearly follows. We shall give now a sketch of a proof of Theorem 3.1 based on a result of J.F. Plante [50]. But first we need some remarks and a lemma.

Let us set $\nu = \mathscr{L}^*\Theta$ and $\Sigma = E^{-1}(k)$. Note that $(d\nu)^{n-1} = d\lambda$, where λ is a 2n-3-form. Let us denote by X the Euler-Lagrange vector field. Take a vector field Y such that $d\nu_{(x,v)}(Y(x,v), X(x,v)) = 1$ for all $(x,v) \in \Sigma$. Such a vector field always exists since Σ is a regular energy level. Then we have

$$i_X i_Y (\underbrace{d\nu \wedge \ldots \wedge d\nu}_{n}) \,|_{\Sigma} = \underbrace{d\nu \wedge \ldots \wedge d\nu}_{n-1}.$$

Let us define a volume form on Σ by $\Omega := i_Y(\underbrace{d\nu \wedge \ldots \wedge d\nu}_{n})$. It follows that

$$i_X \Omega = \underbrace{d\nu \wedge \ldots \wedge d\nu}_{n-1} = d\lambda,$$

and hence ϕ_t preserves the volume form Ω. Let μ_Ω denote the smooth invariant probability measure induced by Ω. Observe that in general there may exist many invariant volume forms on Σ and therefore many smooth invariant probability measures. However, if the energy level is Anosov, there is only one smooth probability measure and we shall call it *the Liouville measure* and it will be denoted by μ_l.

We can associate to the measure μ_Ω an *asymptotic cycle* ρ (cf. [51]) which is the element of $H_1(\Sigma, \mathbb{R})$ defined by

$$\rho(\varphi) = \int_\Sigma \varphi(X)\, d\mu_\Omega,$$

where φ is a closed 1-form in Σ.

LEMMA 3.2. — *The asymptotic cycle of μ_Ω vanishes.*

Proof: We need to show that

$$\int_\Sigma \varphi(X)\Omega = 0,$$

where φ is any closed 1-form. But

$$\int_{\Sigma} \varphi(X)\Omega = \int_{\Sigma} i_X\Omega \wedge \varphi = \int_{\Sigma} (dv)^{n-1} \wedge \varphi = \int_{\Sigma} d\lambda \wedge \varphi = \int_{\Sigma} d(\lambda \wedge \varphi) = 0. \qquad \diamond$$

Let us explain now how derive Theorem 3.1. By Lemma 3.2, the Liouville measure of the Anosov energy level has vanishing asymptotic cycle and by a result of Plante [50] the closure of the set of primitive closed orbits of ϕ_t in $H_1(E^{-1}(k), \mathbb{R})$ is the closure of a convex open set containing the origin in its interior. Thus if $\alpha : H_1(E^{-1}(k), \mathbb{R}) \to \mathbb{R}$ is any non-trivial cohomology class, there exists a closed orbit γ of ϕ_t such that $\alpha(\gamma) < 0$.

Suppose now that for some $(x, v) \in E^{-1}(k)$, $E(x, v) \cap V(x, v) \neq \{0\}$, where E stands for the weak stable or the weak unstable subbundle of ϕ_t. Then (cf. [40, Proposition 4]) the Maslov class $m \in H^1(E^{-1}(k), \mathbb{R})$ associated with E is non-trivial. On the other the convexity of the Lagrangian implies that if γ is any closed orbit of ϕ_t, then $m(\gamma) \geqslant 0$ [5], [13]. This contradiction completes the proof of the theorem. $\diamond$

We note that from Lemma 3.2 (which rules out the case of a suspension) and results in [49] it follows that the Euler-Lagrange flow restricted to an Anosov energy level must be topologically mixing and that the strong stable and strong unstable manifolds must be dense in the energy level.

Motivated by our results, Mañé posed us the following question,

Question. — If the energy level k is Anosov, is it true that $k > c_0(L)$?

In [44] we gave a negative answer to this question, however in [11] we showed,

THEOREM 3.3. — *If the energy level k is Anosov, then*

$$k > c_u(L).$$

Note that it follows from the result of M.J. Dias Carneiro quoted in Section 2 that if μ is a minimizing measure, then its support is contained in a fixed energy level k with $k \geqslant c_0(L)$. Our examples in [44] show that Anosov energy levels on manifolds with non-zero first Betti number **do not** necessarily contain minimizing measures and makes Theorem 1.1 in the introduction meaningful!

Let M be a closed manifold endowed with a Riemannian metric and let θ denote a smooth 1-form on M that is not closed. Consider a Lagrangian of the type kinetic energy plus a magnetic field, i.e.,

$$L(x, v) = \frac{1}{2} \langle v, v \rangle_x + \theta_x(v).$$

Suppose that the geodesic flow associated with the Riemannian metric is Anosov. Then by structural stability the Euler-Lagrange flow of L is Anosov for any sufficiently large value of the energy. However, Theorem 3.3 shows that it cannot be Anosov for *all* values of the energy otherwise $c_u(L) = 0$ and we know by Theorem 2.1 that $c_u(L)$ is positive if θ is not closed. Let $\mathcal{E}$ denote the smallest possible value of the energy such that for all $k' > \mathcal{E}$ the energy level k' is Anosov. Theorem 3.3 immediately implies the following lower bound for $\mathcal{E}$.

COROLLARY 3.4. —

$$\mathcal{E} \geqslant c_u(L).$$

In [42] we obtained lower bounds for $\mathcal{E}$ in terms of $d\theta$ and the curvature tensor of M and we proved through different methods that $\mathcal{E}$ cannot vanish if θ is not closed. Another interesting class of Lagrangians is the following. A Lagrangian L is said to be *simple* if there exist a real number $R > 0$ and a smooth convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ such that

$$L(x, v) = \varphi(||v||_x^2) \quad \text{for } ||v||_x > R.$$

Clearly for high values of the energy, the Euler-Lagrange flow of L is a reparametrization of the geodesic flow of the Riemannian metric $||\ ||$. If the geodesic flow is Anosov then we can consider as before a number $\mathcal{E}$ that is given by the smallest possible value of the energy such that for all $k > \mathcal{E}$ the energy level k is Anosov and we also have that $\mathcal{E} \geqslant c_u(L)$.

We shall prove next several important properties of Anosov energy levels similar to those of geodesic flows. We shall need these properties for the proof of Theorem 1.1. Suppose that the energy level k is Anosov and set $\Sigma \stackrel{\text{def}}{=} E^{-1}(k)$. Observe first that we could rephrase Theorem 3.1 by saying that the weak stable foliation $\mathcal{W}^s$ is *transverse* to the fibres of the fibration by $(n-1)$-spheres given by

$$\pi|_\Sigma : \Sigma \to M.$$

Let $\widetilde{M}$ denote the universal covering of M with projection $p : \widetilde{M} \to M$. Let $\widetilde{\Sigma}$ denote the lifting of Σ to $T\widetilde{M}$ via the map $dp : T\widetilde{M} \to TM$. Observe that $\widetilde{\Sigma}$ coincides with the energy level k of the lifted Lagrangian $\widetilde{L}$. We also have a fibration by $(n-1)$-spheres

$$\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{M}.$$

Let $\widetilde{\mathcal{W}}^s$ be the lifted foliation which is in turn a weak stable foliation for the Euler-Lagrange flow of $\widetilde{L}$ restricted to $\widetilde{\Sigma}$. The foliation $\widetilde{\mathcal{W}}^s$ is also transverse to the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{M}$ since the map dp is a local diffeomorphism. Since the fibres are compact a result of Ehresman (cf. [8]) implies that for every $v \in \widetilde{\Sigma}$ the map

$$\widetilde{\pi}|_{\widetilde{\mathcal{W}}^s(v)} : \widetilde{\mathcal{W}}^s(v) \to \widetilde{M},$$

is a covering map. Since $\widetilde{M}$ is simply connected, $\widetilde{\pi}|_{\widetilde{\mathcal{W}}^s(v)}$ is in fact a diffeomorphism and $\widetilde{\mathcal{W}}^s(v)$ is simply connected. Consequently, $\widetilde{\mathcal{W}}^s(v)$ intersects each fibre of the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{M}$ at just one point and therefore the space of leaves $\mathcal{F}^s$ of the weak stable foliation can be identified topologically with the $(n-1)$-sphere. Similarly the space of leaves $\mathcal{F}^u$ of the weak unstable foliation is also an $(n-1)$-sphere. Note that $\pi_1(M)$ acts on $\mathcal{F}^{s,u}$.

Let us assume now that M admits a Riemannian metric of negative curvature. Then we have the following two lemmas.

LEMMA 3.5. — *Suppose that the energy level k is Anosov. There is no periodic orbit of ϕ_t with energy k whose projection to M is null-homotopic. If σ denotes a non-trivial free homotopy class of M, then there exists a unique closed orbit of ϕ_t with energy k such that its projection to M belongs to the homotopy class σ.*

LEMMA 3.6. — *Let γ denote and element of $\pi_1(M)$ acting on $\mathcal{F}^u$. Then there exists two fixed points a^+ and a^- for γ such that if p is any point in $\mathcal{F}^u$, then*

$$\lim_{n\to+\infty} \gamma^n(p) = a^+,$$
$$\lim_{n\to-\infty} \gamma^n(p) = a^-.$$

Clearly a similar lemma holds for γ acting on $\mathcal{F}^s$.

Let us prove the lemmas. The transversality property and Lemma 3.1 in [21] implies that a solution of the Euler-Lagrange equation with energy k in $\widetilde{M}$ is a *quasi-geodesic* with respect to the background Riemannian metric and as a consequence using exactly the same methods in the proof of Theorem 4.5 in [21] we deduce that the Euler-Lagrange flow of the energy level k is topologically conjugate to the geodesic flow of the Riemannian metric. It is well known that the geodesic flow of a negatively curved manifold has the properties stated in the two lemmas. Using the orbit equivalence between the flows we immediately obtain the same properties for the Euler-Lagrange flow in the level k, thus proving the lemmas.

In [6], P. Boyland and C. Golé proved that under certain hypotheses on the Lagrangian there are minimizers (in the universal covering) which are quasi-geodesics and using them they show the existence of a collection of compact invariant sets of the Euler-Lagrange flow that are semiconjugate to the geodesic flow of an underlying hyperbolic metric.

4. Regularity of the Anosov splitting

We describe in this section the results obtained in [46] for twisted geodesic flows. In the next section we shall prove Theorem 1.1 stated in the Introduction.

Let M^n be a closed n-dimensional manifold endowed with a C^∞ Riemannian metric $\langle\ ,\ \rangle$, and let $\pi : TM \to M$ denote the canonical projection. Let ω_0 denote the symplectic form on TM obtained by pulling back the canonical symplectic form of T^*M via the Riemannian metric. Let $H : TM \to \mathbb{R}$ be defined by

$$H(x, v) = \frac{1}{2}\langle v, v\rangle .$$

The Hamiltonian flow of H with respect to ω_0 gives rise to the geodesic flow of M. Let Ω be a closed 2-form of M which does not vanish identically and consider the new symplectic form ω_λ defined as:

$$\omega_\lambda \stackrel{\text{def}}{=} \omega_0 + \lambda\pi^*\Omega, \qquad \lambda \in \mathbb{R}.$$

Such a form is called a *twisted symplectic structure* [1] and the Hamiltonian flow of H with respect to ω_λ gives rise to a flow $\phi_t^\lambda : TM \to TM$ that we shall call *twisted geodesic flow.* This flow models the motion of a particle of unit mass and charge λ under the effect of a magnetic field, whose Lorentz force $Y : TM \to TM$ is the bundle map uniquely determined by:

$$\Omega_x(u, v) = \langle Y_x(u), v\rangle ,$$

for all u and v in T_xM and all $x \in M$. Observe that ϕ_t^λ preserves all the energy levels H = const, in particular $SM \stackrel{\text{def}}{=} H^{-1}(1/2)$. From now on let us consider the restriction of ϕ_t^λ to SM.

Various properties of these flows were studied in [42], [43]. For example, we showed that if we start with an Anosov geodesic flow ϕ_t^0 and we increase the value of λ we must exit the set of Anosov flows for some critical value $\lambda_c < \infty$ and that the topological entropy presents a strict global maximum at $\lambda = 0$ when restricted to $(-\lambda_c, \lambda_c)$.

In [46] we studied a new feature of the twisted geodesic flows, namely the regularity of the Anosov splitting. If $\lambda \in (-\lambda_c, \lambda_c)$, let us denote by $E_\lambda^0 \oplus E_\lambda^s \oplus E_\lambda^u$ the Anosov splitting of ϕ_t^λ, where E_λ^0 denotes the one dimensional subbundle associated with the flow direction and $E_\lambda^{s,u}$ denote the strong stable and strong unstable bundles respectively.

If $\dim M = 2$ then $E_\lambda^0 \oplus E_\lambda^s$ and $E_\lambda^0 \oplus E_\lambda^u$ are both of class $C^{1,x\log x}$ by results of S.Hurder and A. Katok [28]. In particular, when $\lambda = 0$, i.e. for geodesic flows, this implies that E_0^s and E_0^u are both of class $C^{1,x\log x}$ since the geodesic flow is of contact type. Also, if M has 1/4-pinched negative sectional curvature, E_0^s and E_0^u are both of class C^1 [27]. If one assumes that E_0^s and E_0^u are both of class C^∞ then combining results of Y. Benoist, P. Foulon and F. Labourie [3] with results of G. Besson, G. Courtois and S. Gallot [4] it follows that M must be locally symmetric, thus generalizing and improving previous results of M. Kanai, A. Katok and R. Feres [30], [16], [17], [18]. Most likely the same result is true assuming only that E_0^s and E_0^u are both of class C^2 but this is only known for surfaces [22] and for small deformations of hyperbolic metrics by results of U. Hamenstädt [23] and L. Flaminio [19]. We refer to [24] for more on the regularity of the Anosov splitting.

For twisted geodesic flows assuming C^1 regularity already implies rigidity provided that Ω is an exact form:

THEOREM 4.1. — *Let M be a closed Riemannian manifold whose geodesic flow is Anosov. Suppose Ω is exact. Then E_λ^s and E_λ^u are never both of class C^1 unless $\lambda = 0$.*

If the cohomology class of Ω is not trivial, the theorem is no longer true as it can be easily seen by looking at the case of a surface of constant negative curvature and Ω the area form. More generally consider a compact locally symmetric space of non-constant negative curvature ($n \geqslant 4$). Let $J_1, \dots, J_{d-1}$ be the parallel orthogonal endormorphisms defining the complex ($d = 2$), quaternionic ($d = 4$) or Cayley ($d = 8$) hyperbolic structure of M. If we consider the 2-form Ω naturally associated each J_i ($1 \leqslant i \leqslant d - 1$) then it is straightforward to check that the splitting is C^∞.

Problem. — Are these are the only cases in which the splitting can be C^1 for $\lambda \neq 0$?

Observe that for surfaces, E^s_λ and E^u_λ are both of class C^1 if and only if $E^s_\lambda \oplus E^u_\lambda$ is of class C^1. If τ_λ denotes the one-form that vanishes on $E^s_\lambda \oplus E^u_\lambda$ and takes the value one on the vector field associated with ϕ^λ_t, then the theorem is saying, for the surface case, that τ_λ is of class C^1 if and only if $\lambda = 0$. Note that τ_0 is C^∞ and coincides with the contact form α of the geodesic flow.

J.F. Plante [49] gave the first examples of volume preserving Anosov flows for which the strong stable and unstable bundles are not both of class C^1. His examples are also volume preserving perturbations of Anosov geodesic flows, but he used the fact that the asymptotic cycle (cf. [51]) of the measure induced by the volume form was not zero for the perturbed flows. It is not hard to see that ϕ^λ_t preserves the volume form $\alpha \wedge (d\alpha)^{n-1}$ and therefore the Liouville measure μ_l of SM. Using the same arguments as in Lemma 3.2 it follows right away that the asymptotic cycle of ϕ^λ_t with respect to μ_l vanishes for all λ (provided that M is not a 2-torus). It follows that no argument like in [49] can be used to show the non-smoothness of the bundles E^s_λ and E^u_λ, even in the surface case.

The proof of theorem is based on a combination of a result of U. Hamenstädt in [23] and the theory of convex superlinear Lagrangians described in Section 2. By writing $\Omega = d\theta$, the twisted geodesic flows can also be obtained as the Euler-Lagrange flows of the one-parameter family of Lagrangians

$$L_\lambda(x, v) = \frac{1}{2}\langle v, v\rangle - \lambda\theta_x(v).$$

The energy function of these Lagrangians is $E(x, v) = \frac{1}{2}\langle v, v\rangle$ and we are interested in the level $1/2$. The proof of the theorem splits into the three cases:

- $1/2 > c_0(L_\lambda)$;
- $1/2 = c_0(L_\lambda)$;
- $1/2 < c_0(L_\lambda)$.

As we explained in Section 3, the three cases may indeed occur as long as we do not make any smoothness assumption on the bundles E^s_λ and E^u_λ. Each case gives rise to different variational properties that allow us to prove the theorem. In fact one shows that if E^s_λ and E^u_λ are both of class C^1 then the last two cases cannot occur and that in the first case we must have $\lambda = 0$. In the next section we shall use similar ideas to prove Theorem 1.1.

5. Proof of theorem 1.1

Let us prove the theorem stated in the Introduction. We shall need first two lemmas.

LEMMA 5.1. — *If Θ denotes the canonical 1-form in T^*M and X the Euler-Lagrange vector field associated with the Euler-Lagrange flow, then*

$$\left(\mathscr{L}^*\Theta\right)(X)|_{E^{-1}(k)} = L + k.$$

Proof: Let us denote by $\eta : T^*M \to M$ the canonical projection. Using the definition of the Legendre transform we have

$$(\mathscr{L}^*\Theta)(X)(x,v) = \Theta_{\mathscr{L}(x,v)}(d_{(x,v)}\mathscr{L}(X)) = \frac{\partial L}{\partial v}(x,v)\ (d_{\mathscr{L}(x,v)}\eta(d_{(x,v)}\mathscr{L}(X)))\,.$$

But since

$$d_{\mathscr{L}(x,v)}\eta(d_{(x,v)}\mathscr{L}(X)) = d_{(x,v)}(\eta\circ\mathscr{L})(X) = d_{(x,v)}\pi(X) = v,$$

we have

$$(\mathscr{L}^*\Theta)(X)(x,v) = \frac{\partial L}{\partial v}(x,v)(v).$$

On the energy level k we have

$$L(x,v) + k = \frac{\partial L}{\partial v}(x,v)(v),$$

thus concluding the proof of the lemma. ◇

Set as before $\nu = \mathscr{L}^*\Theta$. Let us recall from Section 3 that each regular energy level $E^{-1}(k)$ possesses an invariant volume form Ω that induces a smooth invariant probability measure μ_Ω with vanishing asymptotic cycle. The volume form is given by $\Omega = i_Y(d\nu)^n$, where Y is a vector field such that $dE(Y) = d\nu(Y,X) = 1$ on $E^{-1}(k)$. Since $dE(Y) = 1$, the vector field Y "points outwards" the manifold with boundary $V_k \overset{\text{def}}{=} E^{-1}(-\infty,k]$. Let us orient TM such that $(d\nu)^n$ is a *positive* volume form. The manifold with boundary $V_k \subset TM$ inherits this orientation and induces a boundary orientation on $E^{-1}(k)$. In other words, $\{u_1,\dots,u_{2n-1}\}$ is a positively oriented basis of a tangent space to

$E^{-1}(k)$ iff $\Omega(u_1, \dots, u_{2n-1}) = (d\nu)^n(Y, u_1, \dots, u_{2n-1}) > 0$. Therefore Ω is positive in the induced orientation of $E^{-1}(k)$.

LEMMA 5.2. — *For any regular energy level $E^{-1}(k)$ we have:*

$$\int_{E^{-1}(k)} (L + k)\, d\mu_\Omega > 0.$$

Proof: We shall consider V_k and $E^{-1}(k)$ oriented as in the previous paragraph. By Lemma 5.1 it suffices to show that

$$\int_{E^{-1}(k)} \nu(X)\Omega > 0.$$

Using Stokes theorem we have:

$$\int_{E^{-1}(k)} \nu(X)\Omega = \int_{E^{-1}(k)} \nu \wedge i_X\Omega = \int_{E^{-1}(k)} \nu \wedge (d\nu)^{n-1} = \int_{V_k} (d\nu)^n.$$

But the last integral is positive because of our choice of orientation. ◇

Let τ denote the one-form that vanishes on $E^s \oplus E^u$ and takes the value one on the vector field X. If the splitting is of class C^1 then τ is also of class C^1 and $d\tau$ is a continuous two-form invariant under the Euler-Lagrange flow. U. Hamenstädt showed in [23], for the geodesic flow case, that any continuous invariant exact two-form must be a constant multiple of the symplectic form provided that the splitting is of class C^1. Hamenstädt's proof carries over to the case of Euler-Lagrange flows *without* major changes using the results from Section 3, particularly Lemma 3.6 if we assume that M admits a Riemannian metric of negative curvature. However, for completeness sake we include a proof of this fact at the end of this section (cf. Theorem 5.5 below).

It follows that there exists a constant χ such that:

$$d\tau = \chi \mathscr{L}^* d\Theta,$$

and thus

$$d(\tau - \chi \mathscr{L}^*\Theta) = 0.$$

Let us write

$$\varphi \stackrel{\text{def}}{=} \tau - \chi \mathscr{L}^*\Theta.$$

Then φ_λ is a smooth *closed* one-form on $E^{-1}(k)$. Using Lemma 5.1 we obtain

$$\varphi(X)(x, v) = 1 - \chi(L(x, v) + k). \tag{5.7}$$

Integrating the last equality with respect to the Liouville measure μ_l of $E^{-1}(k)$ and using that the asymptotic cycle of μ_l vanishes (cf. Lemma 3.2) we have

$$0 = 1 - \chi \int_{E^{-1}(k)} (L + k)\, d\mu_l.$$

Lemma 5.2 implies that $\chi > 0$.

It follows from the Gysin exact sequence for sphere bundles that if $k > e$ then the map

$$\pi^* : H^1(M, \mathbb{R}) \to H^1(E^{-1}(k), \mathbb{R}),$$

is an isomorphism, provided that M is not diffeomorphic to a 2-torus. Therefore there exist a *closed* smooth one-form δ in M and a smooth function $f : E^{-1}(k) \to \mathbb{R}$ such that

$$\varphi = \pi^* \delta + d f,$$

and hence equation (5.7) together with the fact that $d_{(x,v)}\pi(X(x,v)) = v$ gives

$$\delta_x(v) + d f(X)(x, v) = 1 - \chi(L(x, v) + k). \tag{5.8}$$

Let μ be any invariant measure whose support is contained in $E^{-1}(k)$ and whose homology $\rho(\mu)$ vanishes. If we integrate the last equality with respect to μ we obtain:

$$\int_{E^{-1}(k)} (L + k)\, d\mu = 1/\chi > 0 \tag{5.9}$$

We want to show that if the splitting is C^1 in the energy level $E^{-1}(k)$, then $k > c_0(L)$. We shall see that if we suppose $k \leqslant c_0(L)$ we shall obtain a contradiction to inequality (5.9).

- $k = c_0(L)$.

  Take a minimizing measure μ such that $\rho(\mu) = 0$. It satisfies

  $$\beta(0) = \int L\, d\mu.$$

  The result of M.J. Dias Carneiro explained in Section 2 assures that the support of μ is contained in the energy level $-\beta(0) = c_0(L) = k$ and therefore

  $$\int_{E^{-1}(k)} (L + k)\, d\mu = 0.$$

  But this contradicts inequality (5.9) and hence the case $k = c_0(L)$ cannot occur if the splitting is of class C^1.

- $k < c_0(L)$.

  In this case the contradiction to inequality (5.9) is an immediate consequence of the following proposition which has independent interest. Recall that by Theorem 3.3 we know that $k > c_u(L)$.

PROPOSITION 5.3. — *If $c_u(L) < k < c_0(L)$, there exists an invariant measure μ whose support is contained in the energy level k, $\rho(\mu) = 0$ and*

$$\int_{E^{-1}(k)} (L+k)\, d\mu \leqslant 0.$$

Proof: We shall use the following result of Mañé [35], [9] that exhibits the relevance of the critical values for variational problems on fixed energy levels.

THEOREM 5.4. — *Suppose $k > c_u(L)$. Then, given $x_1 \neq x_2$ in $\widetilde{M}$, there exists a solution $x : \mathbb{R} \to \widetilde{M}$ of the Euler-Lagrange equation with energy k such that for some $T > 0$, $x(0) = x_1$, $x(T) = x_2$ and*

$$A_{\tilde{L}+k}(x|_{[0,T]}) = \tilde{\Phi}_k(x_1, x_2).$$

Since $k < c_0(L) = c_a(L)$ there exists $T_0 > 0$ and an absolutely continuous closed curve $u : [0, T_0] \to M$ homologous to zero such that

$$A_{L+k}(u) < 0. \tag{5.10}$$

For $n \geqslant 1$, let us denote by $u^n : [0, nT_0] \to M$ the curve u wraped up n times. Since $k > c_u(L)$, u^n cannot be homotopic to zero otherwise we would contradict (5.10). If $p : \widetilde{M} \to M$ denotes the covering projection, let us pick a point $y \in \widetilde{M}$ such that $p(y) = u(0) = u(T_0)$. Let $\widetilde{u^n} : [0, nT_0] \to \widetilde{M}$ denote the unique lift of u^n such that $\widetilde{u^n}(0) = y$. Set $y_n = \widetilde{u^n}(nT_0)$. Let us provide M with a Riemannian metric and lift it to $\widetilde{M}$. Since $p(y_n) = u(0)$ for all n it follows that $d(y, y_n) \to \infty$ otherwise some power u^n would be homotopic to zero.

By Theorem 5.4 there exists for each n a solution $x_n(t)$ of the Euler-Lagrange equation with energy k such that for some $T_n > 0$, $x_n(0) = y$, $x_n(T_n) = y_n$ and

$$A_{\tilde{L}+k}(x_n|_{[0,T_n]}) = \tilde{\Phi}_k(y, y_n). \tag{5.11}$$

Since the solutions have energy k, there exists a constant a such that $||\dot{x}_n(t)|| < a$ for all n and all t. Therefore

$$d(y, y_n) \leqslant aT_n.$$

It follows that $T_n \to \infty$. Let μ_n denote the probability measure in TM uniformly distributed along $p \circ x_n|_{[0,T_n]}$ and let μ denote a point of accumulation of μ_n. Since all the x_n have energy k, the support of μ is contained in the energy level k. Equality (5.11) implies that

$$A_{\tilde{L}+k}(x_n|_{[0,T_n]}) \leqslant A_{\tilde{L}+k}(\widetilde{u^n}),$$

hence

$$A_{L+k}(p \circ x_n|_{[0,T_n]}) \leqslant nA_{L+k}(u) < 0,$$

which implies by taking limits that

$$\int_{E^{-1}(k)} (L+k)\, d\mu \leqslant 0.$$

To finish the proof observe that μ is clearly invariant and that if ω is a closed 1-form then

$$\langle \rho(\mu), w \rangle = \lim_{T_n \to \infty} \frac{1}{T_n} \int_{p \circ x_n|_{[0,T_n]}} \omega = 0,$$

since the curves $p \circ x_n|_{[0,T_n]}$ are all homologous to zero because they are homotopic to the u^n's. ◇

To finish this section we prove:

THEOREM 5.5. — *Suppose that M admits a Riemannian metric of negative curvature and suppose that the energy level $E^{-1}(k)$ is Anosov with a splitting of class C^1. Let η be a continuous exact 2-form defined on $E^{-1}(k)$ and invariant under the Euler-Lagrange flow ϕ_t. Then η is a constant multiple of the symplectic form $\mathscr{L}^* d\Theta$.*

Proof: This theorem was proved by U. Hamenstädt in [23] for geodesic flows. We shall explain now why her proof extends to the case of Euler-Lagrange flows. Observe that the theorem is a straightforward consequence of ergodicity if $n = 2$.

Let us write $\eta = d\tau$ and $\nu \stackrel{\text{def}}{=} \mathscr{L}^*\Theta$. First note that since η is ϕ_t-invariant, $i_X \eta = 0$. Also there exists a bundle map $G : E^s \oplus E^u \to E^s \oplus E^u$ such that G is ϕ_t-invariant and

$$\eta(x, y) = d\nu(Gx, y),$$

for x and y in $E^s \oplus E^u$.

Let us define

$$A = \int_{E^{-1}(k)} \nu \wedge (d\nu)^{n-1}.$$

Note that $A \neq 0$ since by Stokes theorem, A also equals the integral of the volume form $(d\nu)^n$ on the region of TM bounded by the energy level $E^{-1}(k)$.

Consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(r) = \int_{E^{-1}(k)} (\tau - r\nu) \wedge (d\nu)^{n-1}.$$

Since $F'(r) = -A$, it follows that there exists $\chi \in \mathbb{R}$ such that $F(\chi) = 0$. Let us set $\beta = \tau - \chi\nu$. Clearly $d\beta$ is ϕ_t-invariant and $i_X d\beta = 0$. By ergodicity, there exist constants c_i such that

$$(d\beta)^i \wedge (d\nu)^{n-1-i} = c_i (d\nu)^{n-1},$$

and thus

$$\nu \wedge (d\beta)^i \wedge (d\nu)^{n-1-i} = c_i \nu \wedge (d\nu)^{n-1}.$$

Integrating by parts one finds that

$$c_{i-1} F(\chi) = c_i A,$$

and therefore all the c_i must vanish. It follows that G can be written as $\chi Id + B$, where $B : E^s \oplus E^u \to E^s \oplus E^u$ is a nilpotent map. Next we note that E^s and E^u are invariant subspaces for B. Let B^s and B^u denote the map induced by B on E^s and E^u respectively. We shall show that B^s and B^u vanish. Let $Q(v) \stackrel{\text{def}}{=} \ker B^s(v)$. Choose an open dense ϕ_t-invariant set $U \subset E^{-1}(k)$ on which Q is a continuous subbundle of E^s. Now the key step is Lemma 4.3 in [23] which shows that $Q|_U$ is an *integrable* subbundle (here one uses that the splitting is C^1). Using the holonomy transport along the weak unstable foliation (one also needs here the splitting to be C^1) we can construct as in [23, Lemma 3.4] a C^0-foliation on the energy level k of the universal covering of M. This C^0-foliation descends to the space of leaves $\mathscr{F}^u$ and induces a C^0-foliation, which by construction, is invariant under the induced action of $\pi_1(M)$ on $\mathscr{F}^u$. By Lemma 3.6 an element γ of $\pi_1(M)$ has a dynamics of type "North-South" on the sphere $\mathscr{F}^u$. By a result of P. Foulon [20] a C^0-foliation which is invariant under a map like γ must be trivial and therefore B^s must vanish identically. The argument to prove the vanishing of B^u is completely similar. ◇

6. Expansive energy levels

Recall that a flow $\phi_t : W \to W$ on a compact metric space (W, d) is said to be *expansive* if given $\varepsilon > 0$ there exists $\delta > 0$ such that if there is an homeomorphism $\tau : \mathbb{R} \to \mathbb{R}$, $\tau(0) = 0$, such that

$$d(\phi_{\tau(t)}(y), \phi_t(x)) < \delta,$$

for all $t \in \mathbb{R}$, then $y = \phi_{\bar{t}}(x)$ where $|\ \bar{t}\ | < \varepsilon$. Anosov flows and suspensions of Pseudo-Anosov maps are examples of expansive flows.

In [47] the second author showed that if the geodesic flow on a closed surface M is expansive, then there are no conjugate points and the flow is topologically conjugate to the geodesic flow of a metric of constant negative curvature.

An *expansive energy level* is a regular energy level with a connected component on which the Euler-Lagrange flow is expansive. We shall assume in what follows that M is a closed oriented surface. In this section we show the following theorem that generalizes and completes the results in [41]. Our proof will be based on results of M. Brunella [7] and T. Inaba and S. Matsumoto [29].

THEOREM 6.1. — *If the regular energy level $E^{-1}(k)$ is expansive, then:*

- $k > e$;
- *the energy level is free of conjugate points;*
- *on the energy level, the Euler-Lagrange flow is topologically conjugate to the geodesic flow of a metric of constant negative curvature.*

Recall that a manifold is said to be aspherical if its universal covering is contractible. We need the following lemma.

LEMMA 6.2. — *If $k < e$, the connected components of energy level $E^{-1}(k)$ are not aspherical.*

Proof: Let Σ be a connected component of $E^{-1}(k)$. If $k < e$ then, $\pi(\Sigma)$ is a smooth compact surface with boundary and the boundary is a finite union of circles, let us say k. Let M^* denote the closed surface obtained from $\pi(\Sigma)$ by glueing disks to the boundary circles. Note that we can still regard Σ as a smooth hypersurface in TM^*. Let $\widetilde{M}^*$ denote the universal covering of M^* with covering projection p. Let $\tilde{\Sigma}$ denote the lift of Σ to $T\widetilde{M}^*$ via dp. Since $\tilde{\Sigma}$ is a covering of Σ it suffices to show that Σ is not aspherical. If we still denote by π the projection $T\widetilde{M}^* \to \widetilde{M}^*$, then $\pi(\tilde{\Sigma})$ is a surface with boundary and the boundary is a union of contractible circles in $\widetilde{M}$. There will be at least two circles unless M^* is a sphere and $k = 1$. In this case it is very easy to check that $\tilde{\Sigma}$ is a 3-sphere which is certainly not aspherical. Therefore let C_1 and C_2 denote two *distinct* circles in the boundary of $\pi(\tilde{\Sigma})$. The sets $\Gamma_i \stackrel{\text{def}}{=} \pi^{-1}(C_i) \cap \tilde{\Sigma}$ for $i = 1, 2$ are smooth embedded circles in $\tilde{\Sigma}$. Let $\gamma : [0,1] \to \pi(\tilde{\Sigma})$ denote a simple curve such that $\gamma(0) \in C_1$ and $\gamma(1) \in C_2$ and $\gamma(t) \notin C_1 \cup C_2$ for $t \in (0,1)$. The set $Q \stackrel{\text{def}}{=} \pi^{-1}(\gamma([0,1])) \cap \tilde{\Sigma}$ is an embedded two-sphere in $\tilde{\Sigma}$ and it is quite simple to check that the intersection number of Q with Γ_1 or Γ_2 is ± 1. Therefore Q is not homotopic to a point in $\tilde{\Sigma}$ and therefore $\pi_2(\tilde{\Sigma}) \neq \{0\}$ showing that $\tilde{\Sigma}$ is not aspherical. ◇

Let us describe some important facts about expansive flows. Let W be a closed oriented 3-manifold endowed with a Riemannian metric. Let $\phi_t : W \to W$ be a smooth expansive flow with associated vector field X. Let us suppose that $X(x) \neq 0$ for all $x \in W$. Define

$$H_\varepsilon(x) = \{exp_x v : \ ||v|| < \varepsilon \quad \text{and} \quad \langle X(x), v \rangle = 0\}.$$

For $\varepsilon > 0$ small enough, $H_\varepsilon(x)$ is a family of transverse local sections to the flow. It is easy to see that ϕ_t is expansive if there exists $0 < \alpha < \varepsilon$ such that if there is a continuous increasing surjective function $\tau_{x,y} : [0, \infty) \to [0, \infty)$, $\tau_{x,y}(0) = 0$ for which

$$\phi_t(y) \in H_\alpha(\phi_{\tau_{x,y}(t)}(x)),$$

for all $t \in \mathbb{R}$, then $x = y$.

For $\delta < \alpha$ define the stable sets $S_\delta(x)$ as

$$S_\delta(x) = \Big\{y \in H_\delta(x) : \phi_t(y) \in H_\delta(\phi_{\tau_{x,y}(t)}(x)) \text{ for } t \geqslant 0 \text{ and for some continuous increasing surjective function } \tau_{x,y} : [0, \infty) \to [0, \infty), \tau_{x,y}(0) = 0\Big\}.$$

Analogously for $t \leqslant 0$ we define the unstable set $U_\delta(x)$.

We say that $x \in W$ has a *local product structure* if there exists a homeomorphism of $\mathbb{R}^2$ onto an open neighborhood of x in $H_\alpha(x)$ that maps horizontal (vertical) lines onto open subsets of local stable (unstable) sets. The main consequence of expansivity is the following proposition which is proved in [48] (see [32], [26] for the discrete version).

PROPOSITION 6.3. — *Except for a finite number of periodic orbits, whose points we shall call singular, every point of W has a local product structure. If x is a singular point, $S_\delta(x)$ is a union of r arcs, $r \geqslant 3$, that only meet at x.*

We are now ready for the proof of Theorem 6.1. Inaba and Matsumoto showed [29] that a closed 3-manifold that supports a non-singular expansive flow must be aspherical. Therefore by Lemma 6.2 we must have that $k > e$ (note that e is not a regular value of the energy). When $k > e$, the energy level $\Sigma \stackrel{\text{def}}{=} E^{-1}(k)$ is a circle bundle over M and by a result of Brunella [7] $\phi_t|_\Sigma$ is topologically conjugate to the geodesic flow of a metric on M of constant negative curvature. It follows then, that there are *no* singular points and that the stable sets give rise to a continous foliation on Σ. Hence we can attach to each continuous closed curve $\alpha : S^1 \to \Sigma$ a Maslov type index $m(\alpha)$ just as it was done in [47] for the geodesic flow case. This index defines an integer cohomology class and it is roughly the winding number of the stable foliation around α. As a consequence of the convexity we have the following two basic properties which are proved very much in the same way as in [47] for the geodesic flow:

1. if α is a closed orbit of ϕ_t then $m(\alpha) \geqslant 0$;
2. if α is a closed orbit of ϕ_t then $m(\alpha) > 0$ if and only if the orbit α has conjugate points.

Now note that since $\phi_t|_\Sigma$ is topologically conjugate to the geodesic flow of a metric on M of constant negative curvature, the closure of the set of primitive closed orbits of ϕ_t in $H_1(\Sigma, \mathbb{R})$ is the closure of a convex open set containing the origin in its interior, since the same property holds for the geodesic flow of a compact negatively curved manifold. Thus if $\beta : H_1(\Sigma, \mathbb{R}) \to \mathbb{R}$ is any non-trivial cohomology class, there exists a closed orbit γ of ϕ_t so that $\beta(\gamma) < 0$.

Suppose now that the level Σ has conjugate points. Since the closed orbits of ϕ_t are dense we can find a closed orbit α that possesses conjugate points and hence positive index. Therefore the cohomology class m is non-trivial and has the property that if γ is any closed orbit of ϕ_t, then $m(\gamma) \geqslant 0$. This contradiction completes the proof of the theorem. $\diamond$

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