

# SÉMINAIRE DE THÉORIE SPECTRALE ET GÉOMÉTRIE

ROBERT BROOKS

## Isospectral graphs and isospectral surfaces

*Séminaire de Théorie spectrale et géométrie*, tome 15 (1996-1997), p. 105-113

[http://www.numdam.org/item?id=TSG\\_1996-1997\\_\\_15\\_\\_105\\_0](http://www.numdam.org/item?id=TSG_1996-1997__15__105_0)

© Séminaire de Théorie spectrale et géométrie (Grenoble), 1996-1997, tous droits réservés.

L'accès aux archives de la revue « Séminaire de Théorie spectrale et géométrie » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## ISOSPECTRAL GRAPHS AND ISOSPECTRAL SURFACES

*Robert BROOKS*

*In memory of Hubert Pesce*

In this paper, we investigate the following question: to what extent is there a converse to the Theorem of Sunada [Su] in the context of graphs?

Our experience in dealing with the question, “Can one hear the shape of a drum?” is that the many facets of this question turn out to be surprisingly delicate. The present instance is no exception.

We will first present a partial converse to Sunada’s Theorem, giving a necessary and sufficient condition for two graphs to be isospectral in terms of a Sunada-like condition. We will then present a construction of isospectral graphs, known as Seidel switching [CDGT], which on its face seems to have little to do with the Sunada condition. In fact, it remains an open problem whether graphs constructed this way need arise from a Sunada construction.

Finally, we will sketch a construction from [BGG] of large sets of mutually isospectral Riemann surfaces. While this material is somewhat independent of the rest of the material, it connects with it in the following way: these sets grow in size like  $g^{(\text{const}) \log(g)}$ , where  $g$  is the genus of the surface. On the other hand, a construction based on Seidel switching gives a growth rate of isospectral sets of graphs exponential in  $g$ . This suggests that the

---

\*Partially supported by a grant from the Israel Academy of Sciences, the Fund for the Promotion of Research at the Technion, and the C. Wellner Fund

graphs arising from Seidel switching in fact do not arise from the Sunada construction, because there are too many of them.

My thinking on this subject was prompted by the work of Hubert Pesce [Pe 1,2,3], giving a converse to Sunada's Theorem in the context of manifolds. As a member of the jury of Hubert's *habilitation*, I raised the question of whether he had thought about a converse to Sunada's Theorem in the context of graphs. I had very much hoped to interest him in the problems raised in this paper.

Like so many others, I was shocked to hear of his untimely death six months later. The spectral geometry community has lost a strong and vigorous talent. We have also lost a good friend, who was always willing to share his zest for life with us.

## 1 Sunada's Theorem for Graphs

In this section, we present a version of Sunada's Theorem in the context of  $k$ -regular graphs. We will then show how a generalized version of Sunada's Theorem actually gives a necessary and sufficient condition for two  $k$ -regular graphs to be isospectral.

Let  $\Gamma$  be a  $k$ -regular graph, and let  $G$  be a group of automorphisms of  $\Gamma$ . If  $H_i$  are subgroups of  $G$  acting freely on  $\Gamma$ , then the graphs  $\Gamma_i = \Gamma/H_i$  are defined.

We will say that the quadruple  $(\Gamma, G, H_1, H_2)$  satisfies the Sunada condition if, for all  $g \in G$ ,

$$\#([g] \cap H_1) = \#([g] \cap H_2),$$

where  $[g]$  denotes the conjugacy class of  $g$  in  $G$ .

For every positive integer  $n$ , let  $H_i^{(n)}$  be the set of elements  $h$  of  $H_i$  such that there is a path of length  $\leq n$  in  $\Gamma$  whose endpoints differ by multiplication by  $h$ . We will say that  $(\Gamma, G, H_1, H_2)$  satisfies the Sunada condition up to length  $n$  if

$$\#([g] \cap H_1^{(n)}) = \#([g] \cap H_2^{(n)}).$$

We then have:

**Theorem 1** (a) *Suppose that  $(\Gamma, G, H_1, H_2)$  satisfies the Sunada condition. Then the graphs  $\Gamma_1$  and  $\Gamma_2$  are isospectral.*

(b) Suppose that  $(\Gamma, G, H_1, H_2)$  satisfies the Sunada condition up to length  $N$ , where

$$N \geq \max(\#(\Gamma), \#(\Gamma_2)).$$

Then  $\Gamma_1$  and  $\Gamma_2$  are isospectral.

**Proof:** To prove (a), we make use of the fact that two  $k$ -regular graphs  $H_1$  and  $H_2$  are isospectral if and only if, for any  $l$ , the number of closed paths of length  $l$  is the same for  $H_1$  and  $H_2$ .

The Sunada condition then says that, for a given path  $\gamma$  in  $\Gamma$ , not necessarily closed, the number of times a  $G$ -translate of  $\gamma$  covers a closed path in  $\Gamma_1$  is the same as the number of times  $\gamma$  covers a closed path in  $\Gamma_2$ . It follows that the two graphs are isospectral.

To establish (b), we note that the set of numbers of closed paths of length  $l$  for  $l = 1, \dots, \#(\Gamma_i)$ , determines the number of lengths for all  $l$ . This establishes (b).

One typically uses this theorem in the case where  $\Gamma$  is the Cayley graph of  $G$ , with respect to some set of generators of  $G$ . We will say that the two graphs  $\Gamma_1$  and  $\Gamma_2$  constructed in this way are *simple Sunada equivalent*. We will refer to  $\Gamma_1$  and  $\Gamma_2$  arising in the more generalized case where  $G$  need not act freely transitively on  $\Gamma$  as being *general Sunada equivalent*. This distinction will be important in the next section.

We now sketch a proof of the following converse to Theorem 1:

**Theorem 2** *Let  $\Gamma_1$  and  $\Gamma_2$  be two  $k$ -regular isospectral graphs. Then, for any  $n$ , there is a graph  $\Gamma^{(n)}$  which covers both  $\Gamma_1$  and  $\Gamma_2$ , a group of graph automorphisms  $G^n$ , and two subgroups  $H_1^{(n)}$  and  $H_2^{(n)}$  which act freely on  $\Gamma^{(n)}$ , with*

$$\Gamma_i = \Gamma^{(n)} / H_i^{(n)}, i = 1, 2,$$

*so that  $(\Gamma^{(n)}, G^n, H_1^{(n)}, H_2^{(n)})$  satisfies the Sunada condition up to length  $n$ .*

We begin the proof with an application of Leighton's Theorem ([Le], [AG]) in the  $k$ -regular case, which asserts that any two finite  $k$ -regular graphs have a common finite covering. See [TS] for a discussion and proof of Leighton's Theorem in the  $k$ -regular case along the lines we are following.

We may think of Leighton's Theorem as describing for us a (possibly orbifold) graph  $\Gamma_0^{(0)}$ , such that both  $\Gamma_1$  and  $\Gamma_2$  cover  $\Gamma_0^{(0)}$ . Then the fundamental

groups of  $\Gamma_1$  and  $\Gamma_2$  sit inside the fundamental group of  $\Gamma_0^{(0)}$  (considered as an orbifold) as subgroups of finite index, and hence there is a normal subgroup contained in both of them of finite index. We may set  $\Gamma^{(n)}$  to be the covering determined by this subgroup, and  $G^{(n)}$  the quotient group.

Isospectrality of  $\Gamma_1$  and  $\Gamma_2$  says that for each number  $l$ , the number of lifts of closed paths of length  $l$  in  $\Gamma_0^{(0)}$  to closed paths in  $H_i$ ,  $i = 1, 2$  respectively, agree. The Sunada condition up to length  $n$  would say that, for each closed path of length  $l$ ,  $l \leq n$ , the number of lifts would be the same. This need not be the case if  $\Gamma_0^{(0)}$  had several paths of length  $l$ , but would follow from isospectrality if we knew there were only one path of length  $l$  in  $\Gamma_0^{(0)}$  which lifted to a closed path in  $\Gamma_1$  and  $\Gamma_2$ .

We now make use of the fact that, given any two paths of the same length in a  $k$ -regular tree, there is an automorphism of the tree taking one to the other. We use this to construct an orbifold graph  $\Gamma_0^{(n)}$ , such that  $\Gamma_0^{(0)}$  covers  $\Gamma_0^{(n)}$ , and so that, for each  $l \leq n$ , the closed paths of length  $l$  on  $\Gamma_0^{(0)}$  all project to the same path on  $\Gamma_0^{(n)}$ .

This completes the sketch of the proof.

## 2 Seidel Switching

In this section, we give a method for constructing isospectral  $k$ -regular graphs which do not, at least on the face of it, arise from Sunada's Theorem. This technique, known as Seidel switching, is well-known in the graph-theory literature [CDGT], and has been greatly generalized [GM]. Here, we present a fairly down-to-earth version which has the advantage of giving rather explicit examples.

See [Qu] for a lovely account of Seidel switching.

Let  $G_1$  and  $G_2$  be two  $k$ -regular graphs, with  $\#(G_1) = \#(G_2) = N$  an even number, and  $\Delta$  a subset of  $G_1 \times G_2$  with the following properties:

- (i) for  $g \in G_1$ , the set of  $g_2$  such that  $(g, g_2) \in \Delta$  has order  $N/2$ .
- (ii) for  $g \in G_2$ , the set of  $g_1$  such that  $(g_1, g) \in \Delta$  has order  $N/2$ .

Let  $\Gamma_1$  and  $\Gamma_2$  be the following  $(k + N/2)$ -regular graphs:

- (a) The vertices of  $\Gamma_1$  and  $\Gamma_2$  are precisely the union of the vertices of  $G_1$  and  $G_2$ .

- (b) If  $v_1$  and  $v_2$  are both vertices in  $G_1$  (resp.  $G_2$ ), then for both  $\Gamma_1$  and  $\Gamma_2$ , the set of edges joining  $v_1$  and  $v_2$  is precisely the set of edges joining  $v_1$  and  $v_2$  in  $G_1$  (resp.  $G_2$ ).
- (c) If  $v_1 \in G_1$  and  $v_2 \in G_2$ , then  $v_1$  is joined to  $v_2$  in  $\Gamma_1$  if and only if  $(v_1, v_2) \in \Delta$ .
- (d) If  $v_1 \in G_1$  and  $v_2 \in G_2$ , then  $v_1$  is joined to  $v_2$  in  $\Gamma_2$  if and only if  $(v_1, v_2) \notin \Delta$ .

We illustrate this rather confusing-sounding recipe in the diagrams below.

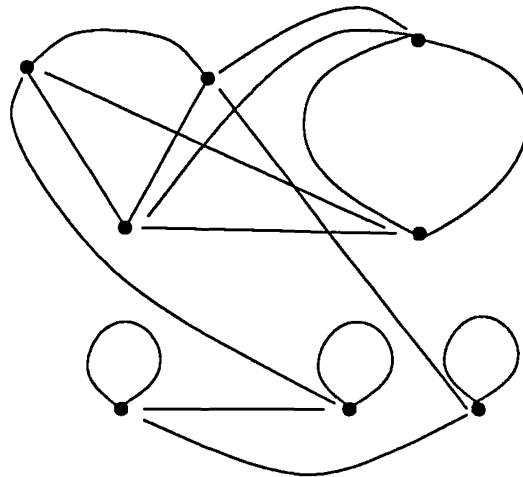


Figure 1: Seidel switching: the graphs  $G_1$  and  $G_2$

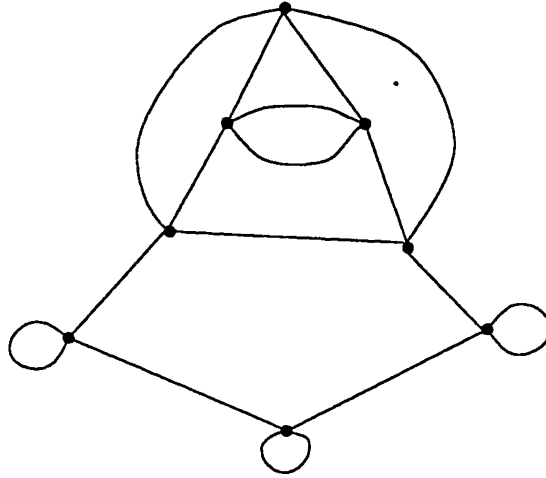
**Theorem 3 (Seidel switching)** *The graphs  $\Gamma_1$  and  $\Gamma_2$  so constructed are isospectral.*

The proof of this theorem is a simple yet clever argument by counting closed paths of a given length.

It is shown by a direct argument in [BL] that the  $\Gamma_1$  and  $\Gamma_2$  given above are not simple Sunada equivalent.

**Question:** Are the two graphs general Sunada equivalent?

We mention that simple Sunada equivalence has the following geometric interpretation: a coloring of type  $(r, s)$  of a  $(2r + s)$ -regular graph is an assignment to each edge of either a color in the set  $\{1, \dots, r\}$  and a direction,

Figure 2: Seidel switching: The graph  $\Gamma_1$ 

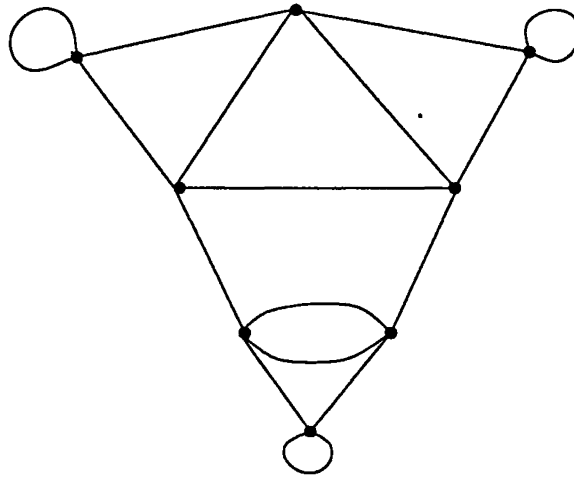
or a color in the set  $\{r + 1, \dots, r + s\}$ , such that each vertex has precisely one outgoing and one incoming edge of color  $i$ , for each  $i \in \{1, \dots, r\}$ , and one edge of type  $i$ ,  $i \in \{r + 1, \dots, r + s\}$ . Then two graphs are simple Sunada equivalent if they admit colorings such that the number of closed paths of a given sequence of colors and directions is the same for the two graphs.

General Sunada equivalence has a similar interpretation, except that we group together patterns under a finite set of pattern equivalences.

### 3 Mutually Isospectral Graphs and Surfaces

We now raise the question: given a  $k$ -regular graph  $\Gamma$ , how many graphs are there isospectral to  $\Gamma$ ? This question becomes meaningful if we let the size of the graph grow in some meaningful way.

One motivation in raising this question is to compare the size of isospectral sets we can achieve with the size of sets of Sunada equivalent graphs. If the numbers we obtain this way are different, then we have presented some evidence to the effect that many isospectral graphs are not Sunada equivalent. While we are at present a long way from this goal, we find the evidence presented rather convincing.

Figure 3: Seidel switching: The graph  $\Gamma_2$ 

We have the following:

**Theorem 4** *For every integer  $n$  divisible by 4, there is a family of cardinality  $2^{n/4-1}$  of mutually isospectral 6-regular graphs with  $n$  vertices.*

The proof is based on stitching together a number of graphs via Seidel switching. The power of 2 comes from the fact that we can perform switching independently at each of the stitchings.

When one allows the regularity to grow, one needs some measure of the complexity of a graph that is finer than the number of vertices. We find a convenient measure to be the genus of the graph, which is given by Euler's formula as

$$g = 1 + \frac{V}{2} \left( \frac{k}{2} - 1 \right),$$

where  $V$  is the cardinality of the graph and  $k$  the regularity.

The following was worked out with Greg Quenell, based on [GM]:

**Theorem 5** (a) *For a set of  $n \rightarrow \infty$ , there are sets of isospectral  $k$ -regular graphs on  $n$  vertices, where  $k \rightarrow \infty$  with  $n$ , which grow in size like  $n^{(\text{const})^n}$ .*  
 (b) *For a set of  $g \rightarrow \infty$ , there are sets of isospectral  $k$ -regular graphs of genus  $g$ , where  $k \rightarrow \infty$ , which grow in size like  $e^{(\text{const})^g}$ .*



Indeed, the graphs of (b) are the sets constructed by Seidel switching. We have not been able to do better than this when one measures growth by the genus.

When we restrict to sets of Sunada isospectral graphs, the situation changes dramatically. Here, we find it easier to work with Riemann surfaces, and then later restrict to graphs.

We show:

**Theorem 6 ([BGG])** *For a set of  $g \rightarrow \infty$ , there are sets of mutually isospectral Riemann surfaces of genus  $g$ , of cardinality  $\sim g^{(\text{const}) \log(g)}$ .*

The construction involves an investigation of Heisenberg groups over finite fields. Roughly speaking, the Sunada triples that enter here are the discrete analogues of the nilpotent isospectral deformations of Gordon and Wilson [GW].

Because these surfaces arise by the Sunada method, they give rise to isospectral graphs which are simple Sunada equivalent.

#### REFERENCES

- [AG] D. Angluin and A. Gardiner, "Finite Common Coverings of Pairs of Graphs," *J. Comb. Theory B* 30 (1981), pp. 184-187.
- [TS] R. Brooks, "Twist Surfaces," to appear *Proc. Cortona Conf.*
- [BGG] R. Brooks, R. Gornet, and W. Gustafson, "Mutually Isospectral Riemann Surfaces," to appear *Adv. Math.*
- [BL] R. Brooks and A. Lubotzky, "Non-Sunada Graphs," preprint.
- [CDGT] D. Cvetkovic, M. Doob, I. Gutman, and A. Targasev, *Recent Results in the Theory of Graph Spectra*, Ann. Disc. Math. 36, North Holland, 1988.
- [GM] C. Godsil and B. McKay, "Constructing Cospectral Graphs," *Aeq. Math.* 25 (1982), pp. 257-268.
- [GW] C. Gordon and E. Wilson, "Isospectral Deformations of Compact Solvmanifolds," *J. Diff. Geom.* 19 (1984), pp. 241-256.

- [Le] F. Leighton, "Finite Common Coverings of Graphs," *J. Comb. Th. B* 33 (1982), pp. 231-238.
- [Qu] G. Quenell, "The Combinatorics of Seidel Switching," preprint.
- [Pe1] H. Pesce, "Une réciproque Générique du Théorème du Sunada," to appear in *Compositio Math.*
- [Pe2] H. Pesce, "Quelques Applications de la Théorie des Représentations en Géométrie Spectrale," Habilitation, Université Joseph Fourier (Grenoble I)
- [Pe3] H. Pesce, "Variétés Isospectrales et Représentations des Groupes," in Brooks, Gordon, and Perry (eds), *Geometry of the Spectrum*, *Contemp. Math.* 173 (1994), pp. 231-240.
- [Su] T. Sunada, "Riemannian Coverings and Isospectral Manifolds," *Ann. Math.* 121 (1985), pp. 169-186.

Robert BROOKS  
Department of Mathematics  
The Technion  
HAIFA (Israël)