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AN APPLICATION OF RAMANUJAN GRAPHS TO C*-ALGEBRA TENSOR PRODUCTS, II

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Let A, B be C*-algebras; denote by $A \otimes B$ the algebraic tensor product of A and B. According to the general theory of tensor products of C*-algebras, C*-norms on $A \otimes B$ lie between a minimal C*-norm, denoted by $\|\cdot\|_{\min}$, and a maximal C*-norm, denoted by $\|\cdot\|_{\max}$ (see [Tak79] for all this). Denote by H an infinite-dimensional separable Hilbert space, and by $B(H)$ the C*-algebra of linear, bounded operators on H. It was an old problem to decide whether or not all C*-norms coincide on $B(H) \otimes B(H)$; in a remarkable paper [JP95], M. Junge et G. Pisier give a negative answer to that question. More precisely, to describe quantitatively the discrepancy between the maximal and minimal C*-norms on $B(H) \otimes B(H)$, they introduce for any $n \in \mathbf{N}$ the number

$$\lambda(n) = \sup\{\frac{\|u\|_{\max}}{\|u\|_{\min}} : u \text{ tensor of rank } n \text{ in } B(H) \otimes B(H)\}$$

and they prove:

THEOREM 1. — *There exists a constant $c > 0$ such that, for any $n \in \mathbf{N}$:*

$$c\sqrt[8]{n} \leq \lambda(n) \leq \sqrt{n}.$$

In [JP95], Junge and Pisier ask for the precise asymptotic behaviour of $\lambda(n)$ for $n \to \infty$. Actually they provide an excellent way of getting lower bounds on $\lambda(n)$ by introducing another constant C_n as follows:

DEFINITION 1. — *Let $\mathbf{F_n}$ denote the free group on n generators $a_1, a_2, , a_n$. Let C_n denote the infimum of all numbers $C > 0$ for which there exists a sequence $(\pi_k)_{k\geq 1}$ of unitary, finite-dimensional representations of $\mathbf{F_n}$ such that:*

$$\|\sum_{i=1}^{n}(\pi_k \otimes \overline{\pi_m})(a_i)\| \leq C \text{ for all } k \neq m$$

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(where $\overline{\pi_m}$ denotes the contragredient representation of π_m).

It is then proved in [JP95] that the constants $\lambda(n)$ and C_n are related through the following inequality:

$$\frac{n}{C_n} \leq \lambda(n) \qquad (*)$$

In [Pis], Pisier proved:

PROPOSITION 1. — *For any $n \geq 2$, one has*

$$C_n \geq 2\sqrt{n-1}.$$

On the other hand, using Ramanujan graphs (see [Lub94]), I proved in [Val]:

PROPOSITION 2. — *Let q be a prime power. Then*

$$C_{q+1} \leq 2\sqrt{q}.$$

From these two results, it is natural to conjecture that $C_n = 2\sqrt{n-1}$ for any $n \geq 2$. I shall confirm this conjecture by proving that it holds asymptotically. I am grateful to U. Haagerup and G. Skandalis for suggesting to me the possibility of such a proof.

PROPOSITION 3.

$$\lim_{n\to\infty} \frac{C_n}{2\sqrt{n}} = 1$$

Proof: It follows from Proposition 3 in [Val] that the sequence $(\frac{C_n}{2\sqrt{n}})_{n\geq 1}$ is bounded, so let K be an upper bound for that sequence. It is also known that the function $n \to C_n$ is sub-additive (see the lemma in [Val]). Denote by p_k the k-th prime. For fixed n, let k be such that $p_k + 1 \leq n < p_{k+1}$. Then:

$$C_n \leq C_{p_k+1} + C_{n-p_k-1}$$

Using Proposition 2, we get:

$$C_n \leq 2\sqrt{p_k} + 2K\sqrt{n-p_k-1} \leq 2\sqrt{p_k} + 2K\sqrt{p_{k+1}-p_k}$$

Hence:

$$\frac{C_n}{2\sqrt{n}} \leq \frac{C_n}{2\sqrt{p_k}} \leq 1 + K\sqrt{\frac{p_{k+1}}{p_k} - 1}.$$

By the prime number theorem of Hadamard and de la Vallée-Poussin, the ratio $\frac{p_{k+1}}{p_k}$ tends to 1 for $k \to \infty$. Thus:

$$\limsup_{n\to\infty} \frac{C_n}{2\sqrt{n}} \leq 1.$$

On the other hand, it follows from Proposition 1 that we also have

$$\liminf_{n\to\infty} \frac{C_n}{2\sqrt{n}} \geq 1,$$

so that the result is proved.

From Proposition 3 and the inequality (*), it immediately follows that:

COROLLARY 1. —

$$\liminf_{n\to\infty} \frac{\lambda(n)}{\sqrt{n}} \geq \frac{1}{2}.$$

References

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