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Compactness of Isospectral Sets

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Let M be a compact Riemannian manifold, and $\Delta = -\mathrm{div}(\mathrm{grad})$ the Laplacian on M. The solutions to the equation

$$\Delta(f) = \lambda \cdot f$$

are called eigenfunctions, and the corresponding λ's are eigenvalues. It is not hard to see that there are countably many λ's, tending to ∞, and for each λ a finite-dimensional family of functions f, for which this equation has a solution. The λ's may be thought of as the "fundamental frequencies" of M, just as a musical instrument has a series of overtones which are the frequencies at which it vibrates.

A fundamental question is: to what extent do the eigenvalues of M determine the geometry and toipology of M? This was framed by Mark Kac in 1966 as the question: Can one hear the shape of a drum?

Our first observation is that the general answer to the question is "no." Indeed, one has, by a construction of Sunada [**Su**], a powerful technique for constructing many different types of isospectral manifolds.

On the other hand, it should be the case that the spectrum of Δ tells us a lot about M. For instance, it has been well-known for many years that the spectrum of Δ of a surface determines the genus of the surface, and recent work of Osgood, Phillips, and Sarnak [**OPS**] shows that the spectrum of Δ of a surface determines the surface up to a compact family of metrics.

Our first result is:

Theorem 1 ([BPP]) *(a) Sets of isosapectral manifolds of negative curvature contain only finitely many topological types.*

(b) Sets of isospectral manifolds whose sectional curvatures are bounded from below contain only finitely many topological types.

Theorem 2 ([BPP]) *(a) Sets of isospectral 3-manifolds of negative curvature are compact.*

(b) Sets of isospectral 3-manifolds whose Ricci curvatures are bounded from below are compact.

The fundamental idea behind these theorems is that if one can decompose a manifold into some fixed number of pieces, all of which are pretty much standard, then one can determine M by the finitely many ways in which these pieces can be assembled. This is an important idea behind the Cheeger Finiteness Theorem [**Ch**].

In part (a) in the theorems above, the finitely many pieces are given by balls of radius less than the injectivity radius of M. which are topologically disks. Once one has a bound on the injectivity radius, which one obtains from the asymptotics of the wave equation, one obtains the desired decomposition of M.

In part 1(b), the finitely many pieces are given by balls of a fixed radius, which may be topologically non-trivial, and may also be of very small volume. A theorem of Cheng [**Cg**] shows how to control the number of such pieces by the low eigenvalues of M, while an argument of Grove and Petersen [**GP**] shows that, even though these balls may be topologically non-trivial, their contribution to the topology of M can be controlled.

To obtain 2, we make use of the decomposition of M into bits to estimate the Sobolev constant of M. Knowing this, and using an estimate of Gilkey [**Gi**] on the leading terms in the asymptotics of the heat equation. one can get better and better bounds on the curvature and its covariant derivatives, thus establishing the desired compactness.

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