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**RENCONTRES DE THEORIE SPECTRALE ET GEOMETRIE  
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# **The Evolution of Harmonic Maps**

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SUISSE**

1. Let  $M$  and  $N$  be compact Riemannian manifolds of dimensions  $m$  and  $\ell$  and with metrics  $\gamma$  and  $g$ , respectively. We may assume that  $N$  is isometrically embedded in some Euclidean  $\mathbb{R}^n$ .

For  $C^1$ -maps  $u : M \rightarrow N \subset \mathbb{R}^n$  let

$$e(u) = \frac{1}{2} |\nabla u|^2 = \frac{1}{2} \sum_{\alpha, \beta=1}^m \sum_{i=1}^n \gamma^{\alpha\beta} \frac{\partial}{\partial x_\alpha} u^i \frac{\partial}{\partial x_\beta} u^i \quad (1.1)$$

be the energy density and - with  $dM = \sqrt{\det(\gamma_{\alpha\beta})} dx$  - let

$$E(u) = \int_M e(u) dM$$

be the energy of  $u$ . If  $M \subset \mathbb{R}^m$  carries the Euclidean metric,  $E$  is nothing but the standard Dirichlet integral

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 dx.$$

A map  $u$  is harmonic if  $E$  is stationary at  $u$ , which for  $u \in C^2$  is equivalent to the condition that the vector  $(\Delta_M u)(x)$  at all points  $x \in M$  is orthogonal to the tangent space  $T_{u(x)}N$  to  $N$  at  $u(x) \in \mathbb{R}^n$ ; that is,

$$\Delta_M u \perp T_u N. \quad (1.2)$$

( $\Delta_M$  is the Laplace-Beltrami operator on  $M$ .) In local coordinates on  $N$ , (1.2) takes the form

$$\Delta_M u = \Gamma(u)(\nabla u, \nabla u) \quad (1.3)$$

with a bilinear form  $\Gamma$  involving the Christoffel symbols of the metric  $g$  on  $N$ . Again, if  $M \subset \mathbb{R}^m$  and if in particular,  $N = S^\ell \subset \mathbb{R}^n$  ( $n = \ell + 1$ ), (1.3) takes the following simple form

$$\Delta u = u |\nabla u|^2$$

The notion of harmonic map generalizes the concepts of closed geodesic (for  $M = S^1$ ) and harmonic function (for  $N = \mathbf{R}$ ).

The study of harmonic maps was initiated by Fuller, Nash, and Sampson; see Eells-Lemaire [12] for further background material and references. The first general existence result is due to Eells and Sampson. Since standard variational techniques fail, in their pioneering work Eells and Sampson [13] introduce the evolution problem

$$\partial_t u - \Delta_M u \perp T_u N \quad \text{on } M \times ]0, \infty[ \quad (1.4)$$

with initial condition

$$u = u_0 \quad \text{at } t = 0. \quad (1.5)$$

Upon multiplying (1.4) by  $\partial_t u \in T_u N$  and integrating by parts, we at once obtain the energy inequality

$$E(u(T)) + \int_0^T \int_M |\partial_t u|^2 dM dt \leq E(u_0), \quad \forall T > 0 \quad (1.6)$$

for any solution  $u$  of (1.4), (1.5). That is, (1.4) is the  $L^2$ -gradient flow for  $E$ . In local coordinates again, (1.4) takes the form

$$\partial_t u - \Delta_M u = \Gamma(u)(\nabla u, \nabla u). \quad (1.7)$$

Upon differentiation the latter expression and multiplying by  $\nabla u$ , we moreover obtain the following Bochner-type differential inequality for the energy density

$$(\partial_t - \Delta_M)\epsilon(u) + c|\nabla^2 u|^2 \leq K_N \epsilon(u)^2 + C_M \epsilon(u), \quad (1.8)$$

where  $K_N$  denotes an upper bound for the sectional curvature of  $N$ ,  $c > 0$ , and  $C_M$  depends on (the Ricci curvature of) the metric  $\gamma$ . (1.6), (1.8) and Moser's weak Harnack inequality for parabolic equations now lead to the following result.

**Theorem 1 (Eells-Sampson [13]):** *Suppose  $K_N \leq 0$ , then for any smooth map  $u_0 : M \rightarrow N$  problem (1.4), (1.5) admits a unique, smooth solution  $u(t)$  which, as  $t \rightarrow \infty$  suitably, converges to a smooth harmonic map  $u_\infty$  homotopic to  $u_0$ .*

Theorem 1 was extended to manifolds with boundary by Hamilton [17]. Moreover, in this case the curvature restriction  $\kappa_N \leq 0$  can be weakened if the initial and boundary data have small range (Jost [21]). However, for general data, the curvature restriction  $K_N \leq 0$  is in some sense optimal as Eells and Wood [14] show that Theorem 1 ceases to be true for  $M = T^2$ ,  $N = S^2$  and an initial map  $u_0$  of topological degree 1.

2. Nevertheless, in two dimensions ( $m = 2$ ) Lemaire [23] and Sacks-Uhlenbeck [29] independently showed that also the topological condition  $\pi_2(N) = 0$  suffices to find harmonic representatives for all homotopy classes of maps  $u_0 : M \rightarrow N$ . In fact, a new proof of this result, using the evolution problem (1.4), can be given [35]. (Since by the result of Eells and Wood we must expect singularities, we consider also maps in the Sobolev space

$$H^{1,2}(M; N) = \left\{ u \in H^{1,2}(U; \mathbb{R}); u(M) \subset N \right\}$$

of measurable, finite energy maps  $u : M \rightarrow N$ ; that is, with distributional derivative in  $L^2$ .) Then we have the following generalization of Theorem 1.

**Theorem 2** (Struwe [39; Theorem 4.2]): *Suppose  $m = 2$ . For any  $u_0 \in H^{1,2}(M; N)$  there exists a global weak solution  $u : M \times ]0, \infty[ \rightarrow N$  of (1.4), (1.5) which satisfies (1.6) and is  $(C^\infty -)$  regular away from finitely many points  $(\bar{x}_k, \bar{t}_k)$ ,  $1 \leq k \leq K$ . The solution  $u$  is unique in this class.*

*At a singularity  $(\bar{x}, \bar{t})$  a "harmonic sphere"  $\bar{u} : S^2 \cong \overline{\mathbb{R}^2} \rightarrow N$  separates in the sense that for suitable  $x_m \rightarrow \bar{x}$ ,  $R_m \searrow 0$ ,  $t_m \nearrow \bar{t}$  we have*

$$u_m(x) := u(\exp_{x_m}(R_m x), t_m) \longrightarrow \bar{u} \quad \text{in } H_{\text{loc}}^{2,2}(\mathbb{R}^2; N), \quad (2.1)$$

where  $\bar{u} \not\equiv \text{const.}$  is harmonic, has finite energy and extends to a smooth harmonic map  $\bar{u} : S^2 \cong \overline{\mathbb{R}^2} \rightarrow N$ .

*Finally, as  $t \rightarrow \infty$  suitably,  $u(t) \rightarrow u_\infty$  weakly in  $H^{1,2}(M; N)$ , where  $u_\infty : M \rightarrow N$  is smooth and harmonic. Convergence is strong away from finitely many points  $(\bar{x}_\ell, \bar{t}_\ell = \infty)$ ,  $1 \leq \ell \leq L$ , where harmonic spheres separate in the sense (2.1).*

*Remark.* Let  $\varepsilon_0 = \inf\{E(u); u : S^2 \rightarrow N \text{ is non-constant and harmonic}\} > 0$ . Then by (2.1) we can bound  $K + L \leq \varepsilon_0^{-1} E(u_0)$ . In particular, for initial data such that  $E(u_0) < \varepsilon_0$  the solution  $u$  constructed in Theorem 2 is smooth and converges uniformly to a harmonic limit.

Theorem 2 was extended to 2-manifolds  $M$  with boundary  $\partial M \neq \emptyset$  by Chang [3] with applications to results by Brezis-Coron and Jost on the Dirichlet problem for harmonic maps into the sphere, and by Chen-Musina [7] to the case of target manifolds with boundary.

Moreover, variants of (1.4) arise if one attempts to solve free boundary problems for minimal surfaces by a deformation method and results completely analogous to Theorem 2 hold; see Struwe [34]. Ma Li [24] has recently generalized these results to free boundary problems for harmonic maps.

To this day it is not known whether in general (in two dimensions) the flow (1.4) will encounter singularities in finite time. (Of course, the result of Eells and Wood provides us with an example where either such singularities exist or the flow fails to converge asymptotically.) However, some recent results of Chang and Ding [4], respectively work by Grayson and Hamilton [16] lend support to the conjecture that for  $m = 2$  the flow (1.4) does not develop singularities in finite time.

3. The situation is quite different in higher dimensions, as there are energy minimizing harmonic maps with singularities. An example is given by the well-known map  $u(x) = \frac{x}{|x|} : B_1(0) \subset \mathbb{R}^m \rightarrow S^{m-1} \subset \mathbb{R}^m$  (Brezis-Coron-Lieb [2], Lin [26]); if  $m \geq 7$ ,  $u$  is minimizing even if we regard  $u$  as a map  $u : B_1(0) \rightarrow S^m \subset \mathbb{R}^{m+1}$  (Jäger-Kaul [20]). For energy-minimizing maps therefore only partial regularity results can be expected. Such a regularity theory was developed by Schoen and Uhlenbeck [30] - and independently by Giaquinta-Giusti [15] in the case that the image is covered by a single chart. A crucial role is played by a subtle monotonicity estimate.

Similarly, progress on the evolution problem (1.4) for  $m \geq 3$  and general targets came as a consequence of a peculiar monotonicity formula for (1.4), discovered in [36; Lemma 3.2] for maps  $u : \mathbb{R}^m \times [0, T] \rightarrow N$  and extended to curved domains by Chen-Struwe [8]. Let

$$G_{z_0}(x, t) = \frac{1}{\sqrt{2\pi(t_0 - t)}^m} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right), \quad \text{if } t < t_0 \quad (3.1)$$

be the fundamental solution to the heat equation on  $\mathbb{R}^m \times \mathbb{R}$  with singularity at  $z_0 = (x_0, t_0)$ , and let  $\varphi \in C_0^\infty(\mathbb{R}^m)$  be a smooth cut-off function such that  $\varphi \equiv 1$  in a neighbourhood of 0 and such that the support of  $\varphi$  is contained in ball of radius  $\rho$  less than the injectivity radius  $\rho_M$  of  $M$ . For a solution  $u : M \times ]0, T[ \rightarrow N$  and  $R^2 < t_0 < T$  let

$$\Phi(R; z_0) = \frac{1}{2} R^2 \int_{B_\rho(x_0)} |\nabla u|^2 \varphi^2 G_{z_0} dx \Big|_{t=t_0-R^2},$$

in local normal coordinates around  $x_0$  on  $M$ .

**Theorem 3** (Chen-Struwe [8; Lemma 4.2]): *There exists a constant  $C$  depending only on  $M$  and  $N$  such that for any  $T > 0$ , any  $0 < R^2 < R_0^2 \leq t_0 < T$  and any regular solution  $u : M \times ]0, T[ \rightarrow N$  of (1.4) there holds*

$$\Phi(R; z_0) \leq \exp(C(R_0 - R)) \Phi(R_0; z_0) + CE(u_0)(R_0 - R). \quad (3.2)$$

A particular consequence of the monotonicity formula (3.2) is the following.

**Theorem 4** (Struwe [36; Theorem 5.1]; Chen-Struwe [8; Lemma 4.4]): *There exists a constant  $\varepsilon_0 > 0$  depending only on  $M$  and  $N$  such that for any solution  $u : M \times [0, T[ \rightarrow N$  of (1.4) the following is true:*

*If  $\Phi(R; z_0) < \varepsilon_0$  for some  $z_0 = (x_0, t_0)$ ,  $0 < R^2 \leq t_0 < T$ ,  $R < \iota_M$ , then  $|\nabla u(z_0)| \leq C$ , with a constant  $C = C(M, N, R)$ .*

Theorem 4 at once leads to a new proof of Mitteau's [27] global existence result for initial data with small energy density. More important, Theorem 4 together with a penalization device to obtain approximate solutions for (1.4), due to Chen [5], Keller-Rubinstein-Sternberg [22], and Shatah [31] led to a global existence and partial regularity result for (1.4) in higher dimensions ( $m \geq 3$ ).

**Theorem 5** (Chen-Struwe [8; Theorem 1.5]): *For any (smooth) map  $u_0 : M \rightarrow N$  there exists a global weak solution  $u : M \times ]0, \infty[ \rightarrow N$  of (1.4), (1.5) satisfying (1.6) and regular off a set of co-dimension  $\geq 2$  (for all  $t > 0$ ; see Cheng [9]). As  $t \rightarrow \infty$  suitably,  $u(t) \rightarrow u_\infty$  weakly in  $H^{1,2}(M, N)$  where  $u_\infty : M \rightarrow N$  is harmonic and regular off a set of co-dimension  $\geq 2$ .*

Moreover,  $u$  satisfies a variant of the monotonicity formula (3.2). This fact was used by Coron [10] to prove that the solution obtained in Theorem 5 is in general not unique - even among partially regular solutions satisfying (1.6).

The estimate on the co-dimension of the singular set very likely can be improved to 3, as for energy-minimizing (stationary) harmonic maps; see Giaquinta-Giusti [15] and Schoen-Uhlenbeck [30]. However, as was first observed by Coron-Ghidaglia [11], in higher dimensions singularities may appear in finite time. Subsequently, Chen and Ding [6] gave an argument relating singularities to the fact that in higher dimensions the infimum of the energy in certain non-trivial homotopy classes of maps may be 0, an observation due to B. White [37].

In fact, their reasoning can be considerably simplified by combining Theorem 4 with Moser's weak Harnack inequality for parabolic equations. First note:

**Theorem 6:** *For any  $T > 0$  there exists  $\varepsilon_1 > 0$  depending only on  $T, M$  and  $N$  such that any smooth solution  $u : M \times [0, T] \rightarrow N$  of (1.4), (1.5) with  $E(u_0) < \varepsilon_1$  can be extended to a global, smooth solution  $u : M \times [0, \infty[ \rightarrow N$ , converging, as  $t \rightarrow \infty$  suitably, to a constant harmonic map.*

*Proof.* Let  $R_0^2 = \inf\{t_M^2, T\}$ . For  $0 < R_0^2 \leq t_0 \leq T$ ,  $x_0 \in M$  we can estimate with constants  $C$  depending on  $M, N$ , and  $T$  only

$$\Phi(R_0; z_0) \leq CR_0^{2-m} E(u(t_0 - R_0^2)) \leq CE(u_0) < \varepsilon_0, \quad (3.3)$$

if  $\varepsilon_1 > 0$  is sufficiently small. Hence by Theorem 4 we have

$$|\nabla u(x, t)| \leq C \text{ uniformly for } t \geq R_0^2, x \in M,$$

and  $u$  can be extended as a smooth solution of (1.4), (1.5) on  $M \times [0, \infty[$ .

By (1.8) and Moser's [28] supremum estimate for weak sub-solutions of linear parabolic equations, moreover we obtain

$$|\nabla u(x, t)|^2 \leq CE(u(t - R_0^2)) \leq CE(u_0) \text{ for } t \geq 2R_0^2, x \in M. \quad (3.4)$$

From this uniform estimate, asymptotic convergence follows as in Eells-Sampson [13] or Jost [21]. Finally, if  $\varepsilon_1 > 0$  is sufficiently small, by (3.4) the image of any map  $u(t)$ ,  $t \geq 2R_0^2$ , and hence also of the limiting harmonic map  $u_\infty$  is contained in a strictly convex geodesic ball on  $N$ . It follows that  $u_\infty \equiv \text{const.}$ ; see Jäger-Kaul [19].  $\square$

By Theorem 6, of course, for homotopically non-trivial initial data  $u_0$  with  $E(u_0) < \varepsilon_1(T)$  the flow (1.4), (1.5) must blow up before time  $T$ . In fact, blow-up time approaches 0 as the initial energy decreases to 0.

Finally, we remark that in dimensions  $m \geq 3$  singularities - as in the case  $m = 2$  - may be related to harmonic spheres or to self-similar solutions  $u(x, t) = w\left(\frac{x-x_0}{\sqrt{t_0-t}}\right)$  of (1.4); see Struwe [36; Theorem 8.1]. (The work of Coron-Ghidaglia strongly suggests that solutions of the latter kind in dimensions  $m \geq 3$  actually exist.)

The approach of [8] in general cannot be extended to initial maps belonging to  $H^{1,2}(M; N)$ , only. A different approach via time-discrete minimization was proposed by Horihata-Kikuchi [18]. Based on their ideas, Bethuel et al. [1] recently established global existence of distribution solutions to (1.4) for finite energy maps into spheres.

Further directions of present research into (1.4) include the study of (1.4) on complete, non-compact manifolds; see Li-Tam [25] for some recent work in this regard.

A subject related to (1.4) is the Cauchy problem for harmonic maps into Minkowski space. See Shatah [31], Sideris [32].

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