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LINES OF CURVATURE AND UMBILIC POINTS ON SURFACES

par *Jorge SOTOMAYOR*

The *principal configuration* on an oriented surface immersed in $\mathbf{R}^3$ is defined by the triple formed by the set of its umbilic points and the foliations by minimal and maximal lines of principal curvature on the complemnt of umbilic points.

The study of principal configurations on surfaces has well established historical roots in Differential Geometry and Differential Equations. Its sources can be traced back to the initial contributions of Monge, Dupin and Darboux, among others.

Monge [M], determined the principal configuration on the ellipsoid with three different axes, which exhibits 4 umbilic points, conneted pairwisely by umbilic separatrices, and closed principal cycles packed into cylinders. This is the first non-trivial global example of principal configuration. It was extended for general surfaces which belong to triply orthogonal families of surfaces by Dupin [St], [Sp]. The beautiful picture which illustrates this configuration can be found in Hilbert Cohn-Vossen [H], [CV] and Fischer [Fi].

The principal configuration near a generic umbilic point of analytic surfaces was determined by Darboux [D].

This paper contains a discussion of recent work focusing on the unifying effect that crucial ideas originating from Differential Equations and Dynamical Systems, such as Structural Stability, Bifurcation and Genericity, have in the study of so classical geometric objects such as the principal configurations on surfaces.

In fact, most of the results and problems discussed here focus on the study of how the principal configuration changes when the surface is slightly deformed.

The first purpose here is to identify the class of surfaces which are structurally stable (or robust) in the sense that their principal configurations remain qualitatively unchanged after such deformations.

The starting point for this research can be linked to Monge's Ellipsoid (which characterizes the structurally stable principal configurations on compact quadrics) and to Darbouxian Umbilics (which characterize, in terms of the 3-jet of the surface, the locally stable principal configurations at umbilic points).

In [G-S;1,8] has been given exhibited a C^3-open class, $\mathcal{S}$, of structurally stable principal configurations on compact smooth surfaces immersed into $\mathbf{R}^3$. This class is defined in terms of a) the umbilic points, assumed Darbouxian, b) the principal cycles, assumed hyperbolic, c) the absence of umbilic separatrix connections and d) the absence of non-trivially recurrent principal lines.

It was proved in [G-S;2,8] that $\mathcal{S}$ is C^2-dense in the space of immersions. To raise the class from 2 to 3 in this density result, seems to be a difficult "Closing Lemma" problem, precisely to achieve condition d) by means of C^3-small deformation.

These results also provide the first examples of surfaces with isolated principal cycles. In fact, in all the examples of classical surface theory these cycles appear packed in cylinders or tori.

Extensions of the structural stability results to surfaces with singularities have been carried out in [G-S;4] and [Ga-S;1].

See [Ga] for a partial extension to principal configurations on smooth hypersurfaces of $\mathbf{R}^4$.

A secondary purpose here is to describe the simplest patterns of qualitative changes (bifurcations) of principal congigurations on families of surfaces depending on a real parameter. This leads to the definition of three hypersurfaces $\mathcal{S}_1(i)$, $i = a, b, c$ in the space of surfaces immersed into $\mathbf{R}^3$, on which condition i is violated in the mildest possible way while the other conditions are respected. Deformations of surfaces along these hypersurfaces preserve the qualitative properties of principal configurations.

Deformations on surfaces, transverse to $\mathcal{S}_1(i)$, produce a qualitative changes on their principal configuration that can be easily described in terms of the splitting, disappearance or exchange of types of umbilic points, principal cycles and umbilic connections. See [G-S;5,6,7,9] and [Ga-S;2].

The systematic study of the simplest non-trivial recurrent principal lines, which amounts to the violation of condition d), and their bifurcations have not been developed yet. See [G-S;2,8] for the simplest examples of this phenomenon on surfaces of genus 0 and 1.

Other results concerning the local and global principal configurations on surfaces with constant mean curvature and Weingarten surfaces have been obtained respectively in [G-S;3], [G-S et al] and [S]. The principal cycles of these surfaces, however, never appear isolated. This fact is due to the existence of an invariant transversal measure for their principal foliations.

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