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MIKHAIL KATZ

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## THE FILLING RADIUS OF HOMOGENEOUS MANIFOLDS

par *Mikhail KATZ*

**ABSTRACT.** — A closed Riemannian manifold  $V$  imbeds isometrically in the Banach space  $L^\infty(V)$ . Here  $x \in V$  goes to  $d_x \in L^\infty$ , where  $d_x(y) = \text{dist}(x, y)$  for all  $y \in V$ . The determination of the homotopy type of the  $\varepsilon$ -neighborhoods  $U_\varepsilon V$  of  $V \subset L^\infty$  allows one to compute a Riemannian invariant called the filling radius of  $V$  (see [1]). Of particular interest is the first change in homotopy type as  $\varepsilon > 0$  increases. Suppose  $V = G/H$  is a homogeneous space. The general conjecture for the new homotopy type  $X$  of  $U_\varepsilon V$  after the first change is as follows. There is a subgroup  $K \subset G$  such that

$$X \subset V * (G/K),$$

where  $*$  denotes the topological join. This conjecture is verified in the case of the circle and the complex projective space. Thus in the case of the circle  $V = S^1$ , one has  $G = SO(2)$ ,  $H = \{1\}$ ,  $K = Z_3$ , and  $X = S^1 * S^1 = S^3$ . In the case of  $V = CP^n$ , the quotient  $G/K$  can be described geometrically as the set of equilateral 4-point sets inscribed in the projective lines  $CP^1 \subset CP^n$ , and  $X$  is a proper subset of the join for  $n \geq 2$ . One conjectures a certain geometric condition on the homogeneous  $V$ , under which one can always specify such an  $X$ . The condition is stated in terms of an equivalence relation  $\sim$ , defined in [4], among the points of a set  $Y \subset V$  (under a certain diameter restriction). The condition is that each equivalence class  $\alpha \subset Y$  of  $\sim$  may be replaced by a connected set containing  $\alpha$ , without increasing the diameter of  $Y$ .

### 1. The isosystolic inequality

In 1949, C. Loewner found a kind of an opposite isoperimetric inequality for the 2-torus : {length of shortest non-contractible loop}  $\leq \sqrt{\frac{2}{\sqrt{3}}} \sqrt{\text{Area}}$ , for an arbitrary metric on  $T^2$ . The case of equality is satisfied by the flat equilateral torus.

Consider the family of flat rectangular metrics on  $T^2$  with area 1.

Then one of the sides may become long, but the other side would have to become short for the area to be 1. Thus the shortest side is always bounded from above by a fixed constant.

Loewner's proof used conformal techniques and integral geometry, and is tailored to the 2-dimensional case. Pu proved a similar result for  $RP^2$  in 1952.

M. Berger introduced the notion of the  $k$ -systol of an  $n$ -dimensional Riemannian

manifold  $(V^n, g)$  in 1972. The first in the family, 1-systol or simply systol, is the length of the shortest noncontractible loop in  $V$ , denoted  $\text{sys}(g)$ . Berger asked, what is the smallest value of a constant  $C(V)$ , possibly depending on the topological type of  $V$ , such that the so-called isosystolic inequality is satisfied for all metrics  $g$  on  $V$  :

$$\text{sys}(g) \leq C(V) \text{vol}(g)^{1/n} .$$

*Example.* — The manifold  $V = S^1 \times S^2$  does not satisfy the isosystolic inequality with any finite constant  $C(V)$  : product metrics of a fixed volume allow arbitrarily large  $S^1$ -factor, whose length is the systol. Thus there is no upper bound on the systol. The problem is the simply connected factor  $S^2$ .

Gromov [1] proved the isosystolic inequality for essential manifolds in 1983. A manifold  $V$  is called essential if there exists a map  $f : V \rightarrow K(\pi, 1)$  from  $V$  to an Eilenberg-MacLane space such that the image of the fundamental homology class of  $V$  is nonzero :  $f_*([V]) \neq 0$ , where  $f_* : H_n(V) \rightarrow H_n(K(\pi, 1))$  is the induced map on top-dimensional homology of  $V$ .

Gromov introduced a new Riemannian invariant called the Filling Radius and denoted  $\text{Fill Rad}(V)$ , and proved that  $\text{sys}(V) \leq 6 \text{Fill Rad}(V)$  for essential manifolds, and that  $\text{Fill Rad}(V) \leq C_n \text{vol}(V)^{1/n}$  for all manifolds, where  $C_n$  is a universal constant depending only on dimension. The combination of the two inequalities gives the isosystolic inequality.

The idea of the filling radius can be illustrated by a simple closed curve  $C \subset \mathbb{R}^2$  in the plane. In this case,  $\text{Fill Rad}(C \subset \mathbb{R}^2) = r$ , where  $r$  is the radius of the biggest circle in the interior of the closed curve.

Consider the tubular  $\varepsilon$ -neighborhoods of  $C \subset \mathbb{R}^2$ . As  $\varepsilon$  increases, the tubular neighborhoods eat up more and more of the interior of  $C$ , until the last point of the interior disappears, and then the curve  $C$  can be shrunk to a point in its  $\varepsilon$ -neighborhood. At this final moment,  $\varepsilon = r$ , the radius of the biggest inscribed circle.

## 2. The Kuratowski imbedding

To define the filling radius of an abstract Riemannian manifold, we need to imbed it isometrically as a metric space in a linear space. To proceed further, we need to point out that the unit circle is imbedded isometrically in the plane as a Riemannian manifold, but it is not imbedded isometrically as a metric space. Indeed, the intrinsic distance between two opposite points of the circle is  $\pi$ , and the ambient distance is 2. On the other hand, if we imbed the circle in the 2-sphere as the equator, we obtain an imbedding which is isometric in the strong sense that we need.

C. Kuratowski in 1935 defined an isometric imbedding of a metric space  $(V, \text{dist}_V)$  in the Banach space  $L^\infty(V)$  of bounded functions on  $V$  with the sup-norm  $\| \cdot \|$ . Let  $x \in V$ . We define a function  $d_x \in L^\infty$  by setting for all  $y \in V$ ,  $d_x(y) = \text{dist}_V(x, y)$ . Then the Kuratowski imbedding is defined by  $V \rightarrow L^\infty$ ,  $x \mapsto d_x$ .

Denote by  $U_r(V) \subset L^\infty$  the tubular  $r$ -neighborhood of  $V \subset L^\infty$ . Then

$$\text{Fill Rad}(V) := \inf \{ r > 0 : \iota_r([V]) = 0 \},$$

where  $\iota_r; H_n(V) \rightarrow H_n(U_r V)$  is induced by the inclusion  $V \subset U_r V$ .

The choice of the coefficients in homology does not affect the proof of the isosystolic inequality, so long as  $V$  has a nonzero fundamental class. We will write  $\text{Fill Rad}(V)$  to denote the filling radius defined using integer coefficients, and  $\text{Fill Rad}(V, \mathbb{Q})$ , using rational coefficients.

The calculation of the filling radius of two-point homogeneous spaces, or the rank one symmetric spaces, was initiated by me in 1983 and will be the topic of the rest of this talk. I showed [2] that  $\text{Fill Rad}(S^n, g_{\text{can}}) = \frac{1}{2} \arccos(-\frac{1}{n+1})$ , or half the spherical diameter of the regular  $(n+1)$  simplex inscribed in  $S^n$ . Also  $\text{Fill Rad}(\mathbb{R}P^n, g_{\text{can}}) = \pi/6$  regardless of dimension.

The next step is the calculation of the filling radius of the complex projective space with the Fubini-Study metric,  $CP^n$ . The complex projective space is of course not essential, but the techniques used in the calculation of its filling radius can also be used to compute the filling radius of certain lens spaces, which are essential.

### 3. The unit speed cone

The calculation of the filling radius breaks up into two parts : upper bounds and lower bounds. The key construction used for all upper bounds is the unit speed deformation in  $L^\infty$ . Given two functions  $f, g \in L^\infty$ , we deform  $f$  to  $g$  as follows. We let the value of  $f$  at  $x \in V$  go to  $g(x)$  with unit speed and stop upon reaching it. In symbols,

$$US(f, g, t)(x) = \begin{cases} \max(f(x) - t, g(x)) & \text{if } f(x) \geq g(x) \\ \min(f(x) + t, g(x)) & \text{otherwise.} \end{cases}$$

Given a function  $f \in L^\infty$ , the family  $US(d_x, f, y)$ , for all  $x \in V$  and  $t > 0$ , defines a cone  $C(V, f) \subset L^\infty$  on  $V$  with vertex  $f \in L^\infty$ .

We now describe how to choose a particularly good cone and obtain an estimate for the filling radius. Given a subset  $Y \subset V$ , we let  $d = \text{diam}(V)$  and  $e = \max_{x \in V} \text{dist}(x, Y)$ . Set  $r = \frac{1}{2} \max(d, e)$ . We define  $f_Y \in L^\infty$  by  $f_Y(x) = \text{dist}_V(x, Y) + r$ .

LEMMA. — For every set  $Y \subset V$ , we have the following.

- (a) the unit speed cone  $C(V, f_Y)$  lies in  $\bar{U}_r(V)$ ;
- (b) if we use a linear interval joining  $d_x$  with  $f_Y$  in the definition of the cone, then the cone may not belong to  $\bar{U}_r(V)$ .

COROLLARY. —  $\text{Fill Rad}(V) \leq r$ .

*Proof.* — The fundamental class of  $V$  goes to 0 when  $V$  is included in the cone.

*Application.* — Let  $V = S^n$ . Let  $Y \subset S^n$  be the set of vertices of the regular inscribed  $(n + 1)$ -simplex. Then the lemma gives the correct upper bound for  $\text{Fill Rad}(S^n)$ .

Let  $V = \mathbf{R}P^n$ . Let  $Y \subset \mathbf{R}P^n$  be a ball of radius  $\pi/6$ . We obtain the filling radius of  $\mathbf{R}P^n$ .

#### 4. The diameter Morse function

To compute the filling radius of complex projective space, we need to go beyond the cone construction, and actually determine the homotopy type of tubular neighborhoods of  $\mathbf{C}P^n$  in the Kuratowski imbedding. We determine this homotopy type using a kind of Morse theory.

To illustrate the main idea, consider the case of the circle  $V = S^1$ . Denote by  $S^1/Z_3$  the set of equilateral triangles  $Y$  inscribed in  $S^1$ . Here by an equilateral triangle we mean the set of vertices of such a triangle. Then  $S^1/Z_3$  is homeomorphic to the circle, but we would like to distinguish the two circles. Note that the topological join of  $S^1$  and  $S^1/Z_3$  is homeomorphic to the 3-sphere:  $S^1 * (S^1/Z_3) \sim S^3$ .

Then the set  $\{US(d_x, f_Y, t)\}$ , where  $x \in S^1$ ,  $Y \in S^1/Z_3$ ,  $t > 0$  imbeds the topological join  $S^3$  into  $\bar{U}_r(S^1) \subset L^\infty$ , where  $r = \frac{1}{6} \text{length}(S^1)$ .

LEMMA.

- (a) The homotopy type of  $U_r(S^1) \subset L^\infty$  is  $S^1$  for  $r \leq \frac{1}{6} \text{length}(S^1)$ ;
- (b) The homotopy type of  $U_r(S^1) \subset L^\infty$  is  $S^3$  for  $\frac{1}{6} \text{length}(S^1) < r < \frac{1}{3} \text{length}(S^1)$ .

*Remark.* — Thus a tubular neighborhood of a circle is homotopic to the 3-sphere. Our difficulty in visualizing such an imbedding stems from our Euclidean intuition.

*Proof.* — Proof is by Morse theory. The Morse functional in question is the diameter functional.

DEFINITION. — Denote by  $2^V$  the set of closed subsets of  $V$ . Typically one imposes an implicit uniform upper bound on the number of points in a set  $Y \in 2^V$ , which is large enough to accommodate all subsequent constructions. Then  $2^V$  is a metric space with respect to the Hausdorff distance among sets.

The diameter functional  $\delta : 2^V \rightarrow \mathbf{R}_+$  associates to each  $Y \subset V$ , its diameter  $\max_{x,y \in Y} \text{dist}_V(x, y)$ .

A set  $Y \subset V$  is a local minimum of  $\delta$  if no perturbation of  $Y$  (with respect to the Hausdorff distance) increases its diameter. One can also define a more general critical point of  $\delta$ , but we will not enter into this.

The first nontrivial minimum of  $\delta$  on  $2^{S^1}$  is the equilateral triangle. This is the

reason for the appearance of the circle  $S^1/Z_3$  in the expression for the homotopy type of  $U_r S^1$ .

At this point one could ask two questions.

QUESTION 1. — One is interested in the homotopy types of certain neighborhoods in the function space  $L^\infty(V)$ . How do the critical points of  $\delta$  on another space,  $2^V$ , come into play?

QUESTION 2. — If I only talk about local minima of  $\delta$ , how can Morse theory yield nontrivial homotopy types?

The answers to both questions are related. In fact, the space of subsets  $2^V$  is used to model only parts of the function space  $L^\infty$ ; the rest is done using the Mayer-Vietoris homology exact sequence.

The relation between the space of subsets and the function space has already appeared in passing from  $Y \subset V$  to  $f_Y \in L^\infty(V)$ , defined by  $f_Y(x) = \text{dist}_V(x, Y) + r$ . To go from  $f$  to  $Y$ , we roughly take the set of points where  $f$  achieves its minimum.

I would like to illustrate this connection further by showing how one passes from a discontinuous family of sets to a continuous family of functions. This can be done conveniently in the context of a construction using the cut locus.

## 5. The cut locus

Let  $V$  be a Riemannian manifold with a chosen point  $p \in V$ . Let  $Q \subset V$  be the cut locus of  $p$ . Let  $q \in Q$ , and denote by  $T(p, q) \subset V$  the union of all minimizing geodesics joining  $p$  with  $q$ .

LEMMA. — Let  $Y \subset V$  be a closed subset, and let  $Y_q = Y \cap T(p, q)$ . Let  $r > 0$ . Assume that for all  $q \in Q$ , the following two conditions are satisfied :

- (i)  $\text{diam}(Y_q) \leq 2r$ .
- (ii)  $\text{dist}_V(x, Y_q) \leq 2r$  for all  $x \in T(p, q)$ .

Then  $\text{Fill Rad}(V) \leq r$ .

*Remark.* — This is a refinement of the basic estimate with the unit speed cone, which I discussed earlier.

*Proof.* — The main point of the proof is to pass from the (typically discontinuous) family of sets  $\{Y_q\}$ ,  $q \in Q$  in  $2^V$  to a continuous family of functions  $\{f_q\}$  in  $L^\infty$ , defined as follows :

$$f_q(x) = \inf_{q' \in Q} (\text{dist}(x, Y_{q'}) + N \text{dist}(q', q)) + r .$$

Here  $N > 0$  is a large parameter.

As an application, we obtain very explicit upper bounds for the integer filling radius of complex projective spaces and lens spaces.

### 6. Flags, joins, and complex projective space

In the case of the rational filling radius, the situation is much simpler.

**THEOREM.** —  $\text{Fill Rad}(\mathbb{C}P^n, Q) = \frac{1}{2} \arccos(-\frac{1}{32})$ .

*Proof.* — We replace the cone construction by that of a “partial join”. Denote by  $S(\mathbb{C}P^n)$  the unit tangent sphere bundle of  $\mathbb{C}P^n$ . We will make use of a somewhat nonstandard fibration :  $SO(3) \rightarrow S(\mathbb{C}P^n) \rightarrow Gr_2(\mathbb{C}^{n+1})$ . Let  $\pi : S(\mathbb{C}P^n) \rightarrow Gr_2(\mathbb{C}^{n+1})$ . Here we view the Grassmanian of 2-planes in  $\mathbb{C}^{n+1}$  as the set of complex projective lines in  $\mathbb{C}P^n$  in homogeneous coordinates. The projection  $\pi$  can be understood as the composition of the fiberwise projectivization  $S(\mathbb{C}P^n) \rightarrow P(T\mathbb{C}P^n)$ , and the “forgetful” map between flag manifolds  $F_{1,2,n+1} \rightarrow F_{2,n+1} = Gr_2(\mathbb{C}^{n+1})$ , by noting that  $P(T\mathbb{C}P^n) = F_{1,2,n+1}$ . The fiber of  $\pi$  is  $SO(3)$ .

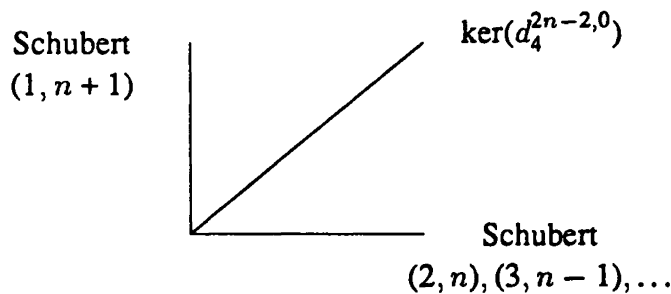
Let  $X \subset \mathbb{C}P^n * S(\mathbb{C}P^n)$  be the subset of the join of  $\mathbb{C}P^n$  and  $S(\mathbb{C}P^n)$  consisting of intervals joining  $x \in \mathbb{C}P^n$  and  $v \in S(\mathbb{C}P^n)$  such that  $x$  lies in the unique complex projective line tangent to  $v$  :

$$X = \{x * v \subset \mathbb{C}P^n * S(\mathbb{C}P^n) | x \in \pi(v)\} .$$

To each  $v \in S(\mathbb{C}P^n)$  we associate a set  $Y_v \subset \pi(v)$  which is the set of vertices of a certain regular tetrahedron inscribed in  $\pi(v)$ . This is the tetrahedron such that one of its vertices is the basepoint of  $v$ , and such that  $v$  is tangent to one of the three (spherical) edges emanating from the vertex.

Let  $\alpha_1 = \text{diam}_{\mathbb{C}P^1}(Y_v) = \arccos(-\frac{1}{3})$ . Let  $f_v = \text{dist}(x, Y_v) + \frac{1}{2}\alpha_1$ . This gives us a map  $S(\mathbb{C}P^n) \rightarrow \bar{U}_{\alpha_1/2}\mathbb{C}P^n$ . We map the partial join  $X$  to  $\bar{U}_{\alpha_1/2}\mathbb{C}P^n$  by  $US(d_x, f_v, t)$ ,  $x \in \mathbb{C}P^n$ ,  $v \in S(\mathbb{C}P^n)$ ,  $t > 0$ .

To prove the theorem-(or at least its upper bound), it suffices to verify that the inclusion of  $\mathbb{C}P^n$  in  $X$  sends the fundamental homology class of  $\mathbb{C}P^n$  to 0. This turns out to be equivalent to fact that the kernel of the differential  $d_4^{2n-2,0}$  in the Serre spectral sequence of the fibration  $SO(3) \rightarrow S(\mathbb{C}P^n) \rightarrow Gr_2(\mathbb{C}^{n+1})$  is an oblique line with respect to the standard basis for the homology of the Grassmannian (in other words, it does not lie in a certain coordinate hyperplane).



I would like to state a result about the situation in the case of integer coefficients.

Let  $\lambda_i$  be the  $i$ -th critical value of  $\delta : 2^{CP^n} \rightarrow \mathbf{R}_+$ . Then  $\lambda_1 = \alpha_1$ , but the value of  $\lambda_2 > \lambda_1$  is unknown.

**THEOREM.** —  $\text{Fill Rad}(CP^3) \geq \frac{1}{2}\lambda_2$ .

**COROLLARY.** — *The space  $CP^n$  can be retracted to a point in its closed  $\alpha_1/2$ -neighborhood in  $L^\infty(CP^n)$  for  $n = 1, 2$  but not for  $n = 3$ .*

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Mikhail KATZ  
URA 750 (CNRS)  
Département de Mathématiques  
Université de NANCY 1  
BP 239  
54506 VANDOEUVRE (France)