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ON BLASCHKE MANIFOLDS AND HARMONIC MANIFOLDS

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0. — A compact riemannian manifold M is called a Blaschke manifold if the diameter of M and the injectivity radius of M coincide. It is known that if M is a Blaschke manifold, then M is diffeomorphic to S^n or $\mathbf{R}P^n$, or $\pi_1(M) = \{0\}$ and $H^*(M, \mathbf{Z}) \cong$ the $\mathbf{Z}$–cohomology ring of $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathbf{Ca}P^2$.

The main problem about Blaschke manifolds is to know if the following conjecture, the Blaschke conjecture, is true or not : if M is a Blaschke manifold, then it would be a compact rank one symmetric space.

There are classes of riemannian manifolds related to Blaschke manifolds. A riemannian manifold M is called a globally harmonic manifold if the determinant of $d(\exp_p)_x : T_pM \to T_{\exp_p x}M$ ($p \in M$, $x \in T_pM$) depends only on the norm $|x|$. A compact riemannian manifold is called a C_1–manifold if all of its geodesics are closed and have the same length 1. The relation is as follows :

$$\begin{array}{c}\text{compact, simply connected,}\\ \text{globally harmonic}\end{array} \Longrightarrow \text{Blaschke} \Longrightarrow C_1 .$$

The following results are known :

1. (Green, Berger et al.). — If (S^n, g) is a Blaschke manifold, then it is isometric to the standard one.

2. (Green, Berger et al.). — If $(\mathbf{R}P^n, g)$ is a C_1–manifold, then it is isometric to the standard one.

3. (Kiyohara). — Let P be one of the projective spaces $\mathbf{C}P^1$, $\mathbf{H}P^n$ ($n \geqslant 2$) , $\mathbf{Ca}P^2$, and let (P, g) be a C_π–manifold. If the metric g is sufficiently close to the standard C_π–metric g_0 , then (P, g) is isometric to the standard one (P, g_0) .

4. (Zoll,Weinstein). — There are non–standard C_1–manifolds (S^n, g) for any dimension $n \geqslant 2$.

1. — From now on we assume M is a Blaschke manifold, $\pi_1(M) = \{0\}$, $H^*(M,\mathbf{Z}) \cong H^*(\mathbf{C}P^n,\mathbf{Z})$ $(\dim M = 2n\ ,\ n \geqslant 2)$, and the diameter of M is $\pi/2$. The followings are known about M :

1) For any $p \in M$ and any $q \in \mathrm{Cut}(p)$ (the cut locus of p), the distance $d(p,q) = \pi/2$.

2) Every cut locus is a submanifold of codimension 2.

3) Let ρ be the bundle projection $TM \to M$, and let $\{\zeta_t\}$ be the geodesic flow on SM. Then $\rho \circ \zeta_{\pi/2} : S_p(M) \to \mathrm{Cut}(p)$ is a fibre bundle whose fibres are great circles on S_pM.

4) For $p, q \in M$ with $d(p,q) = \pi/2$, we denote by $\Sigma(p,q)$ the union of geodesic orbits through p and q. Then $\Sigma(p,q)$ is a 2–dimensional submanifold diffeomorphic to S^2.

Now we define a mapping $I : SM \to SM$ as follows : since $H_2(M,\mathbf{Z}) \cong \mathbf{Z}$, we fix a positive generator. Then on each $\Sigma(p,q)$ the orientation is determined. Hence we have an orientation on each fibre S^1 of the fibre bundle $\rho \circ \zeta_{\pi/2} : S_pM \to \mathrm{Cut}(p)$, because the fibre S^1 over $q \in \mathrm{Cut}(p)$ is nothing but the unit sphere of $T_p\Sigma(p,q)$. So $I : SM \to SM$ is defined by the conditions :

1) If $v \in S_pM$, then $Iv \in S_pM$ and $\rho(\zeta_{\pi/2}v) = \rho(\zeta_{\pi/2}Iv)$.

2) $\langle v, Iv \rangle = 0$.

3) $\{v, Iv\}$ is positive in this order.

We extend the mapping I to $TM\backslash\{0\text{–section}\}$ homogeneously, and let $I_{*v} : T_{\rho(v)}M \to T_{\rho(v)}M$ be the differential of $I|T_{\rho(v)}M\backslash\{0\}$ at v. From the definition the mapping I satisfies $I \circ I = (-1)$ identity. So it looks like an almost complex structure, and we have the following

PROPOSITION A. — *Assume* $I_{*v}^2 + 1 = 0$ *for all* $v \in SM$. *Then* $I : T_pM\backslash\{0\} \to T_pM\backslash\{0\}$ *can be extended to a linear mapping on* T_pM *for every* $p \in M$, *i.e. I is an almost complex structure and it is integrable. Therefore* (M, I) *is a hermitian manifold. Moreover each cut locus is a complex submanifold and is holomorphically isomorphic to* $\mathbf{C}P^{n-1}$.

PROPOSITION B. — *Assume* $\dim M = 4$. *If* $I_{*v}^2 + 1 = 0$ *for all* $v \in SM$ *and if every cut locus is minimal, then* M *is isometric to* $(\mathbf{C}P^2, g_0)$.

LEMMA C. — *If* M *is moreover globally harmonic, then* $(I_{*v}^2 + 1)^{n-1} = 0$ *for every* $v \in SM$ *and every cut locus is minimal* $(\dim M = 2n)$.

COROLLARY D. — *If* $\dim M = 4$ *and* M *is globally harmonic, then* M *is isometric to* $(\mathbf{C}P^2, g_0)$.

Remarque. — This corollary is already known by a different method. See [1].

2. — For the proof of propositions we need some lemmas.

LEMMA 1. — *There is a Jacobi field $Y(t)$ along the geodesic $\gamma_v(t) = \rho(\zeta_t v)$ such that*

$$\begin{bmatrix} Y(0) \\ Y'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ Iv \end{bmatrix} \quad , \quad \begin{bmatrix} Y(\pi/2) \\ Y'(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ -I\overline{v} \end{bmatrix} \quad , \quad \overline{v} = \zeta_{\pi/2} v \, .$$

Moreover if a Jacobi field $X(t)$ along $\gamma_v(t)$ satisfies $X(0) = X(\pi/2) = 0$ then $X(t)$ is a constant multiple of $Y(t)$.

For $X, Y \in T_pM$, $Y \neq 0$, we put $\nabla_X I \cdot Y = \nabla_{\partial/\partial t}(IY_t)_{|t=0}$, where we take a curve $c(t)$ in M such that $c'(0) = X$, and Y_t is the parallel displacement of Y along $c(t)$. $\nabla_X I \cdot Y$ is linear in X , but not necessarily in Y .

LEMMA 2. — *Let $Y(t)$ be a periodic Jacobi field along the geodesic $\gamma_v(t)$, $v \in SM$. Then we have a periodic Jacobi field $Z(t)$ along the geodesic $\gamma_{e^{sI}v}(t)$ ($e^{sI}v = v\cos s + Iv \sin s$) such that*

$$\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} Y(0) \\ (\cos s + \sin s I_{*v})Y'(0) + \sin s(\nabla I \cdot v)Y(0) \end{bmatrix}$$

$$\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} Y(\pi/2) \\ (\cos s + \sin s I_{*v})Y'(\pi/2) - \sin s(\nabla I \cdot \overline{v})Y(\pi/2) \end{bmatrix} ,$$

where $(\nabla I \cdot v)Y(0) = \nabla_{Y(0)} I \cdot v$, etc.

LEMMA 3. —

1) There are Jacobi fields $Y_1(t)$, $Y_2(t)$ along $\gamma_v(t)$ such that

$$\begin{bmatrix} Y_1(0) \\ Y_1'(0) \end{bmatrix} = \begin{bmatrix} Iv \\ -\nabla_v I \cdot v \end{bmatrix} \quad , \quad \begin{bmatrix} Y_1(\pi/2) \\ Y_1'(\pi/2) \end{bmatrix} = \begin{bmatrix} -I\overline{v} \\ \nabla_{\overline{v}} I\overline{v} \end{bmatrix}$$

$$\begin{bmatrix} Y_2(0) \\ Y_2'(0) \end{bmatrix} = \begin{bmatrix} 2\nabla_v I \cdot v \\ R(Iv, v)v - \nabla_v^2 I \cdot v \end{bmatrix} \quad , \quad \begin{bmatrix} Y_2(\pi/2) \\ Y_2'(\pi/2) \end{bmatrix} = \begin{bmatrix} -2\nabla_{\overline{v}} I \cdot \overline{v} \\ -R(I\overline{v}, \overline{v})\overline{v} + \nabla_{\overline{v}}^2 I \cdot \overline{v} \end{bmatrix} .$$

2) $\nabla_{e^{sI}v} I \cdot d^{sI}v = \nabla_v I \cdot v$.

LEMMA 4. — *Let $Y(t)$ be a periodic Jacobi field along $\gamma_v(t)$. Then there is a periodic Jacobi field $Z(t)$ along $\gamma_v(t)$ such that*

$$\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} {}^tI_{*v}Y(0) \\ I_{*v}Y'(0) + (\nabla I \cdot v - {}^t\nabla I \cdot v)Y(0) + \{\langle Y(0), \nabla_v I \cdot v\rangle + \langle Y'(0), Iv\rangle\}v \end{bmatrix}$$

$$\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} {}^tI_{*\overline{v}}Y(\pi/2) \\ -I_{*\overline{v}} - Y'(\pi/2) - (\nabla I\overline{v} - {}^t\nabla I\overline{v})Y(\pi/2) - \{\langle Y(\pi/2), \nabla_{\overline{v}} I\overline{v}\rangle + \langle Y'(\pi/2), I\overline{v}\rangle\}\overline{v} \end{bmatrix} .$$

3. Proof of Proposition A. — Fix $p \in M$ and consider the S^1–principal bundle $\rho \circ \zeta_{\pi/2} : S_pM \to \mathrm{Cut}(p)$, where the S^1-action is given by e^{sI} , $0 \leqslant s \leqslant 2\pi$.

We define a 1–form ω on S_pM by

$$\omega(X) = \langle X, Iv\rangle \ , \ X \in T_v(S_pM) = \{Y \in T_pM | \langle v, Y\rangle = 0\} \ .$$

As is easily seen, ω is a connection form, *i.e.* invariant under the S^1–action. We have

$$d\omega(X,Y) = \langle (I_{*v} - {}^tI_{*v})X, Y\rangle \ .$$

So there is a unique closed 2–form Ω on Cut(p) such that $(\rho \circ \zeta_{\pi/2})^*\Omega = d\omega$. We can see that $[(1/2\pi)\Omega]$ is a generator of $H^2(\mathrm{Cut}(p), \mathbf{Z}) \cong \mathbf{Z}$. Therefore

$$(1/2\pi)^{n-1} \int_{\mathrm{Cut}(p)} \Omega^{n-1} = 1$$

under a proper orientation of Cut(p) , and thus

$$\int_{S_pM} \omega \wedge (d\omega)^{n-1} = (2\pi)^n \ .$$

Now put $J_v = I_{*v} - {}^tI_{*v}$, $S_v = I_{*v} + {}^tI_{*v}$. Then $2I_{*v} = J_v + S_v$ and

$$I_{*v}^2 + 1 = 0 \Longleftrightarrow J_v^2 + S_v^2 + 4 + J_vS_v + S_vJ_v = 0 \ . \tag{\#}$$

Let $e_1, \dots, e_{2n-2}$ be an orthonormal basis of the orthogonal complement to $\mathbf{R}v + \mathbf{R}Iv$ in T_pM such that $J_ve_{2i-1} = \lambda_ie_{2i}$, $J_ve_{2i} = -\lambda_ie_{2i-1}$, $\lambda_i \geqslant 0$, $i = 1, \dots, n-1$. By (#) we have

$$-\lambda_i^2 + |S_ve_{2i}|^2 + 4 = 0 \ .$$

Hence $\lambda_i \geqslant 2$, and $\lambda_i = 2$ for every i if and only if $S_v = 0$. Then

$$\big(\omega \wedge (d\omega)^{n-1}\big)(Iv, e_1, \dots, e_{2n-2}) = (n-1)! \prod_{i-1}^{n-1} \lambda_i \geqslant 2^{n-1}(n-1)! \ ,$$

and the equality holds if and only if $S_v = I_{*v} + {}^tI_{*v} = 0$. Therefore we have

$$(2\pi)^n = \int_{S_pM} \omega \wedge (d\omega)^{n-1} \geqslant 2^{n-1}(n-1)!\mathrm{vol}(S_pM) \ .$$

But vol(S_pM) is just $2\pi^n/(n-1)!$. So the equality holds in the above inequality. Hence we have $S_v = I_{*v} + {}^tI_{*v} = 0$ for any $v \in SM$. Since $I_{*v}^2 + 1 = 0$, it follows that ${}^tI_{*v}I_{*v} = 1$. This implies that the mapping $I : S_pM \to S_pM$ is an isometry, and therefore the restriction of a linear orthogonal transformation of T_pM . Hence I is extended as a tensor field of type (1,1) with $I^2 = -1$,*i.e.* an almost complex structure on M , and (M, I) is an almost hermitian manifold.

By using the square of the endomorphisms on the space of Jacobi fields in Lemma 4, one gets

$$\langle \{I(\nabla I \cdot v - {}^t\nabla I \cdot v) - (\nabla I \cdot v - {}^t\nabla I \cdot v)I\}X, Y\rangle = 0 \ , \ X, Y \perp v, Iv \ .$$

Moreover Lemma 3 (2) gives

$$\langle \{I(\nabla I \cdot v + {}^t\nabla I \cdot v) - (\nabla I \cdot v + {}^t\nabla I \cdot v)I\}X, Y\rangle = 0 \ , \ X, Y \perp v, Iv \ .$$

These formula gives

$$\nabla_{IX} I = I \nabla_X I$$

for any vector X . By this it is easy to see that the Nijenhuis'tensor vanishes, and (M, I) turns out to be a hermitian manifold.

It is now clear that each cut locus is a complex submanifold of M and the S^1–fibration $\rho \circ \zeta_{\pi/2} : S_p M \to \mathrm{Cut}(p)$ is nothing but the standard Hopf fibration : $S^{2n-1} \to \mathbf{C}P^{n-1}$. Hence the last statement of the proposition follows.

Proof of Proposition B. — For $v \in SM$ we define the symmetric endomorphism Φ_v of $T_{\rho(v)}M$ by $\Phi_v v = \Phi_v I v = 0$ and

$$\langle \Phi_v X, Y \rangle = -\langle h(X,Y), v \rangle \ , \ X, Y \in T_\rho(v) M \ , \ X, Y \perp v, Iv \ ,$$

where h is the second fundamental form of $\mathrm{Cut}(\rho\zeta_{\pi/2} v)$ in M at $\rho(v)$. If we take a curve $c(t)$ in $\mathrm{Cut}(\rho\zeta_{\pi/2} v)$ such that $c'(0) = X$, and a normal vector field v_t to $\mathrm{Cut}(\rho\zeta_{\pi/2} v)$ along $c(t)$, we have

$$\langle \Phi_v X, Y \rangle = \langle \nabla_{\partial/\partial t} v_t|_{t=0}, Y \rangle \ .$$

So the following lemma is clear.

LEMMA 5. — $\Phi_{Iv} X = I \Phi_v X + (\nabla_X I) v \ , \ X \in T_{\rho(v)} M \ , \ X \perp v, Iv$.

Since every cut locus is minimal, it follows that $\mathrm{tr}\ \Phi_v = 0$ for any $v \in Sm$, tr being the trace. Hence in view of Lemma 5 one gets

$$\mathrm{tr}\ (\nabla I) v = 0 \ .$$

This together with the formula $\nabla_{IX} I = I \nabla_X I$, shown in the proof of Proposition A, implies that $\nabla I = 0$, *i.e.* (M, I) is kählerian.

By applying Lemma 1 to the Jacobi field Y_2 in Lemma 3,

$$R(Iv, v)v = c(v) Iv \ , \ v \in SM \ ,$$

where c is a function on SM satisfying $c(\zeta_{\pi/2} v) = c(v)$. As is easily seen, $c(v)$ is pointwise constant, *i.e.* if v_1 and v_2 are based at the same point on M , then $c(v_1) = c(v_2)$. Using the fact that for any two points p and q on M, there is a point m such that $d(p,m) = d(q,m) = \pi/2$, we see the constancy of $c(v)$.

Since (M, I) is kählerian and has constant holomorphic sectional curvature, it must be holomorphically isometric to $(\mathbf{C}P^2, g_0)$.

Lemma C is an immediate consequence of Lemma 4.

Reference

[1] BESSE A. — *Manifolds all of whose geodesics are closed*, Springer, 1978.

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