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THE MODELS OF A NON-MULTIDIMENSIONAL ω-STABLE THEORY

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I give a (self-contained) account of the classification of the models of a non-multidimensional ω-stable theory. This result is the generalisation of the Baldwin-Lachlan-Morley classification of the models of an $\aleph_1$-categorical theory, and includes of course the possible spectra that can occur. (Remember that the spectrum of a theory T is given by the function $I(- , T)$, where for κ a cardinal, $I(\kappa , T)$ is the number of models of T of power κ, up to isomorphism.) The crude idea is that, instead of a model of T being determined by the cardinality of one indiscernible set (as when T is $\aleph_1$-categorical), a model of T is now determined by the cardinalities of each member of a fixed "independent" family of indiscernible sets.

I assume the basic facts about stability, forking, definability, rank, etc., which can be found in [4] or even [5].

T will be a countable complete ω-stable theory. The ω-stability of T furnishes us with several nice properties. The most important of these will be :

(i) for any subset A of a model M of T, there is a (real) prime model of $Th(M , a)$, $a \in A$,

(ii) if $M \models T$ and $p \in S(M)$, then there is a finite $A \subset M$ such that p is definable over A (thus p does not fork over A and $p \upharpoonright A$ is stationary),

(iii) all types over arbitrary subsets are ranked by Morley rank.

I will also follow the usual practice of working in a large sufficiently saturated model of T.

I. Strongly regular types.

Strongly regular types are generalisations of types of Morley rank 1, degree 1. If $p \in S(M)$, I denote by $M(p)$, the model which is prime over $M \cup \{\bar{a}\}$, where $tp(\bar{a}/M) = p$. This model might also be denoted by $M(\bar{a})$, and is unique up to M-isomorphism.

Definition 1.1. - Let $p \in S_1(M)$, p not algebraic and $\varphi(x) \in p$ (φ might countain parameters from M). The pair (p , φ) is said to be strongly regular if whenever $b \in M(p)$, $b \notin M$ and $M(p) \models \varphi(b)$, then $tp(b/M) = p$. p is said to be

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strongly regular if there is $\varphi \in p$ such that (p , φ) is strongly regular.

LEMMA 1.2. - Suppose that p and $q \in S(M)$, p is strongly regular and q is realised in $M(p)$. Then p is realised in $M(q)$, (We assume q is not algebraic).

Proof. - Suppose that (p , φ) is strongly regular. Let a realise p, and $\bar{b} \in M(a)$ such that $\bar{b}$ realises q. It is clear that a and $\bar{b}$ are note independant over M $(\bar{b} \notin M)$. Thus there is a formula $\alpha(x , \bar{y})$ over M such that $M(a) \models \alpha(a , \bar{b})$, but $M(a) \models \neg \alpha(m , \bar{b})$ for all $m \in M$. Note that " $\varphi(x) \wedge \alpha(x , \bar{b})$ " is consistent. Now $M(q) = M(\bar{b}) \prec M(a)$, and let $c \in M(\bar{b})$ such that $M(b) \models \varphi(c) \wedge \alpha(c , \bar{b})$. Then $c \notin M$, $c \in M(a)$ and $M(a) \models \varphi(c)$. Thus $tp(c/M) = p$, and so p is realised in $M(q)$.

Definition 1.3. - Let p and q be strongly regular types over M such that q is realised in $M(p)$. Then we say that p and q are equivalent, $p \sim q$.

(By lemma 1.2, this definition makes sense).

The next lemma shows that "enough" strongly regular types exist.

LEMMA 1.4. - Suppose that $M \prec N$, the $L(M)$ formula $\varphi(x)$ is "augmented" in N, and a is chosen in $\varphi^N - M$ such that $tp(a/M)$ has least possible Morley rank. Then $tp(a/M)$ is strongly regular.

Proof. - (Let $R(-)$ denote Morley rank). Let $R(tp(a/M)) = \alpha$, and pick $L(M)$-formula $\psi(x)$ such that $\vdash \psi(x) \longrightarrow \varphi(x)$, $N \models \psi(a)$ and $R(\psi(x)) = \alpha$, and degree $(\psi(x)) = 1$. Now $M(a) \prec N$, and so it is clear that $(tp(a/M) , \psi)$ is strongly regular.

Definition 1.5. - Let $p(\bar{x})$ and $q(\bar{y})$ be types over M. p and q are said to be perpendicular $(p \perp q)$ if $p(\bar{x}) \cup q(\bar{y})$ determines a complete $\bar{x} \frown \bar{y}$ type over M.

Note. - If $p(\bar{x})$ and $q(\bar{y})$ are types over a model M, then $p \perp q$ if, and only if, whenever $\bar{a}$ and $\bar{b}$ realise p and q respectively, then $\bar{a}$ and $\bar{b}$ are independent over M.

Fact 1.6. - Let $\bar{a}$ and $\bar{b}$ be independent over M. Let A be a atomic over $M \cup \{\bar{a}\}$ and B atomic over $M \cup \{\bar{b}\}$. Then A and B are independent over M.

Now the proof of lemma 1.2 actually implies that if $tp(a/M)$ is strongly regular and $\bar{b} \in M(a) - M$, then $tp(a/M \cup \{\bar{b}\})$ is isolated. A simple consequence of this and fact 1.6 is the following :

Observation 1.7. - Let p_1, p_2, q_1, q_2 be all strongly regular types over M such that $p_1 \sim p_2$ and $q_1 \sim q_2$. Then $p_1 \perp q_1$ if, and only if, $p_2 \perp q_2$.

PROPOSITION 1.8. - Let p and q be strongly regular types over M. Then p and q are perpendicular if, and only if, p and q are not equivalent.

Proof. – It is clear that if p and q are equivalent then they are not perpendicular. Conversely, assume that p and q are not equivalent. We wish to show that they are perpendicular. By 1.7, we can assume that $R(p) = \alpha$ is minimal among strongly regular types over M equivalent to p, and similarly for q, with $R(q) = \beta$. So we can find formulae $\varphi(x)$ and $\psi(x)$, both of degree 1, and of rank α and β respectively, such that (p , φ) and (q , ψ) are strongly regular. Suppose (without loss) that $\alpha \leqslant \beta$. Now if p and q are not perpendicular, then there are realisations a and b of p and q respectively, such that a and b are not independent over M. As in the proof of 1.2, if follows that $\varphi(x)$ is "augmented" in $M(b)$ (i. e. $\varphi^{M(b)} - M$ is nonempty). By lemma 1.4, there is $c \in \varphi^{M(b)} - M$, such that $tp(c/M)$ is strongly regular. Clearly, $tp(c/M)$ is equivalent to q, and $R(tp(c/M)) \leqslant \alpha$. If $R(tp(c/M)) = \alpha$, then clearly $tp(c/M) = p$, which contradicts the non-equivalence of p and q. On the other hand, if $R(tp(c/M)) < \alpha$, then we contradict the minimal choice of $R(q)$. Thus the proposition is proved.

PROPOSITION 1.9. – Let $M \prec M'$, $p \in S_1(M)$ and p' the nonforking extension (or heir) of p over M'. Then p is strongly regular if, and only if, p' is strongly regular.

Proof. – First suppose that p' is strongly regular. Then there is an $L(M)$ formula $\varphi(x)$ such that (p' , φ) is strongly regular (Any $L(M')$ formula $\varphi(x) \in p'$ such that degree $(\varphi) = 1$, and $R(\varphi) = R(p')$ will suffice. But $p \subseteq p'$, and $R(p) = R(p')$. Thus φ can be chosen over M). We show that (p , φ) is strongly regular. Let a realise p'. So $tp(a/M) = p$, and $M(p) = M(a) \prec M'(a)$. Let $b \in M(a)$, $b \notin M$ and b satisfy φ. Now b and a are not independent over M. Thus $b \notin M'$. But then $tp(b/M') = p'$ (by strong regularity of (p' , φ) .) Thus $tp(b/M) = p$. So (p , φ) is strongly regular.

Conversely, suppose that $\varphi(x) \in p$, and (p , φ) is strongly regular. Let a realise p'. If (p' , φ) is not strongly regular, then there is b in $\varphi^{M'(a)} - M'$ such that $tp(b/M') \neq p$. Now p' is definable by a schema d, over M (where d also defines p), and also a and b are not independent over M'. Thus there are L-formulae $\psi(y , \bar{z})$ and $\alpha(x , y , \bar{w})$, and $\bar{c}$ and $\bar{d}$ in M' such that

$$M'(a) \models (\exists y)(\varphi(y) \wedge \psi(y , \bar{c}) \wedge \neg d(\psi)(\bar{c}) \wedge \alpha(a , y , \bar{d})) ,$$

where the formula $\alpha(x , y , \bar{w})$ is not represented in p' (so neither in p). But $tp(a/M')$ is the heir of $tp(a/M)$. Thus we can find $\bar{c}'$ and $\bar{d}'$ in M such that

$$M(a) \models (\exists y)(\varphi(y) \wedge \psi(y , \bar{c}') \wedge \neg d(\psi)(\bar{c}') \wedge \alpha(a , y , \bar{d}')) .$$

If we let b' be such a y in $M(a)$, then $b' \notin M$, $M(a) \models \varphi(b')$ and $tp(b'/M) \neq p$. This contradicts the fact that (p , φ) is strongly regular, and completes the proof.

Definition 1.10. - Let A be a subset (of the big model), $p \in S_1(A)$ a stationary type, and $\varphi(x) \in p$. We call (p , φ) strongly regular if there is a model M countaining A and nonforking extension p' of p over M such that (p',φ) is strongly regular. Again p will be called strongly regular if there is $\varphi(x)$ such that (p , φ) is strongly regular.

Note. - If follows immediately from 1.9 that for $p \in S_1(A)$, p is strongly regular if, and only if, for all M extending A and nonforking extension p' of p to M, p' is strongly regular.

PROPOSITION 1.11. - Let p and q be strongly regular types over M, and let p' and q' be their respective heirs over $M' \prec M$. Then $p \perp q$ if, and only if, $p' \perp q'$.

Proof. - Suppose that p dans q are not perpendicular. Then there are realisations a and b of p and q respectively, such that a and b are not independent over M. Let $a' \wedge b'$ realise the heir of $tp(a \wedge b/M)$ over M'. Then $tp(a'/M') = p'$, $tp(b'/M') = q'$, and a' and b' are not independent over M'. Thus $p' \not\perp q'$.

Conversely, suppose that p and q are perpendicular. We may again suppose that p and q are chosen with minimal rank in their equivalence classes. So we have (p , φ) strongly regular, with $R(p) = R(\varphi) = \alpha$, and (q , ψ) strongly regular, with $R(q) = R(\psi) = \beta$, and suppose $\alpha \leq \beta$. So (p' , φ) and (q' , ψ) are strongly regular. If p' and q' are not perpendicular, then again if follows that $\varphi(x)$ is augmented in $M'(q')$. As q' is the heir of q, it is easy to prove that $\varphi(x)$ is augmented in $M(q)$, but this will again contradict the minimal choice of $R(q)$. So the proposition is proved.

By propositions 1.8 and 1.11, we have :

COROLLARY 1.12. - Let p and q be strongly regular types over M, and $M \prec M'$, and p', q' the heirs of p and q over M'. Then $p \sim q$ if, and only if, $p' \sim q'$.

Definition 1.13. - Let $p(\bar{x})$ and $q(\bar{y})$ be in $S(A)$, where A is an arbitrary subset. Then p and q are said to be orthogonal if for all $B \supset A$ and nonforking extensions p' and q' of p and q over B, $p'(\bar{x}) \cup q'(\bar{y})$ determines a complete type over B.

PROPOSITION 1.14. - Let p and q be strongly regular types over A. Then the following are equivalent :

(i) p and q are orthogonal,

(ii) for some $M \supset A$, (M a model), and nonforking extensions p', q' of p, q over M, p' and q' are perpendicular.

Proof. – By proposition 1.11.

Note. – It was shown in [3] that if p and q are any types over M, and p', q' their heirs over some $M' > M$, then $p \perp q$ if, and only if, $p' \perp q'$. It follows that proposition 1.14 holds without the hypothesis that p and q are strongly regular. However 1.14 in its present form will suffice for our needs.

Given strongly regular types p and q over A, we will call p and q equivalent if they are not orthogonal. By 1.8 and 1.14, this is consistent with def. 1.3.

I complete this section with a couple of observations which will be of use later on.

LEMMA 1.15. – Let $\{p_i \ ; \ i \in I\}$ be a set of stationary pairwise orthogonal types over A. For each $i \in I$, let $\{\bar{a}^i_j \ ; \ j < \gamma_i\}$ be an independent set of realisations of p_i over A. Then $\{\bar{a}^i_j \ ; \ i \in I, \ j < \gamma_i\}$ is independent over A.

Proof. – It suffices to show that if $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$ is independent over A, and $tp(\bar{b}/A)$ and $tp(\bar{a}_i/A)$ are orthogonal for $i = 1, \dots, n$, then $\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n, \bar{b}\}$ is independent over A. This we show by induction. So suppose that we already have $\{\bar{a}_1, \dots, \bar{a}_r, \bar{b}\}$ is independent over A, where $r < n$. Thus $tp(\bar{b}/\{\bar{a}_1, \dots, \bar{a}_r\} \cup A)$ does not fork over A, and we know anyway that $tp(\bar{a}_{r+1}/\{\bar{a}_1, \dots, \bar{a}_r\} \cup A)$ does not fork over A. Thus by the orthogonality of $tp(\bar{b}/A)$ and $tp(\bar{a}_{r+1}/A)$, $\bar{a}_{r+1}$ and $\bar{b}$ are independent over $A \cup \{a_1, \dots, a_r\}$. Thus $\{\bar{a}_1, \dots, \bar{a}_r, \bar{a}_{r+1}, \bar{b}\}$ is independent over A.

LEMMA 1.16. – Let M be a model, and $\{p_i \ ; \ i \in I\}$ a maximal collection of pairwise orthogonal strongly regular types over M. Let $A \subseteq M$ be such that each p_i is definable over A, and for each $i \in I$, let $\{a^i_j \ ; \ j < \gamma_i\}$ be a maximal independent set of realisations of $p_i \restriction A$ in M. Then $J = \{a^i_j \ ; \ i \in I, \ j < \gamma_i\}$ is independent over A, and moreover M is minimal over $A \cup J$.

Proof. – By 1.14, the types $p_i \restriction A$ are strongly regular and pairwise orthogonal. Thus the independence of J over A follows by 1.15.

Suppose that M were not minimal over $A \cup J$. Then there would be a model N such that $A \cup J \subseteq N \subsetneqq M$. By 1.4, we can find $a \in M - N$ such that $tp(a/N)$ is strongly regular. Let $p = tp(a/N)$, and let p' be the heir of p over M. So p' is strongly regular (1.9), and by the choice of the p_i' s there is $s \in I$ such that p' and p_s are not orthogonal. But p_s does not fork over N, and so $p_s \restriction N$ is strongly regular and not orthogonal to p (by prop. 1.9 and prop. 1.11). Thus $p_s \restriction N$ and p are equivalent, and so $p_s \restriction N$ is realised in $N(a)$, where we can assume that $N(a) < M$. Let $c \in N(a)$ realise $p_s \restriction N$. Then, as $p_s \restriction N$ does not fork over A, if follows that c and $\{a^s_j \ ; \ j < \gamma_s\}$ are independent over A. But this contradicts the maximal choice of the independent set $\{a^s_j \ ; \ j < \gamma_s\}$ of realisations of $p_s \restriction A$ in M. So the lemma is proved.

II. Dimension.

Let M be a model, $A \subset M$ and $p \in S(A)$. A set I of tuples from M will be called a basis for p in M, if I is a set of realisations of p in M, independent over A, and maximal such (Note that if p is stationary, then I is also indiscernible over A).

PROPOSITION II.1. - Suppose that $p \in S(A)$, $A \subset M$ and p has some infinite basis in M. Then all bases for p in M have the same cardinality.

Proof. - If not, then it is clear that there are bases I and J of p in M with J infinite and $|I| < |J|$. As I is maximal, for each $\bar{c} \in J$, $tp(\bar{c}/I \cup A)$ forks over A. So there is some finite $I_{\bar{c}} \cup I$ such that $tp(\bar{c}/I_{\bar{c}} \cup A)$ forks over A.

So by the cardinality difference, there is finite $I' \subset I$ and $\bar{c}_n \in J$ for $n<\omega$, such that $tp(\bar{c}_n/I' \cup A)$ forks over A, for each $n < \omega$. But then, as the c_n's are independent over A, we have for each $n < \omega$, $tp(\bar{c}_{n+1}/\{\bar{c}_0,\dots,\bar{c}_n\} \cup I' \cup A)$ forks over $A \cup \{\bar{c}_0, \dots, \bar{c}_n\}$, and thus $tp(I'/\{\bar{c}_0, \dots, \bar{c}_n, \bar{c}_{n+1}\} \cup A)$ forks over $\{\bar{c}_0, \dots, \bar{c}_n\} \cup A$. But this contradicts superstability.

Definition II.2. - If all bases of p in M have the same cardinality, then we define $\dim(p, M)$ to be this cardinality.

Note. - We will see later on that if $p \in S(A)$ is strongly regular and $A \subset M$, then $\dim(p, M)$ is always defined.

Let I be an infinite indiscernible set (maybe of tuples), and B an arbitrary set. Recall that $Av(I/B)$ is defined as follows: for $\bar{b} \in B$, $\varphi(\bar{x}, \bar{b}) \in Av(I/B)$ if, for cofinitely many $\bar{c}$ in I, we have $\models \varphi(\bar{c}, \bar{b})$. Then $Av(I/B)$ is a complete and consistent type over B. Moreover, suppose that p is a stationary type over A, and I is an infinite independent set of realisations of p over A (so I is indiscernible over A), and $B \supset A$. Then $Av(I/B)$ is precisely p' the nonforking extension of p over B.

LEMMA II.3.

(i) Let I be an infinite indiscernible set over A, and M prime over $A \cup I$. Then I is a maximal indiscernible set over A in M.

(ii) Let $I \cup \{c\}$ be an infinite indiscernible set over A, and let M be prime over $A \cup I$. Then $tp(c/A \cup I) \vdash Av(I/M)$.

(iii) Let p be a stationary type over A, and I an independent set of realisations of p over A. Let I_1 be an infinite subset of I, and M be prime over $A \cup I_1$, and let p' denote the nonforking extension of p over M. Then $I - I_1$ is an independent (over M) set of realisations of p'.

Proof.

(i) If I is not maximal indiscernible over A in M, extend it by c in M. Now $tp(c/A \cup I)$ is isolated by a formula $\alpha(x, \bar{a}, \bar{d})$, where $\bar{a} \in A$ and $\bar{d} \subseteq I$. In particular, $M \models \alpha(x, \bar{a}, \bar{d}) \rightarrow x \neq c'$ for all $c' \in I$ (as $c \notin I$). But $M \models \alpha(c, \bar{a}, \bar{d})$, $I \cup \{c\}$ is indiscernible over A and I is infinite. Thus we can find c' in I such that $M \models \alpha(c', \bar{a}, \bar{d})$, and this is a contradiction.

(ii) I show that if $I \cup \{c\}$ is indiscernible over A, then $tp(c/M) = Av(I/M)$ (where M is prime over $A \cup I$). So let $\varphi(x, \bar{m}) \in Av(I/M)$, where $\bar{m} \in M$. I will show that this formula is satisfied by c. Now $tp(\bar{m}/A \cup I)$ is isolated by a formula $\varphi(\bar{y}, \bar{a}, \bar{d})$ where $\bar{a} \in A$ and $\bar{d} \subset I$. Now as $\varphi(x, \bar{m})$ is satisfied by cofinitely many members of I, there is $c' \in I$, $c' \notin d'$ such that $M \models \varphi(c', \bar{m})$. Thus $M \models \forall \bar{y}(\varphi(\bar{y}, \bar{a}, \bar{d}) \rightarrow \varphi(c', \bar{y}))$. But $tp(c \frown \bar{d}/\bar{a}) = tp(c \frown \bar{d}/\bar{a})$. So we have $\models \forall \bar{y}(\varphi(\bar{y}, \bar{a}, \bar{d}) \rightarrow \varphi(c, \bar{y}))$, whereby $\models \varphi(c, \bar{m})$, and we finish.

(iii) Let $c_1, \ldots, c_n$ be in $I - I_1$. We must show that $c_1, \ldots, c_n$ is an independent set of realisations of p' over M. Let p^n denote $tp(c_1 \frown \ldots \frown c_n/A)$. Then I can be considered (by partitioning it into n-tuples) as an independent set of realisations over A of p^n. But then by (ii) and the remarks preceding this lemma, $tp(c_1 \frown \ldots \frown c_n/M)$ does not fork over A, and this is just what we want.

LEMMA II.4.

(i) Let p and q be equivalent strongly regular types over a model M, and let $N \succ M$. If I is a basis of p in N, then there is a basis J of c in N with $|I| \leqslant |J|$.

(ii) Let p and q be equivalent strongly regular types over a set A, and let $N \supset A$. If p has an infinite basis in N, then so does q, and moreover $\dim(p, M) = \dim(q, M)$.

Proof.

(i) Let I be a basis of p in N, and write I as $\{a_\alpha ; \alpha < \kappa\}$. Define models M_α in N for $\alpha < \kappa$, and elements b_α for $\alpha < \kappa$, as follows : $M_0 = M$, $M_{\alpha+1} = M_\alpha(a_\alpha)$, and $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$. Clearly $tp(a_\alpha/M_\alpha)$ is the heir of p over M_α and so strongly regular and equivalent to a_α, the heir of c over M_α (by 1.12). Thus c_α is realised in $M_{\alpha+1}$, and let b_α be such a realisation. By fact 1.6, $\{b_\alpha ; \alpha < \kappa\}$ is an independent set of realisations of q over M, and so can be extended to a maximal such set in N.

(ii) It is enough by II.1 and symmetry to show that if p has an infinite basis I in N, then q has a basis J in N with $|I| \leqslant |J|$. So let I be an infinite basis of p in N. Partition I as $I_1 \cup I_2$, where I_1 is infinite and $|I| = |I_2|$. Let M' be an elementary substructure of N which is prime over

$A \cup I_1$. Let p' be the nonforking extension of p over M' . Then by II.3 (iii), I_2 is a basis of p' in N . But p' is strongly regular and equivalent to q' , the (strongly regular) nonforking extension of q over M' . So by (i) there is a basis J' of q' in N , with $|J'| \geq |I_2| = |I|$. But J' is clearly an independent set of realisations of q in N and so can be extended to a basis J of q in N , and clearly $|I| \leq |J|$.

Let p be a stationary type over A , and let $\bar{a}_1 , \bar{a}_2 , \ldots , \bar{a}_n$ an independent set of realisations of p over A . Then I will denote $tp(\bar{a}_1 \frown \ldots \frown \bar{a}_n/A)$ by p^n .

PROPOSITION II.5. – <u>Let</u> p <u>and</u> q <u>be strongly regular types over a set</u> A , <u>and suppose that, for all</u> $n , m < \omega$, $p^n(\bar{x}) \cup q^m(\bar{y})$ <u>determines a complete type over</u> A . <u>Then</u> p <u>and</u> q <u>are orthogonal</u>.

<u>Proof</u>. – So suppose that, for all $n , m < \omega$, $p^n(\bar{x}) \cup q^m(\bar{y})$ is complete. It follows that if I is an independent set of realisations of p over A , and J is an independent set of realisations of q over A , then I and J are independent over A . Now pick I and J as in the last sentence, and such that both are infinite and $|I| < |J|$. Let M be prime over $A \cup J \cup I$. I assert that I is a basis for p in M . Note first that I is indiscernible over $A \cup J$. Now if c were a realisation of p in M such that $I \cup \{c\}$ were independent over A , then by our hypothesis, $I \cup \{c\}$ and J would be independent over A , and thus $I \cup \{c\}$ would be indiscernible over $A \cup J$, contradicting lemma II.3 (i). Thus I is a basis for p in M , and so $\dim(p , M) = |I|$. But clearly $\dim(q , M) \geq |J| > |I|$. So by lemma II.4, p and q are not equivalent, that is, p and q are orthogonal.

<u>Note</u>. – Proposition II.5 is actually true without the restriction that p and q be strongly regular (although we will not need this here). This fact, together with lemma 1.15 characterises orthogonality for types over sets.

LEMMA II.6. – <u>Let</u> $p \in S_1(A)$ <u>and</u> (p , φ) <u>strongly regular</u>. <u>Suppose that</u> $B \supseteq A$ <u>and that</u> p' <u>and</u> q <u>are in</u> $S_1(B)$, <u>where</u> p' <u>is the nonforking extension of</u> p <u>over</u> B , $q \neq p'$, <u>and</u> $\varphi \in q$. <u>Then</u> p' <u>and</u> q <u>are orthogonal</u>.

<u>Proof</u>. – It is enough to prove this in the case where B is a model, say M , and in this case it is enough to show that p' and q are perpendicular. So let a and b be realizations of p' and q respectively. I show that a and b are independent over M . Now as $q \neq p'$, there is some $L(M)$ formula $\psi(x)$ such that $\psi(x) \in q$ but $\neg \psi(x) \in p$. Suppose that $\alpha(x , y)$ is an **L(M)-formula such** that $\models \alpha(b , a)$. Thus $\models (\exists x)(\varphi(x) \wedge \alpha(x , a))$. So

$$M(a) \models (\exists x)(\varphi(x) \wedge \varphi(x) \wedge \alpha(x , a)) .$$

Let $c \in M(a)$ be such that $M(a) \models \varphi(c) \wedge \psi(c) \wedge \alpha(c\ ,\ a)$. So c satisfies $\varphi(x)$ but c does not realise p' . Thus $c \in M$ (by strong regularity of $(p'\ , \varphi)$). Thus we have shown that $tp(a/M \cup \{b\})$ is the heir of p' , whereby a and b are independent over M .

If follows from lemma II.6 that if p is strongly regular then p is regular ($p \in S(A)$ is said to be <u>regular</u> if whenever $B \supset A$, p' is the nonforking extension of p over B and q is a forking extension of p over B , then p' and q are orthogonal). Now for regular types the "nonforking" notion of independence on realisations of such types satisfies the familiar exchange principle. Namely : let $p \in S(A)$ be regular, $A \subset M$, and let $\bar{a}_i$, for $i < n$, and $\bar{b}$ realise p in M , where $\{\bar{a}_i\ ;\ i \leqslant n\}$ is a basis for p in M . Let $\bar{a}_m$ be the first element such that $tp(\bar{b}/\{\bar{a}_i\ ;\ i \leqslant m\})$ forks over M . Then $\{\bar{a}_0\ , \ldots , \bar{a}_{m-1}\ , \bar{b}\ , \bar{a}_{m+1}\ , \ldots , \bar{a}_{n-1}\}$ is a basis for p in M . (This is a simple consequence of regularity and the basis properties of forking). Thus we have :

PROPOSITION II.7. - <u>Let</u> $p \in S_1(A)$ <u>be strongly regular, and</u> $A \subset M$. <u>Then all bases for</u> p <u>in</u> M <u>have the same cardinality (and thus we can speak of</u> $\dim(p\ ,\ M)$)

PROPOSITION II.8. - <u>Let</u> p <u>and</u> q <u>be equivalent strongly regular types over a model</u> M , <u>and let</u> $N > M$. <u>Then</u> $\dim(p\ ,\ N) = \dim(q\ ,\ N)$.

<u>Proof</u>. - By lemma II.4 and proposition II.8.

I recall the following :

<u>Fact II.9</u>. - Let $p \in S(M)$ and $\varphi(\bar{x}) \in p$. Then p does not fork over $\cup\, \varphi^M$.

LEMMA II.10. - <u>Let</u> $p \in S_1(A)$, $(p\ ,\ \varphi)$ <u>strongly regular, and</u> $A \subset M \prec N$. <u>Let</u> p' <u>denote the nonforking extension of</u> p <u>over</u> M . <u>Let</u> I_1 <u>be a basis for</u> p <u>in</u> M , <u>and let</u> I_2 <u>be an independent over</u> M <u>set of realisations of</u> p' <u>in</u> N , <u>and finally let</u> $c \in N$ <u>and</u> $tp(c/I_1 \cup I_2 \cup A)$ <u>is the nonforking extension of</u> p <u>over</u> $I_1 \cup I_2 \cup A$. <u>Then</u> $tp(c/I_2 \cup M)$ <u>does not fork over</u> A (<u>and thus</u> $I_2 \cup \{c\}$ <u>is an independent set of realisations of</u> p' <u>in</u> N , <u>over</u> M .)

<u>Proof</u>. - It is enough to show that $tp(\{c\} \cup I_2/M)$ does not fork over A . By fact II.9, it is enough to show that $tp(\{c\} \cup I_2/\varphi^M \cup A)$ does not fork over A . Now, by hypothesis, $tp(I_2/\varphi^M \cup A)$ does not fork over A , and thus it suffices to prove that $tp(c/I_2 \cup \varphi^M \cup A)$ does not fork over $I_2 \cup A$. But $I_1 \subset \varphi^M$, and we know that $tp(c/I_2 \cup I_1 \cup A)$ does not fork over $I_2 \cup A$. So this leaves us having to prove that

$$(*) \qquad tp(c/I_2 \cup \varphi^M \cup A) \text{ does not fork over } I_2 \cup I_1 \cup A .$$

Let $\bar{d} \subset \varphi^M$, and $d \in \varphi^M$, and suppose that we already know that $tp(c/I_2 \cup I_1 \cup \bar{d} \cup A)$ does not fork over $I_2 \cup I_1 \cup A$. Now it is clear that $tp(d/I_2 \cup I_1 \cup \bar{d} \cup A) \neq tp(c/I_2 \cup I_1 \cup \bar{d} \cup A)$ (either $tp(d/A) \neq p$, or d and I_1 are dependent over A). But d satisfies $\varphi(x)$. So by strong regularity of (p , φ) and lemma II.6, c and d are independent over $I_2 \cup I_1 \cup \bar{d} \cup A$. Thus $tp(c/I_2 \cup I_1 \cup \bar{d} \wedge d \cup A)$ does not fork over $I_2 \cup I_1 \cup A$. So (*) is proved, and so also the lemma.

PROPOSITION II.11. - Let $p \in S(A)$ be strongly regular, $A \subseteq M \prec N$, and p' the nonforking extension of p over M . Then $\dim(p , N) = \dim(p , M) + \dim(p' , N)$

Proof. - By lemma II.10, if I_1 is a basis for p in M , and I_2 is a basis for p' in N , then $I_1 \cup I_2$ is a basis for p in N .

III. Non-multidimensional theories.

Definition III.1.

(i) Let M be a model of T . Then $\mu(M)$ denotes the maximum number of pair-wise orthogonal strongly regular types over M .

(ii) T will be said to be multidimensional if for any λ there is a model M of T with $\mu(M) \geqslant \lambda$. Otherwise T is said to be non-multidimensional.

I now give some background on material to come. Firstly, if p_1 is a type over a finite set $\bar{a}$, then p_1 can be written in the form $p(\bar{x} , \bar{a})$ (so $p(\bar{x} , \bar{y})$ is a type over $\emptyset$). Moreover, if $tp(\bar{a}) = tp(\bar{b})$, then $p(\bar{x} , \bar{b})$ is in $S(\bar{b})$, and for example, $p(\bar{x} , \bar{a})$ is strongly regular if, and only if, $p(\bar{x} , \bar{b})$ is strongly regular.

Secondly, suppose that $p \in S(A)$, and $q \in S(B)$ (A and B subsets of the big model). Then, because p and q are not types over the same set it does not make immediate sense to speak of, for example, p and q being orthogonal or not orthogonal. However we can interpret this to mean that for some C which includes A and B , any nonforling extensions of p and q over C are orthogonal (or not orthogonal, as the case might be). (We assume p and q to be stationary). Then by the results in section I, p and q will be orthogonal if, and only if, for any $C \supseteq A \cup B$ n the nonforking extensions of p and q over C are orthogonal.

Finally, we assume familiarity with the notion of strong type (denoted stp). The important facts are the following assuming ω-stability. If $p \in S_n(A)$, then there is $E \in FE_n(A)$ (that is, $E(\bar{x} , \bar{y})$ is an equivalence relation on n-tuples, definable over A , and with a finite number of classes), such that if $\bar{a}$ and $\bar{b}$ realise p then $\bar{a}$ and $\bar{b}$ have the same strong type over $A (stp(\bar{a}/A) = stp(\bar{b}/A))$ if, and only if, $\models E(\bar{a} , \bar{b})$. Also, if I is independent over A , and all elements of I have the same strong type over A , then I is indiscernible over A . Moreover, if I and J are two such sets, and the elements of I and J have the same type over A , then $tp(I/A) = tp(J/A)$. (In the cases in which we shall be interested, A will be the empty set and so will be omitted.) (Also $stp(\bar{a}/A) = stp(\bar{b}/B)$ implies $tp(\bar{a}/A) = tp(\bar{b}/A)$.)

PROPOSITION III.2. - The following are equivalent (for the theory T).

(i) For all M , $\mu(M) \leq \aleph_0$.

(ii) T is non-multidimensional.

(iii) If $p(x, \bar{a}) \in S(\bar{a})$ is strongly regular, and $stp(\bar{a}) = stp(\bar{b})$, then $p(x, \bar{a})$ and $p(x, \bar{b})$ are not orthogonal (that is equivalent).

Proof. -

(i) implies (ii) is immediate.

(ii) ==> (iii) : Suppose that $p(x, \bar{a}) \in S(\bar{a})$ is strongly regular, $stp(\bar{a}) = stp(\bar{b})$, but $p(x, \bar{a})$ and $p(x, \bar{b})$ are orthogonal. First we can assume that $\bar{a}$ and $\bar{b}$ are independent (For if not, then choose $\bar{c}$ such that $\bar{c}$ and $\bar{a} \frown \bar{b}$ are independent, and $stp(\bar{c}) = stp(\bar{a}) = stp(\bar{b})$. Then $p(x, \bar{a})$ and $p(x, \bar{c})$ are orthogonal). Let λ be any cardinal, and let $\{\bar{a}_\alpha ; \alpha < \lambda\}$ be an independent set of realisations of $tp(\bar{a})$, such that $\bar{a}_0 = \bar{a}$, $\bar{a}_1 = \bar{b}$, and, for all $\alpha < \lambda$, $stp(\bar{a}_\alpha) = stp(a)$. So $\{\bar{a}_\alpha ; \alpha < \lambda\}$ is indiscernible, and, for $\alpha < \beta < \lambda$, $p(x, \bar{a}_\alpha)$ and $p(x, \bar{a}_\beta)$ are orthogonal, (and strongly regular). Let M be a model containing all the $\bar{a}_\alpha$. For each $\alpha < \lambda$, let p_α be the nonforking extension of $p(x, \bar{a}_\alpha)$ over M . Then the p_α are pairwise orthogonal strongly regular types over M . Thus T is multidimensional.

(iii) ==> (i) : Let M be a model, and $q \in S(M)$ strongly regular. There is finite $\bar{a}$ in M such that q is definable over $\bar{a}$. So $p = q \upharpoonright \bar{a}$ is strongly regular, and q is the unique nonforking extension of p over M . Thus it suffices to show that there are at most $\aleph_0$ pairwise orthogonal strongly regular types over finite sets. Now there are only $\aleph_0$ many possible types of finite sets. Moreover for any $\bar{a}$, there are at most $\aleph_0$ types in $S_1(\bar{a})$. Also for any $\bar{a}$ and strongly regular $p(x, \bar{a}) \in S_1(\bar{a})$, there can be only finitely many pairwise orthogonal types of the form $p(x, \bar{b})$, where $tp(\bar{b}) = tp(\bar{a})$ (by (iii) and the paragraph preceding this proposition). Thus we finish.

PROPOSITION III.3. - Let T be non-multidimensional and N a model of T . Then there is a countable $M \prec N$, and a set $J \subset N$, J independent over M such that N is minimal over $M \cup J$.

Proof. - By III.2, $\mu(N)$ is countable. So we can find countable $M \prec N$ such that each of some maximal collection of pairwise orthogonal strongly regular types over N, is definable over M. Now use lemma 1.16.

T will be maid to be unidimensional if, for each $M \models T$, $\mu(M) = 1$.

PROPOSITION III.4. - T is unidimensional if, and only if, T is $\aleph_1$-categorical.

Proof. - Suppose that T is not unidimensional and let M be a model and p, q orthogonal strongly regular types over M. Assume that p and q are chosen with least possible Morley ranks in their respective equivalence classes, say $R(p)=\alpha$, $R(q) = \beta$, $\alpha \leqslant \beta$, and (p , φ) is strongly regular, where $R(\varphi) = \alpha$. As in the proof of 1.8, $\varphi(x)$ is not augmented in $M(q)$, and this, as is well known contradicts $\aleph_1$-categoricity.

Conversely, suppose that T is unidimensional. Let M_0 be the prime model of T. Then there is a strongly regular type p over M_0. If N is any model of T, then M_0 is elementarily embedded in N, and p' the heir of p over N, is strongly regular, and so is essentially the only strongly regular type over N. So N is prime over M_0 and a basis for p' in N. Such a basis is just a Morley sequence of p over M, and its type is determined. Thus if $|N_1| = |N_2| = \lambda > \aleph_0$, then N_1 is prime over $M_0 \cup I$ and N_2 is prime over $M_0 \cup J$, where I and J must both have cardinality λ, and have the same type over M_0. So $N_1 \cong N_2$.

PROPOSITION III.5. - Let T be non-multidimensional, and $p(x , \bar{a})$ a strongly regular type in $S(\bar{a})$. Suppose that $stp(\bar{b}) = stp(\bar{a})$ and M contains $\bar{a}$ and $\bar{b}$. Then $\dim(p(x , \bar{a}) , M) = \dim(p(x , \bar{b}) , M)$.

Proof. - Suppose first that $\bar{a}$ and $\bar{b}$ are independent (over $\emptyset$). Let $M_1 \prec M$ be prime over $\bar{a} \frown \bar{b}$, and let p_1, q_1 be the nonforking extensions of $p(x , \bar{a})$ and $p(x , \bar{b})$ over M_1. Now $tp(\bar{a} \frown \bar{b}) = tp(\bar{b} \frown \bar{a})$, and thus $(M_1 , \bar{a} , \bar{b}) \cong (M_1 , \bar{b} , \bar{a})$, whereby $\dim(p(x , \bar{a}) , M_1) = \dim(p(x , \bar{b}) , M_1)$. By III.2, p_1 and q_1 are equivalent, and thus $\dim(p_1 , M) = \dim(q_1 , M)$. Thus by II.11, $\dim(p(x ,\bar{a}) , M) = \dim(p(x , \bar{b}) , M)$.

Now in the general case, let $\bar{c}$ be such that $stp(\bar{c}) = stp(\bar{a}) = stp(\bar{b})$, and $\bar{c}$ and $\bar{a} \frown \bar{b}$ are independent (over $\emptyset$). Let $M' = M(\bar{c})$, and p', q' the non-forking extensions of $p(x , \bar{a})$ and $p(x , \bar{b})$ over M. Then $\dim(p' , M') = \dim(q' , M')$ (as p' and q' are strongly regular and equivalent), and both these dimensions are finite (otherwise $M' - M$ contains an infinite independent set over M, each element of which is dependent on $\bar{c}$ over M; which contradicts superstability). But by the first part of the proof,

$$\dim(p(x , \bar{c}) , M') = \dim(p(x , \bar{a}) , M') = \dim(p(x , \bar{b}) , M') ,$$

and we know that

$$\dim(p(x , \bar{a}) , M') = \dim(p(x , \bar{a}) , M) + \dim(p' , M')$$

and

$$\dim(p(x , \bar{b}) , M') = \dim(p(x , \bar{b}) , M) + \dim(q' , M') \qquad (II.11) .$$

Thus $\dim(p(x , \bar{a}) , M) = \dim(p(x , \bar{b}) , M)$, and we finish.

I now proceed to show that in the non-multidimensional case, all strongly regular types can be taken as being definable over the prime model of T (and thus in proposition III.3, M can be taken to be M_0 the prime model of T).

LEMMA III.6. - <u>Let</u> T <u>be non-multidimensional</u>. <u>Let</u> $M \prec M' \not\succ N$ <u>be models</u>. <u>Then there is</u> $c \in N - M'$, <u>such that</u> $tp(c/M')$ <u>is strongly regular</u>, <u>and</u> $tp(c/M')$ <u>does not fork over</u> M .

<u>Proof</u>. - Choose $c \in N - M'$ such that $tp(c/M)$ is of least possible Morley rank. Thus clearly there is $\bar{a} \in M$ and $\varphi(x , \bar{a}) \in tp(c/M)$, and for all $d \in (\varphi(x , a))^N - M'$, $tp(d/M) = tp(c/M)$. Let us denote $tp(c/M)$ by p . Now if $tp(c/M')$ does not fork over M (and so is the nonforking extension of p), then it is clear that $(tp(c/M') , \varphi)$ is strongly regular, and we finish. So let us assume that $tp(c/M')$ forks over M , and we <u>seek a contradiction.</u> Now, as $tp(c/M')$ forks over M (by our assumption), $R(tp(c/M')) < R(p)$. We can clearly assume that c has been chosen also to satisfy $R(tp(c/M'))$ being as small as possible (among those x in $N - M'$ for which $tp(x/M) = p$). So $tp(c/M')$ is strongly regular (I.4). Now let $\bar{b}_0$ be chosen in M' such that $tp(c/M')$ is definable over $\bar{b}_0$, and let $q(x , \bar{b}_0)$ denote $tp(c/\bar{b}_0)$. Thus $q(x , \bar{b}_0)$ is strongly regular. Now let $\bar{b}_1$ be such that $tp(\bar{b}_1/M) = tp(\bar{b}_0/M)$ and $\bar{b}_0$ and $\bar{b}_1$ are independent over M .

Thus $stp(\bar{b}_0) = stp(\bar{b}_1)$ (this is easy), and so by III.2, $q(x , \bar{b}_0)$ and $q(x , \bar{b}_1)$ are equivalent. Let q_0 and q_1 be the nonforking extensions f , $q(x , \bar{b}_0)$ and $q(x , \bar{b}_1)$ respectively over $M \cup \{\bar{b}_0 , \bar{b}_1\}$. (So in particular $q_0 \restriction M \cup \bar{b}_0 = tp(c/M \cup \bar{b}_0)$.) So q_0 and q_1 are strongly regular types over the same set which are not orthogonal. Thus by II.5, there are $n , m < \omega$ such that $q_0^n(\bar{x}) \cup q_1^m(\bar{y})$ is not a complete type over $M \cup \{\bar{b}_0 , \bar{b}_1\}$. Thus (as q_0 and q_1 are stationary), there are $c_1 , \dots , c_n$ independent realisations of q_0 over $M \cup \{\bar{b}_0 , \bar{b}_1\}$, and $d_1 , \dots , d_m$ independent realisations of q_1 over $M \cup \{\bar{b}_0 , \bar{b}_1\}$ such that $\{c_1 , \dots , c_n\}$ and $\{d_1 , \dots , d_m\}$ are not independent over $M \cup \{\bar{b}_0 , \bar{b}_1\}$. By minimalising m , we can assume that $\{c_1 , \dots , c_n\}$ and $\{d_1 , \dots , d_{m-1}\}$ are independent over $M \cup \{\bar{b}_0 , \bar{b}_1\}$. Let us denote $\langle c_1 , \dots , c_n\rangle$ by $\bar{c}$ and $\langle d_1 , \dots , d_{m-1}\rangle$ by $\bar{d}$. I assert that

(*) $\qquad \bar{b}_0 \frown \bar{c}$ and $\bar{b}_1 \frown \bar{d}$ are independent over M .

First note that $tp(\bar{d}/\bar{b}_0 \cup \bar{b}_1 \cup M)$ does not fork over $\bar{b}_1 \cup M$, and that $tp(\bar{b}_1/\bar{b}_0 \cup M)$ does not fork over M. Thus $tp(\bar{b}_1 \frown \bar{d}/\bar{b}_0 \cup M)$ does not fork over M, and so

(i) $$tp(\bar{b}_0/\bar{b}_1 \frown \bar{d} \cup M) \text{ does not fork over } M .$$

Also $tp(\bar{c}'/\bar{b}_0 \cup \bar{b}_1 \frown \bar{d} \cup M)$ does not fork over $M \cup \bar{b}_0$. This together with (i) yields $tp(\bar{b}_0 \frown \bar{c}'/\bar{b}_1 \frown \bar{d} \cup M)$ does not fork over M, which means (*)

Note also that $tp(c_n/\{c_1, \ldots, c_{n-1}\} \cup \bar{d} \cup \bar{b}_0 \cup \bar{b}_1 \cup M)$ does not fork over $M \cup \{\bar{b}_0, \bar{b}_1\}$, but that

(**) $tp(c_n/\{c_1, \ldots, c_{n-1}\} \cup \bar{d} \frown d_n \cup \bar{b}_0 \cup \bar{b}_1 \cup M)$ does not fork over $M \cup \{\bar{b}_0, \bar{b}_1\}$

Now $tp(c_n/M \cup \bar{b}_0) = tp(c/M \cup \bar{b}_0)$. So we can assume that $c_n = c$ (leave $\bar{b}_0$ fixed but shift around the other c_i's, the d_i's and $\bar{b}_1$ so as to preserve the type of everything over M), let us denote d_m by d. So $tp(d \frown \bar{b}_1/M) = tp(c \frown \bar{b}_0/M)$, whereby $tp(d/M) = p$, and $tp(d/M \cup \bar{b}_1)$ forks over M, and so there is finite $\Delta \subset L$ such that

(***) $$R(tp(d/M \quad \bar{b}_1), \Delta, 2) < R(p, \Delta, 2) = r .$$

Let us now sum up the information obtained ; denoting now $\langle c_1, \ldots, c_{n-1}\rangle$ by $\bar{c}$, and as before $\langle d_1, \ldots, d_{m-1}\rangle$ by $\bar{d}$.

(a) c and $\bar{c}$ are independent over $M \cup \bar{b}_0$.

(b) $\bar{b}_0 \frown \bar{c} \frown c$ and $\bar{b}_1 \frown \bar{d}$ are independent over M (by (*)).

(c) There is a formula $\chi(x, \bar{z})$ and $\bar{e} \in M$ such that $\models \chi(c, d \frown \bar{d} \frown \bar{c} \frown \bar{b}_0 \frown \bar{b}_1 \frown \bar{e})$, but $\chi(x, \bar{z})$ is not in bound $(tp(c/\bar{b}_0))$ (and so $\chi(x, \bar{z})$ is not represented in $tp(c/M')$) (by (**)).

(d) There is an $L(M)$ formula $\psi(x, \bar{w})$ such that d satisfies $\psi(x, \bar{b}_1)$ and $R(\psi(x, b_1), \Delta, 2) < r$ (by (***)).

(Remember for any type q and finite $\Delta \subset L$, there is finite subtype of q, say q' such that $R(q, \Delta, 2) = R(q', \Delta, 2)$.)

Remember that d also satisfies the formula $\varphi(x, \bar{a})$. Thus by (c) and (d), we have

$$\models (\exists y)(\varphi(y, \bar{a}) \wedge \chi(c, y \frown \bar{d} \frown \bar{c} \frown \bar{b}_0 \frown \bar{b}_1 \frown \bar{e}) \wedge \psi(y, \bar{b}_1) \quad \text{"}R(\psi(x, \bar{b}_1), \Delta, 2) < r\text{"}) .$$

By (b) we can find $\bar{b}_1'$ and $\bar{d}'$ in M such that

$$\models (\exists y)(\varphi(y, \bar{a}) \wedge \chi(c, y \frown \bar{d}' \frown \bar{c} \frown \bar{b}_0 \frown \bar{b}_1' \frown \bar{e}) \wedge \psi(y, \bar{b}_1') \wedge \text{"}R(\psi(x, \bar{b}_1'), \Delta, 2) < r\text{"}).$$

Now by (a) and the fact that $tp(c/M')$ is definable over $M \cup \bar{b}_0$, we can find $\bar{c}' \in M'$ such that

$N \models (\exists y)(\varphi(y , \bar{a}) \wedge \chi(c , y \bar{d}'\hat{}\bar{c}\hat{}\bar{b}_0\hat{}\bar{b}_1\hat{}\bar{e}) \wedge \psi(y , \bar{b}_1') \wedge "R(\psi(x , \bar{b}_1') , \Delta , 2) < r")$.

Pick $a \in N$ to be such a y as given above. First note that $a \notin M'$, for if not then $\chi(x , \bar{z})$ would be represented in $tp(c/M')$, contradicting (c). Thus as a satisfies $\varphi(x , \bar{a})$, we must have $tp(a/M) = p$ (by choice of p and $\varphi(x , \bar{a})$). But now, as a satisfies $\psi(x , \bar{b}_1')$ and $R(\psi(x , \bar{b}_1') , \Delta , 2) < r = R(p , \Delta , 2)$, we must have that $tp(a/M) \neq p$. This contradiction proves the lemma.

PROPOSITION III.7. - Let $M < M'$ be models of T, where T is non-multidimensional, and let $p \in S(M')$ be strongly regular. Then there is $q \in S(M')$, such that q is strongly regular, q is equivalent of p, and q does not fork over M.

Proof. - Lemma III.6 gives us c in $M'(p) - M'$ such that $tp(c/M')$ is strongly regular, and does not fork over M. Clearly $tp(c/M')$ is equivalent to p.

COROLLARY III.8. - Let T be non-multidimensional. Let M be a model, A a set, and N prime over $M \cup A$. Then N is minimal over $M \cup A$.

Proof. - If not, there is model M' such that $M \cup A \subseteq M' \not\prec N$. Lemma III.6 gives us $c \in N - M'$ such that $tp(c/M')$ does not fork over M. But $tp(c/M)$ is not isolated, and $tp(c/M \cup A)$ is isolated, whereby $tp(c/M \cup A)$ forks over M, and so $tp(c/M')$ forks over M. Contradiction.

Let me now state a few obvious things. Let us assume T to be non-multidimensional, and let M_0 be the prime model of T. Let $\{p_i ; i < \mu \leq \aleph_0\}$ be a maximal collection of pairwise orthogonal strongly regular types over M_0. Let N be any model of T. So M_0 is elementarily embedded in N, and let p_i' for $i < \mu$, be the heirs of the p_i over N. Then $\{p_i' ; i < \mu\}$ is a maximal collection of pairwise orthogonal strongly regular types over N. For choose strongly regular $q \in S(N)$. By III.7, q is equivalent of $p \in S(N)$, where p is strongly regular and does not fork over M_0. But there is $i < \mu$ such that $p \restriction M_0$ is equivalent to p_i and so p is equivalent to p_i', and so q is equivalent to p_i'.

IV. The spectrum.

In this section T will be assumed to be non-multidimensional, and M_0 will denote the prime model of T.

First, some more preliminary results.

LEMMA IV.1. - Let M be a model, $\bar{a} \in M$, $p(x , \bar{a}) \in S(\bar{a})$ be strongly regular, and $tp(\bar{a})$ isolated. Suppose that $\bar{b} \in M$, $tp(\bar{b}) = tp(\bar{a})$ and $p(x , \bar{b})$ is equivalent to $p(x , \bar{a})$. Then $\dim(p(x , \bar{a}) , M) = \dim(p(x , \bar{b}) , M)$.

Proof. - Let $M_0 < M$ be a copy of the prime model such that $\bar{a} \in M_0$. It is easy to find $\bar{c} \in M_0$ such that $stp(\bar{c}) = stp(\bar{b})$. By III.2, $p(x, \bar{b})$ and $p(x, \bar{c})$ are equivalent. Thus $p(x, \bar{a})$ and $p(x, \bar{c})$ are equivalent. Let p_1 and p_2 be the nonforking extensions of $p(x, \bar{a})$ and $p(x, \bar{c})$ over M_0. So p_1 and p_2 are equivalent and strongly regular, and thus by II.8, $\dim(p_1, M) = \dim(p_2, M)$. But it is clear that $(M_0, \bar{a}) \cong (M_0, \bar{c})$, and so $\dim(p(x,\bar{a}),M_0)=\dim(p(x,\bar{c}),M_0)$. Thus by II.11, we have

$$\dim(p(x, \bar{a}), M) = \dim(p(x, \bar{c}), M) .$$

But by III.5,

$$\dim(p(x, \bar{c}), M) = \dim(p(x, \bar{b}), M) ,$$

and so we have

$$\dim(p(x, \bar{a}), M) = \dim(p(x, \bar{b}), M) ,$$

as desired.

LEMMA IV.2. (which does not need non-multidimensionality). - Let $p \in S(M_0)$, p definable over $\bar{a} \in M_0$, $p_1 = p \upharpoonright \bar{a}$, and p_1 has an infinite basis in M_0 (thus $\dim(p_1, M_0) = \aleph_0$). Then $M_0(p) \cong M_0$.

Proof. - $M_0(p)$ is countable, and thus it is enough to show that $M_0(p)$ is atomic (i. e. realises only isolated types). Let $\bar{c} \in M_0(p)$ be such that $tp(\bar{c}/M_0) = p$ and $M_0(p)$ is atomic over $M_0 \cup \bar{c}$. It is enough to show that $M_0 \cup \bar{c}$ is atomic. So let $\bar{b} \in M_0$. I show that $tp(\bar{b} \frown \bar{c})$ is isolated, in fact that $tp(\bar{a} \frown \bar{b} \frown \bar{c})$ is isolated. Let $\bar{c}_i$, for $i < \omega$, be a basis for $p_1 = p \upharpoonright \bar{a}$ in M_0. Then by superstability, there must be $i < \omega$ such that $\bar{c}_i$ and $\bar{b}$ are independent over $\bar{a}$. Then clearly $tp(\bar{a} \frown \bar{b} \frown \bar{c}_i) = tp(\bar{a} \frown \bar{b} \frown \bar{c})$, and $tp(\bar{a} \frown \bar{b} \frown \bar{c}_i)$ is isolated, as it is realised in the prime model M_0. So we finish.

Note. - An extension of the above proof shows that if $p \in S(M)$ and for some $\bar{a} \in M$ over which p is definable, $p \upharpoonright \bar{a}$ has an infinite basis in M, then for all $\bar{a} \in M$ over which p is definable $p \upharpoonright \bar{a}$ has an infinite basis in M.

COROLLARY IV. 3. - Let $\{p_i ; i < \kappa (\leq \aleph_0)\}$ be a set of pairwise orthogonal strongly regular types over M_0, such that for each i there is $\bar{a}_i \in M_0$ such that p_i is definable over $\bar{a}_i$, and $\dim(p_i \upharpoonright \bar{a}_i, M_0)$ is infinite. For each $i < \kappa$, let J_i be an independent set of realisations of p_i over M_0, such that $|J_i| \leq \omega$. Then $M_0(\bigcup_{i<\kappa} J_i) \cong M_0$.

Proof. - It is easy, using IV.2, induction and fact 1.6, to show that $M_0(J_0)$ is isomorphic to M_0 (let $J_0 = \{c_n ; n < \lambda\}$, let $M_1 = M_0(c_0)$, and in general $M_{n+1} = M_n(c_n)$. Then $tp(c_n/M_n)$ is the heir of p_0 over M_n, and $M_{n+1} \cong M_0$. So $\bigcup_{n<\lambda} M_n$ is isomorphic to M_0, and is also easily see to be the same as

$M_0(J_0)$). Then it is easy to see that $tp(J_1/M_0(J_0))$ does not fork over M_0, and so we can repeat the process to get $M_0(J_0)(J_1) \cong M_0$. Carry on, and putting $M^0 = M_0$, and $M^{n+1} = M^n(J_n)$, we see that $\bigcup_{n<\kappa} M^n$ is isomorphic to M_0 and is the same as $M_0(\bigcup_{i<\kappa} J_i)$.

LEMMA IV.4. - *Let $\{p_i \,;\, i < \kappa\}$ be pairwise orthogonal types over a model M, and let for each $i < \kappa$, J_i be a set of independent realisations of p_i over M.*

Let N be prime over $M \cup \bigcup_{i<\kappa} J_i$. Then for each $i < \kappa$, J_i is a basis for p_i in N.

Proof. - Consider J_0 for example. Let us define $M_i \prec N$ for $1 \leq i \leq \kappa$, such that M_1 is prime over $M \cup J_1$, and for $i \geq 1$, M_{i+1} is prime over $M_i \cup J_{i+1}$ and $M_\delta = \bigcup_{i<\delta} M_i$ for δ limit. Let M' be M_κ. Let p_0^i be the heir of p_0 over M_i for $i \leq \kappa$. Then it is easy to show by induction, using the orthogonality of p_0 and the p_i' s and fact 1.6, that $p_0 \vdash p_0^i$ for $1 \leq i \leq \kappa$. Thus J_0 is a basis for p_0^κ in N if, and only if, J_0 is a basis for p_0 in N, and clearly J_0 is an independent set of realisations of p_0^κ over M' in N. By III.8 for example, N is prime over $M' \cup J_0$, and so J_0 is easily seen to be a basis for p_0^κ in N. So the lemma is proved.

LEMMA IV.5. - *Let $p \in S(\bar{a})$ be strongly regular, where $tp(\bar{a})$ is isolated, and for some copy of M_0 which contains $\bar{a}$, $\dim(p, M_0) = 0$. Let A be any countable set which is atomic over $\bar{a}$, and let p' be the nonforking extension of p over $A \cup \bar{a}$. Then $p \vdash p'$.*

Proof. - Let A be as given. Then $A \cup \bar{a}$ is an atomic countable set, and we can find a copy of the prime model M_0 such that $A \subset M_0$. By isomorphism, p is not realised in M_0. So by lemma II.11, for any c realising p, $tp(c/M_0)$ does not fork over $\bar{a}$, and thus $tp(c/A \cup \bar{a})$ does not fork over $\bar{a}$. So clearly $p \vdash p'$

We can now begin on the classification. First let μ be the maximum number of pairwise orthogonal strongly regular types over M_0, the prime model of T. (We call μ the number of dimensions of T). Let p_i for $i < \mu$, be pairwise orthogonal and strongly regular types over M_0, and a maximal such collection. Now let N be any model. So $M_0 \prec N$, and (by 1.16, III.7 and remarks at the end of III) N is prime over (in fact minimal over) $M_0 \cup \bigcup_{i<\mu} J_i$ where J_i is a basis for p_i in N, and moreover (by 1.15) $tp(\bigcup_{i<\mu} J_i/M_0)$ is determined just by $\langle \lambda_i \,;\, i < \mu \rangle$ where $\lambda_i = |J_i|$. Conversely, given a sequence $\langle \lambda_i \,;\, i < \mu \rangle$ of cardinals, there is a model N prime over $M_0 \cup \bigcup_{i<\mu} J_i$ where J_i is an independent set of realisations of p_i, and thus by IV.4, a basis for p_i in N. So if we are considering the models of T up to isomorphism over some fixed copy of the prime model M_0 (which we could do by for example adding names for the elements of M_0 to the langage, and replacing T by $Th(M_0)$ in this new language), then the models would correspond exactly to the possible sequences of cardinals $\langle \lambda_i \ ; \ i < \mu \rangle$. However in the general case, one model might contain different copies of M_0 and correspond to different sequences of cardinals. So we have to be more careful in the choices of the p_i , and use some material developed in this section and section III. This we proceed to do, summing up the results later on in a theorem.

First let K_i , for $i < \mu$, be the equivalence classes (or non-rothogonality classes) of strongly regular types over M_0 . We choose, for each $i < \mu$, $p_i \in K_i$ and $\bar{a}_i \in M_0$, such that p_i is definable over $\bar{a}_i$, and also satisfying the following two conditions, where $q_i(x , \bar{a}_i)$ denotes $p_i \upharpoonright \bar{a}_i$ (so $q_i(x , \bar{y}_i)$ is over $\emptyset$) :

(i) $\dim(q_i(x , \bar{a}_i) , M_0)$ is 0 or infinite (i. e. $\aleph_0$), for all $i < \mu$, and

(ii) if $i < j < \mu$, then <u>either</u> $tp(\bar{a}_i) = tp(\bar{a}_j)$ and $q_i(x , \bar{y}_i) = q_j(x , \bar{y}_j)$, <u>or</u> for <u>no</u> $p \in K_j$ is there $\bar{a} \in M_0$ such that p is definable over $\bar{a}$, $tp(\bar{a}) = tp(\bar{a}_i)$ and $p \upharpoonright \bar{a} = q_i(x , \bar{a})$.

(Note that if the second disjunct of (ii) holds, then we also have that for <u>no</u> $p \in K_i$ is there $\bar{a} \in M_0$ such that p is definable over $\bar{a}$, $tp(\bar{a}) = tp(\bar{a}_j)$ and $p \upharpoonright \bar{a} = q_j(x , \bar{a})$.)

This is achieved quite easily. To get (i) for example, suppose p_i has been chosen in K_i , and, for some $\bar{a} \in M_0$, p_i is definable over $\bar{a}$ and $\dim(p_i \upharpoonright \bar{a} , M_0) = n < \omega$. Let $c_1 , \ldots , c_n$ be a basis for $p_i \upharpoonright \bar{a}$ in M_0 , and put $\bar{a}_i = \bar{a} \frown \langle c_1 , \ldots , c_n \rangle$. Then clearly $\dim(p_i \upharpoonright \bar{a}_i , M_0) = 0$. (ii) can easily be obtained by defining the p_i and $\bar{a}_i$ inductively.

This having been done, pick some particular $i < \mu$, and let us put $p = p_i$, $a = \bar{a}_i$, and $q(x , \bar{y}) = q_i(x , \bar{y}_i)$. For how many $j < \mu$, do we have $tp(\bar{a}_j) = tp(\bar{a})$ and $q_j(x , \bar{y}_j) = q(x , \bar{y})$ (and thus $p_j \upharpoonright \bar{a}_j = q(x , \bar{a}_j)$) ? I assert that there can be only finitely many such j . For if not, then there is infinite $J \subseteq \omega$, such that the types $\{q(x , \bar{a}_j) \ ; \ j \in J\}$ are pairwise orthogonal, and $tp(\bar{a}_j) = tp(\bar{a})$ for all $j \in J$. Thus (see background at the beginning of section III), there is $j_1 < j_2$ in J such that $stp(\bar{a}_{j_1}) = stp(\bar{a}_{j_2})$. But by III.2, this contradicts the orthogonality of $q(x , \bar{a}_{j_1})$ and $q(x , \bar{a}_{j_2})$. (Remember $q(x , \bar{a})$ is strongly regular). Thus there are only finitely many such j .

Thus by renumbering the q_i and renaming the p_i and $\bar{a}_i$, we have :

LEMMA IV.6. – <u>There is</u> $\mu' \leq \aleph_0$, <u>and for each</u> $i < \mu'$, <u>some finite</u> n_i , <u>and</u> $q(x , \bar{y}_i)$ <u>over</u> $\emptyset$, <u>and for each</u> $i < \mu'$ <u>and</u> $j < n_i$, <u>types</u> p_i^j <u>over</u> M_0 <u>and tuples</u> $\bar{a}_i^j$ <u>in</u> M_0 <u>such that</u>

(i) $\{p_i^j \ ; \ i < \mu' , \ j < n_i\}$ <u>is a maximal collection of pairwise orthogonal strongly regular types over</u> M_0 .

(ii) p_i^j <u>is definable over</u> $\bar{a}_i^j$,

(iii) <u>for any</u> $i < \mu'$, <u>for</u> $j_1 < j_2 < n_i$, $tp(\bar{a}_i^{j_1}) = tp(\bar{a}_i^{j_2}) = r_i$, and, for each $j < n_i$, $p_i^j \upharpoonright a_i^j = q_i(x , \bar{a}_i^j)$,

(iv) <u>for each</u> i <u>and</u> j , $\dim(a_i(x , \bar{a}_i^j) , M_0) = 0$ <u>or</u> $\aleph_0$,

(v) <u>if</u> $i_1 < i_2 < \mu'$, <u>then there are no</u> $\bar{a}_1$, $\bar{a}_2$ <u>in</u> M_0 <u>such that</u> $tp(\bar{a}_1) = r_{i_1}$ <u>and</u> $tp(\bar{a}_2) = r_{i_2}$, <u>and</u> $q_{i_1}(x , \bar{a}_1)$ <u>is equivalent to</u> $q_{i_2}(x , \bar{a}_2)$.

(vi) $\mu' = \aleph_0$ <u>if, and only if,</u> $\mu = \aleph_0$, <u>and</u> $\mu' = 1$ if, <u>and only if,</u> $\mu = 1$.

LEMMA IV.7. - <u>Let</u> N <u>be any model of</u> T , <u>and let</u> $i_1 < i_2 < \mu'$. <u>Then there are no</u> $\bar{a}_1$ <u>and</u> $\bar{a}_2$ <u>in</u> N <u>such that</u> $tp(\bar{a}_1) = r_{i_1}$, $tp(\bar{a}_2) = r_{i_2}$, <u>and</u> $q_{i_1}(x , \bar{a}_1)$ <u>is equivalent of</u> $q_{i_2}(x , \bar{a}_2)$.

<u>Proof</u>. - Suppose that there are $\bar{a}_1$ and $\bar{a}_2$ in N as described, and we get a contradiction. Let M_0 be some copy of the prime model in N . Now both r_{i_1} and r_{i_2} are isolated types, and so it is easy to find $\bar{a}_1'$ and $\bar{a}'_2$ in M_0 such that $stp(\bar{a}_1') = stp(\bar{a}_1)$ and $stp(\bar{a}_2') = stp(\bar{a}_2)$. Thus by III.2, $q_{i_1}(x , \bar{a}_1')$ is equivalent to $q_{i_1}(x , \bar{a}_1)$, and $q_{i_2}(x , \bar{a}_2')$ is equivalent to $q_{i_2}(x , \bar{a}_2)$. But then $q_{i_1}(x , \bar{a}_1')$ is equivalent to $q_{i_2}(x , \bar{a}'_2)$, **which contradicts lemma IV.6 (v).**

Now we go through the cases depending on the number of dimensions.

<u>Case 1</u>. - μ is finite. So also μ' is finite. Let $A = \bigcup\{\bar{a}_i^j ; i < \mu' , j < n_i\}$ and let q_i^j be the nonforking extension of $q_i(x , \bar{a}_i^j)$ over A . Let λ_i^j for $i < \mu'$ and $j < n_i$ be cardinals chosen arbitrarily subject to the proviso that $\lambda_i^j \geq \aleph_0$ if $\dim(q_i(x , \bar{a}_i^j) , M_0) = \aleph_0$. Let $A(\langle\lambda_i^j ; i < \mu' , j < n_i\rangle)$ denote the model prime over $A \cup \bigcup(I_i^j ; i < \mu' , j < n_i)$, where I_i^j is an independent set of realisations of q_i^j over A of cardinality λ_i^j . Note that $A(\bar{\lambda})$ (where $\bar{\lambda} = \langle\lambda_i^j ; i < \mu' , j < n_i\rangle)$ is well defined by 1.15 and uniqueness of prime models.

<u>Observation IV.8</u>.

(i) $\dim(q_i^j , A(\bar{\lambda})) = \lambda_i^j$.

(ii) $\dim(q_i(x , \bar{a}_i^j) , A(\bar{\lambda})) = \lambda_i^j$.

<u>Proof</u>.

(i) Let M be prime over $M_0 \cup \bigcup\{X_i^j ; i < \mu' , j < n_i\}$ where X_i^j is an independent set of realisations of p_i^j over M_0 of cardinality λ_i^j . Then $\dim(p_i^j , M) = \lambda_i^j$, by lemma IV.4. It is easily seen that M is isomorphic (over A) to $A(\bar{\lambda})$, and that (by II.11 and choice of p_i^j and $\bar{a}_i^j$) that $\dim(q_i^j , M) = \lambda_i^j$.

(ii) We use (i). First suppose that $\dim(q_i(x , \bar{a}_i^j) , M_0) = 0$. Then as $tp(A/\bar{a}_i^j)$ is isolated, we have by IV.5 that $q_i(x , \bar{a}_i^j) \vdash q_i^j$, and thus

$$\dim(q_i(x , \bar{a}_i^j) , A(\bar{\lambda})) = \dim(q_i^j , A(\bar{\lambda})) = \lambda_i^j .$$

Secondly, suppose that $\dim(q_i(x , \bar{a}_i^j) , M_0)$ is infinite. Then so must be $\dim(q_i(x , \bar{a}_i^j) , A(\bar{\lambda}))$. But only finitely many members of a basis for $q_i(x , \bar{a}_i^j)$ in $A(\bar{\lambda})$ can be made to fork by $A - \bar{a}_i^j$ (remember that A is finite at the moment). Thus clearly $\dim(q_i(x , \bar{a}_i^j) , A(\bar{\lambda})) = \dim(q_i^j , A(\bar{\lambda})) = \lambda_i^j$.

Conversely we know that <u>any</u> model N of T can be written as (i. e. is isomorphic to) $A(\langle\lambda_i^j ; i < \mu' , j < n_i\rangle)$, where λ_i^j must be infinite if $\dim(q_i(x , a_i^j) , M_0)$ is infinite (by I. 16, III.7, and remarks at the end of section III). It is also clear that $|A(\bar{\lambda})| = \max(\{\lambda_i^j ; i < \mu' , j < n_i\} \cup \{\aleph_0\})$. When is $A(\lambda) \cong A(\bar{\lambda}^*)$.

<u>Case 1 (i)</u>. – $\mu = 1$. So $\mu' = 1$, and $n_0 = 1$. Also $A = \bar{a}_0^0$. Let us write $\bar{a}_0^0$ as $\bar{a}$ and $q_0(x , \bar{y}_0)$ as $q(x , \bar{y})$. Now suppose that $M = \bar{a}(\lambda) \cong \bar{a}(\lambda^*)$. Then there is $\bar{a}^* \in M$, $tp(\bar{a}^*) = tp(\bar{a})$, and $M = a^*(\lambda^*)$. So $\dim(q(x , \bar{a}),M)=\lambda$, and $\dim(q(x , \bar{a}^*) , M) = \lambda^*$ (by IV.8 (ii)). But as $\mu = 1$, we must have that $q(x , \bar{a})$ and $q(x , \bar{a}^*)$ are equivalent, but then by lemma IV.1, we have that $\lambda = \lambda^*$. So we have $\bar{a}(\lambda) \cong \bar{a}(\lambda^*)$ if, and only if, $\lambda = \lambda^*$. Thus in this case

$$I(\kappa , T) = 1 \quad \text{if} \quad \kappa > \aleph_0 .$$

If $\dim(q(x , \bar{a}) , M_0) = 0$, then

$$I(\aleph_0 , T) = \aleph_0 \quad \text{(as all finite dimensions can occur),}$$

and if $\dim(q(x , \bar{a}) , M_0)$ is infinite, then

$$I(\aleph_0 , T) = 1 .$$

<u>Case 1 (ii)</u>. – $\mu > 1$ (but still finite).

Let $\bar{\mu}$ denote $\langle\mu_i^j ; i < \mu' , j < n_i\rangle$ (no connection with μ, the number of dimension). Suppose that $N = A(\bar{\lambda}) \cong A(\bar{\mu})$. Thus there is A^* in N with $tp(A^*) = tp(A)$, and $N = A^*(\bar{\mu})$. Denote by $\bar{a}_i^{j*}$ the copy of $\bar{a}_i^j$ in A^*. Then $\{q_i(x , \bar{a}_i^{j*}) ; i < \mu' , j< n_i\}$ is a set of pairwise orthogonal strongly regular types, and by IV.8 (ii), $\dim(q_i(x , \bar{a}_i^{j*}) , M_0) = \mu_i^j$. So as the $q_i(x , \bar{a}_i^j)$ are a maximal collection of pairwise orthogonal strongly regular types, and by lemma IV.7, there is σ such that for each $i < \mu'$, $\sigma(i , -)$ is a permutation of n_i and $q_i(x , a_i^{j*})$ is equivalent to $q_i(x , a_i^{\sigma(i,j)})$. Thus by lemma IV.1, $\mu_i^j = \lambda_i^{\sigma(i,j)}$. Thus $A(\bar{\lambda}) \cong A(\bar{\mu})$ implies that $\bar{\mu} = \sigma(\bar{\lambda})$, where σ is a permutation of the sequence $\bar{\lambda}$ (As the number of dimensions is finite, there can only be finitely many such permutations).

<u>Case 1 (ii) (a)</u>. – For some $i < \mu'$, $j < n_i$, $\dim(q_i(x , \bar{a}_i^j) , M_0) = 0$. Then all cardinals (including finite ones) are possible for λ_i^j. Thus the number of sequences of cardinals $\leq \aleph_\alpha$, $\lambda_i^j ; i < \mu' , j < n_i\rangle$ at least one member of which is $\aleph_\alpha$, is $|\alpha| + \aleph_0$. (Note that in this case $|A(\bar{\lambda})| = \aleph_\alpha$.) But by the

above there can be only finitely many other sequences $\bar{\mu}$ giving rise to the same model, and thus we have

$$I(\aleph_\alpha , T) = |\alpha| + \aleph_0 \text{ , for all } \alpha \geqslant 0 .$$

Case 1 (ii) (b). - For all $i < \mu'$, $j < n_i$, $\dim(q_i(x , \bar{a}_i^j) , M_0) = \aleph_0$. But then the countable models of T are just models isomorphic to $A(\bar{\aleph}_0)$, and thus T is $\aleph_0$-categorical, i. e. $I(\aleph_0 , T) = 1$. Now suppose that $A(\bar{\lambda}) \cong A(\bar{\mu})$ as above and thus that there is $A^* \subset N = A(\lambda)$, with $N = A^*(\bar{\mu})$, and σ with $q_i(x , \bar{a}_i^{j*})$ equivalent to $q_i(x , \bar{a}_i^{\sigma(i,j)})$. Then as T is $\aleph_0$-categorical, all types are isolated, and thus $tp(A \frown A^*)$ is realised in every model of T . Clearly the fact that $q_i(x , \bar{a}_i^j)$ is equivalent ot $q_i(x , \bar{a}_i^{j'})$, say, depends only on $tp(\bar{a}_i^j \wedge \bar{a}_i^{j'})$. So we let G denote the group of permutations σ of μ , induced as above, and clearly $A(\bar{\lambda}) \cong A(\bar{\mu})$ if, and only if, there is $\sigma \in G$ with $\sigma(\bar{\lambda}) = (\bar{\mu})$. By our case hypothesis, only infinite values are possible for the λ_i^j . Let us denote by $(|\alpha + 1|^{\mu})^*$ the number of sequences of length μ of ordinals $\leq \alpha$, at least one of which is α . Thus it is clear that

$$I(\aleph_\alpha , T) = (|\alpha + 1|^{\mu})^*/G \text{ , for all } \alpha \geqslant 0 ;$$

Case 2. - $\mu = \aleph_0$, and so μ' is also $\aleph_0$.

Let me denote by $M_0(\lambda_i^j)_{i,j}$ the model prime over $M_0 \cup \bigcup\{I_i^j ; i < \mu' , j < n_i\}$ where I_i^j is an independent set of realisations of p_i^j over M_0 . We know that any λ_i^j can occur. I first want to observe that if $\dim(q_i(x , \bar{a}_i^j) , M_0) = \aleph_0$, then we can assume that λ_i^j is always 0 or uncountable.

LEMMA IV. 9. - Let $N = M_0(\lambda_i^j)_{i,j}$, where, for $(i , j) \in X$, $\dim(q_i(x , \bar{a}_i^j) , M_0) = \aleph_0$, and $\lambda_i^j \leqslant \aleph_0$. Then $N \cong M_0(\lambda_i^{j*})_{i,j}$ where $\lambda_i^{j*} = \lambda_i^j$ if $(i , j) \notin X$, and $\lambda_i^{j*} = 0$ if $(i , j) \in X$.

Proof. - Easy using IV.3 and IV.5.

Thus the models of T are all of the form $M_0(\bar{\lambda})$ where λ_i^j can be anything, if $\dim(q_i(x , a_i^j) , M_0) = 0$, and is 0 or uncountable otherwise. Moreover, it is easy to see, using II.11 and IV.5, that $\dim(q_i(x , \bar{a}_i^j) , M_0(\bar{\lambda})) = \lambda_i^j$, if $\dim(q_i(x , \bar{a}_i^j) , M_0) = 0$, and $= \aleph_0 + \lambda_i^j$ otherwise. It is also clear by IV.4, that $\dim(p_i^j , M_0(\bar{\lambda})) = \lambda_i^j$. Thus, as in case 1, it follows that if $M_0(\bar{\lambda}) \cong M_0(\bar{\mu})$, then there is σ such that for $i < \mu'$, and $j < n_i$, $\sigma(i , j) < n_i$, and for all i , j , $\mu_i^j = \lambda_i^{\sigma(i,j)}$. But μ' is infinite, and if $i_1 < i_2 < \aleph_0$, we can vary $\lambda_{i_1}^{j_1}$ and $\lambda_{i_2}^{j_2}$ (j_1 , j_2 arbitrary), to get different models. Thus it is clear that $I(\aleph_\alpha , T) = \prod_{i<\aleph_0} \varkappa_i$, where $\varkappa_i = |\alpha| + \aleph_0$; if $\dim(q_i(x , a_i^0) , M_0) = 0$, and $\varkappa_i = |\alpha| + 1$ otherwise.

Thus we have proved :

THEOREM IV.10. - Let T be non-multidimensional ω-stable. Let $I(\aleph_\alpha , T)$ denote the number of models of T of power $\aleph_\alpha$ up to isomorphism. Then there is $\mu \leqslant \aleph_0$, where μ is called the number of dimensions of T , such that :

1° if $\mu = 1$, then $I(\aleph_\alpha , T) = 1$ for all $\alpha > 0$, and $I(\aleph_0 , T) = 1$ or $\aleph_0$.

2° If $\mu > 1$ but finite, then either $I(\aleph_\alpha , T) = |\alpha + \omega|$, for all $\alpha \geqslant 0$, or $I(\aleph_0 , T) = 1$ and there is G a group of permutations of μ such that for $\alpha > 0$ $I(\aleph_\alpha , T) = (|\alpha + 1|^{\mu})^{*}/G$, where $(|\alpha + 1|^{\mu})^{*}$ is the number of sequences of length μ of ordinals $\leqslant \alpha$ at least one of which is α , and $\rho_i ; i < \mu\rangle \sim \gamma_i ; i < \mu\rangle$ if, and only if, $\rho_{\sigma(i)} = \gamma_i$ for each $i < \mu$, for some $\sigma \in G$.

3° If $\mu = \aleph_0$, then $I(\aleph_\alpha ; T) = |\alpha + 1|^{\aleph_0}$, for all $\alpha > 0$ and $I(\aleph_0 , T) = 1$, $\aleph_0$ or $2^{\aleph_0}$.

A few final comments ; It can be shown fairly easily that if T is (ω-stable) and multidimensional, then for $\alpha > 0$, $I(\aleph_\alpha , T) \geqslant 2^{|\alpha|}$. Thus there is some content to the multidmensional/non-multidimensional dichotomy.

SHELAH has classified in a similar manner as above, the $F^a_{\aleph_0}$-saturated models of a superstable non-multidimensional theory.

The main result in this paper, and the main notions employed are due to S. SHELAH, "appearing" in [5]. The bulk of our section I parallels the development of the material in LASCAR [3] (sections 2 and 3). The important proposition III.5 is due to BOUSCAREN and LASCAR [1]. Some results on the spectrum were also obtained by LACHLAN [2].

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