Séminaire de théorie des nombres de Bordeaux

Stephen Wainger

Error Estimates in Renewal Theory

Séminaire de théorie des nombres de Bordeaux (1969-1970), exp. nº 3, p. 1-6
<http://www.numdam.org/item?id=STNB_1969-1970____A3_0>

ERROR ESTIMATES IN RENEWAL THEORY

by

Stephen WAINGER

-:-:-:-

The purpose of this lecture is to describe results of Charles Stone and myself on renewal Theory which appeared under the title "One sided error estimates in renewal theory" in the Journal d'Analyse, vol. 20, 1967.

I shall not assume the audience is familiar with probability theory, and hence I shall try to motivate the probabalistic definitions.

It is convenient to consider a person jumping along the real line from one integer point to another. We assume that this person jumps from one integer n to another integer $n+k$ with probability p_k. We assume he starts at 0 lands at some integer ℓ then jumps from ℓ etc and we shall be interested in what happens after large numbers of successive jumps. Since the p_k's are probabilities we assume

$$p_k \geq 0 ,$$

and

$$\sum_{k=-\infty}^{\infty} p_k = 1 .$$

We also assume

$$\sum_{k=-\infty}^{\infty} |k| \, p_k < \infty .$$

We define

$$\mu = \sum_{k=-\infty}^{\infty} k\, p_k ,$$

and we intuitively think of μ as the average jump. The intuitive reason for this definition of μ is as follows. Suppose we imagine N jumps being taken, with N large and $p_{-\ell} = 0$ for $\ell > 0$. Then there would be $\sim p_k N$ jumps of length k and hence the total length of the N jumps would be $\sim \Sigma\, k\, p_k N$. We would then find that the length of the average jum is

$$\sim \frac{1}{N} (\Sigma\, k\, p_k N) = \Sigma\, k\, p_k = \mu .$$

We a ssume throughout that $\mu > 0$. Then if the average jump is n units to the right, we would expect after a large number of jumps we would land in the half open interval $[k, k+1[$ about $1/\mu$ times. The renewal theorem asserts that the expected number of times we land in the interval $[k, k+1[$ (or equivalantly at k) (starting from 0) approaches $1/\mu$ as k approaches plus infinity, provided that the p_k are not supported on a proper subgroup of the integers. (Note that if for example $p_k = 0$ for k odd, there is zero probability of landing at an odd integer).

We now turn to a precise definition of "Expected number of visits to k". We let $P_m^{(n)}$ be the probability of going from the integer ℓ to $\ell + m$ in n jumps. (It will follow later that this is independant of ℓ). Then we define

$$\nu(m) = \text{Expected number of visits to } m \text{ starting from zero}$$

$$= \sum_{n=0}^{\infty} p_m^{(n)} .$$

That this is a reasonable definition may be seen by an argument analagous to the argument justifying the definition of μ.

We next wish to note that $p_m^{(n)}$ is the n-fold convolution of the original probabilities p_m. For simplicity assume $n = 2$. Then

Prob (jumbing from ℓ to $k+\ell$ in 2 steps)

= Prob (jumbing from ℓ to $\ell + m$ in one step) $\times$ Prob (jumping from $\ell + m$ to $k+\ell$ in one step)

$= p_m \, p_{k-m}$.

We are now in a position to give a precise mathematical formulation of the renewal theorem.

THEOREM. Let $p_k \geq 0$, $\sum_{k=-\infty}^{\infty} p_k = 1$, and $\sum_{k=-\infty}^{\infty} |k| \, p_k < \infty$. Assume $\mu = \sum_{k=-\infty}^{\infty} k \, p_k > 0$, and that the p_k's are not supported on a proper subgroup of the integers. Define

$$\nu(k) = \sum_{n=0}^{\infty} p_k^{(n)} ,$$

where $p_k^{(n)}$ is the n-fold convolution of the p_k with it self. Then

$$\lim_{k \to +\infty} \nu(k) = \frac{1}{\mu} .$$

After the proof of the renewal theorem, people proved various theorems showing that if one assumes

$$\sum_{-\infty}^{\infty} |k|^{1+\delta} p_k < \infty ,$$

one can make a conclusion about the rate at which $\nu(k)$ approaches $1/\mu$. Stone and I obtained two types of results along these lines, which I now describe.

For a large class of functions $M(k)$ increasing to infinity (but slower than exponentially)

I) $$\sum_{k=0}^{\infty} p_k \, k \, M(k) < \infty$$

implies $\left(\nu(k) - \frac{1}{\mu} \right) M(k) \to 0$.

II) With some aditional hypothesis on M

$$\nu(k) - \frac{1}{\mu} = r(k) + e(k)$$

where $r(k)$ can be expressed easily in terms of the p_k, and $e(k)$ tends to zero essentially as fast as the p_k.

Our hypothesis on $M(k)$ allow $M(k)$ to be of the form k^{α} $\alpha > 0$ or k^{α} $\exp(k^{\beta})$ $0 < \beta < 1$. (The theoreme would be false for for $M(k) = \exp(\lambda_k)$).

For precise statements we refer you to our paper.

There are 3 interesting features of our results :

1) The generality of $M(k)$,

2) The extra assumptions on p_k to make $\nu_k - \frac{1}{\mu}$ tend to zero quickly are only needed for $k > 0$.

3) The conclusions are essentially best possible. That is a conclusion $e_k = 0(p(k))$ would be false.

We now give an idea of the proof. Let

$$f^{(n)}(\theta) = \sum_{k=-\infty}^{\infty} p_k^{(n)} e^{ik\theta} .$$

Then since $p_k^{(n)}$ is the n-fold convolution of p_k with itself we have

$$f^{(n)}(\theta) = [f(\theta)]^n .$$

Then

$$\nu(k) = \lim_{r \nearrow 1} \sum_{n=0}^{\infty} r^n p_k^{(n)}$$

$$= \frac{1}{2\pi} \lim_{r \to 1} \int_{-\pi}^{\pi} e^{-ik\theta} \sum_{n=0}^{\infty} r^n f^{n}(\theta)\, d\theta$$

$$= \frac{1}{2\pi} \lim_{r \to 1} \int_{-\pi}^{\pi} e^{-ik\theta} \frac{1}{1 - r f(\theta)} d\theta .$$

Now since $f(0) = 1$, $(1-f(\theta))^{-1}$ has a bad singularity at $\theta = 0$, but only at $\theta = 0$ since the p_k are not supported on a proper subgroup of the integers. In particular one may not take the limit as $r \to 1$ under the above integral sign. As $f'(0) = i\mu$, one might expect

$$\frac{1}{1-r\,f(\theta)} - \frac{1}{\mu(1-r\,e^{i\theta})}$$

to be a little nicer than $\frac{1}{1-r\,f(\theta)}$. In fact, with a little care one can show

$$\lim_{r\to 1} \int_{-\pi}^{\pi} e^{-ik\theta} \left\{\frac{1}{1-r\,f(\theta)} - \frac{1}{\mu(1-r\,e^{i\theta})}\right\} d\theta$$

$$= \int_{-\pi}^{\pi} e^{-ik\theta} \left\{\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right\} d\theta .$$

As

$$\lim_{r\to 1} \int_{-\pi}^{\pi} e^{-ik\theta} \frac{1}{\mu(1-r\,e^{i\theta})} d\theta = \frac{1}{\mu} ,$$

we have

$$\nu(k) - \frac{1}{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} \left\{\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right\} d\theta .$$

Now the main idea of the proof is the following.

We would like to think of the integral above as an integral in the complex plane and move the contour of integration into the lower half plane. Note that if $\text{Im}(\theta) = -\tau$ with $\tau > 0$;

$$|e^{ik\theta}| = e^{-k\tau} .$$

We are of course prevented from moving the contour into the lower half plane because $f(\theta)$ is not an analytic function.

Hence, we approximate $f(\theta)$ by

$$f_k(\theta) = \sum_{-\infty}^{k} p_j\, e^{ij\theta} .$$

We then write

$$\nu(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\{[\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}] - [\frac{1}{1-f_k(\theta)} - \frac{1}{\mu_k(1-e^{i\theta})}]\}\,d\theta$$

$$(\mu_k = \sum_{-\infty}^{k} j\,p_j)$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}[\frac{1}{1-f_k(\theta)} - \frac{1}{\mu_k(1-e^{i\theta})}]d\theta = I_k + II_k\,.$$

We may the try to estimate I_k and in II_k move the contour of integration into the half plane $Im(\theta) < 0$, though we must be careful with the zeros of $1-f_k(\theta)$. (In fact the distance we may move our contour into the lower half plane then depends on k).

To obtain the results of our theorems we must make a number of complicated technical modifications of the above idea.

-:-:-:-

REFERENCE

STONE and WAINGER, One-sided error estimates in renewal theory, Journal d'Analyse mathematique, (Jerusalem), 1967, 20, p. 325-352.

-:-:-:-

Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53 706 U. S. A.