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ON A TRIPLET OF EXPONENTIAL BROWNIAN FUNCTIONALS

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Abstract. We study the three-dimensional joint distribution of a Brownian motion and the integrals of its exponential and its exponential squared from some points of view. We show an explicit expression of the Laplace transform of the distribution, which gives an extension of Yor's result on the two-dimensional one. We apply the result to some problems, in particular, to the calculation of an explicit form of the heat kernel of the semigroup generated by the Maass Laplacian on the Poincaré upper half plane.

1. Introduction

Let $B = \{B_t, t \geqq 0\}$ be a one-dimensional standard Brownian motion starting from 0. In this paper we are concerned with the Brownian functionals of exponential type a_t and A_t defined by

$$(1.1) \qquad a_t = \int_0^t \exp(B_s)\, ds \qquad \text{and} \qquad A_t = \int_0^t \exp(2B_s)\, ds,$$

respectively. These functionals have recently been studied extensively by many authors (see, e.g., [1], [3], [11], [15], [22] and [24]) in relation to mathematical finance, Brownian motions on hyperbolic spaces, some disordered systems, generalized Bessel processes and so on. In particular, the explicit form of the two-dimensional joint distribution of (A_t, B_t) is known by Yor [22] (see (2.3) in the next section) and it plays important roles in those domains.

The purpose of this paper is to discuss the three-dimensional joint distribution of (A_t, a_t, B_t) and to show some applications; in particular, we will show an explicit formula, which is of simpler form than Fay's original expression ([6]), for the heat kernel of the semigroup generated by the Schrödinger operator H_k with constant magnetic field or the Maass Laplacian on the Poincaré upper half plane $\mathbf{H}^2$.

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We will show explicit expressions of the Laplace transform

$$(1.2)\qquad E[\exp\left(-\frac{1}{2}\lambda^2 A_t\right); a_t \in dv, B_t \in dy], \qquad t>0, \lambda>0, v>0, y\in\mathbf{R},$$

and, equivalently,

$$(1.3)\qquad E[\exp\left(\lambda k a_t - \frac{1}{2}\lambda^2 A_t\right)|B_t = y], \qquad k\in\mathbf{R},$$

by means of the density function of the Hartman–Watson law (cf. Yor [21]). By the Feynman–Kac formula, the latter gives the heat kernel corresponding to the Schrödinger operator $H^M_{\lambda,k}$ with the Morse potential $V^M_{\lambda,k}$ on $\mathbf{R}$ defined by

$$H^M_{\lambda,k} = -\frac{1}{2}\frac{d^2}{dx^2} + V^M_{\lambda,k}, \qquad V^M_{\lambda,k}(x) = \frac{1}{2}\lambda^2 e^{2x} - \lambda k e^x.$$

In [11], the close relation between H_k and $H^M_{\lambda,k}$ has been shown and, in particular, it has been shown that $H^M_{\lambda,k}$ is obtained by a separation of variables from the operator H_k on $\mathbf{H}^2$.

The three-dimensional distribution has been considered by Leblanc [13] for the study of some models, in particular of a stochastic volatility model introduced by Hull and White [10], in mathematical finance and by Ikeda–Matsumoto [11] for the study of H_k and $H^M_{\lambda,k}$ mentioned above. They have considered the Laplace transform (in time t) of the distribution or the Green functions for $H^M_{\lambda,k}$ by several methods. Alili [1] has also studied it and has given an alternative decomposition via an exponential formula of the excursion theory.

We will show a closed form for (1.2) by reducing the computations on the three-dimensional distribution to those on the two-dimensional one and by using Yor's result. We need not consider the Laplace transform in time or the Green functions for $H^M_{\lambda,k}$. However, it should be noted that, starting from the explicit expressions of the Green functions for $H^M_{\lambda,k}$ which are obtained by the general theory of the Sturm–Liouville operators (see [11] and also Section 4) and using the integral representation of a product of the Whittaker functions, we can also show our result for (1.2) by analytic methods on the basis of (1.3). Furthermore we are able to carry out the explicit calculations for the Laplace transform by using time change argument instead of Yor's result and to see the correspondence between the probabilistic and analytic methods. Thus we approach our problem from several points of view and, in the course of study, we obtain other proofs of Yor's fundamental result.

We end this Introduction by mentioning the organization of this paper. An explicit expression for (1.2) will be given in the next Section 2 and the proof will be given in Section 3. In Section 4 we will obtain a closed form of the heat kernel for $H^M_{\lambda,k}$ by using the result in Section 2. An important feature is that we can trace our way back, as we shall see in Sections 3 and 4. We will show a closed form of the heat kernel for H_k in Section 5. Other applications of our results, to the studies on the distributions of a hyperbolic drifted diffusion process which appears in relation to generalized Bougerol identity and of the asset process in the Hull–White model mentioned above, will be given in Section 6.

2. Main Result

We first recall the Hartman–Watson distribution. For details, see Yor [21]. For $r > 0$, the probability distribution $\eta_r(du)$ on $[0,\infty)$ characterized by

$$\int_0^\infty \exp(-\frac{1}{2}\nu^2 t)\,\eta_r(dt) = \frac{I_\nu(r)}{I_0(r)}, \qquad \nu > 0 \tag{2.1}$$

is called the Hartman-Watson distribution, where I_ν is the usual modified Bessel function. The probability measure $\eta_r(dt)$ has the density with respect to the Lebesgue measure given by

$$\eta_r(dt) = \frac{\theta_r(t)}{I_0(r)}\,dt,$$

where

$$\theta_r(t) = \frac{r}{(2\pi^3 t)^{1/2}}\,e^{\pi^2/2t}\int_0^\infty e^{-\xi^2/2t}e^{-r\cosh(\xi)}\sinh(\xi)\sin(\pi\xi/t)\,d\xi. \tag{2.2}$$

For the distribution of the exponential functional A_t given in the Introduction, Yor [22] has shown

$$P(A_t \in dv, B_t \in dx) = \frac{1}{v}\exp(-\frac{1+e^{2x}}{2v})\theta_{e^x/v}(t)\,dvdx, \qquad v > 0, x \in \mathbf{R}, \tag{2.3}$$

which is also proved by modifying the arguments in Section 4. By the self-similarity of Brownian motion, we may rewrite (2.3) in the following form which is more convenient for our purpose:

$$P(a_t \in dv, B_t \in dx) = \psi_t(v,x)\,dvdx, \tag{2.4}$$

where

$$\psi_t(v,x) = \frac{1}{2v}\exp(-\frac{2(1+e^x)}{v})\,\theta_{4e^{x/2}/v}(t/4).$$

We set

$$\phi \equiv \phi(v,x;\lambda) = \frac{2\lambda \exp(x/2)}{\sinh(\lambda v/2)}, \quad \lambda > 0, v > 0, x \in \mathbf{R}. \tag{2.5}$$

Then we show the following:

Theorem 2.1. *For any $t > 0, \lambda > 0, v > 0, x \in \mathbf{R}$, it holds that*

$$\begin{aligned} &E[\exp(-\frac{1}{2}\lambda^2 A_t)|a_t = v, B_t = x]\, \psi_t(v,x) \\ &= \frac{\lambda}{4\sinh(\lambda v/2)} \exp\left(-\lambda(1+e^x)\coth(\lambda v/2)\right) \theta_\phi(t/4). \end{aligned} \tag{2.6}$$

3. Proof of Theorem 2.1

In order to prove Theorem 2.1, we consider

$$I_t(\gamma,\beta,\lambda) \equiv E[\exp\left(\gamma B_t - \beta a_t - \frac{1}{2}\lambda^2 A_t\right)], \qquad \gamma,\beta \in \mathbf{R}, \lambda > 0,$$

and show that we can reduce the calculation for this Laplace transform of the three-dimensional distribution to some formula which are calculated by using Yor's result (2.3) on the two-dimensional distribution. For this purpose we prepare two lemmas.

Lemma 3.1. *For any locally bounded Borel function $\varphi : [0,\infty) \to \mathbf{R}$ and $\lambda > 0$, the solution of the stochastic differential equation*

$$dX_t = dB_t - \lambda \exp(X_t + \varphi(t))\, dt$$

is explicitly given by

$$X_t = X_0 + B_t - \log\left(1 + \lambda \int_0^t \exp(X_0 + B_s + \varphi(s))\, ds\right).$$

Proof. We can prove the lemma by using Itô's formula, but we give another proof based on the ordinary differential equation method. In fact, only the continuity of Brownian motion plays a role here. We set $Y_t = X_t - X_0 - B_t$. Then Y_t satisfies

$$dY_t = -\lambda \exp(\beta_t + Y_t)\, dt,$$

where $\beta_t = X_0 + B_t + \varphi(t)$. Hence the process $\{Y_t, t \geqq 0\}$ is absolutely continuous in t and solves the equation

$$\exp(-Y_t) dY_t = -\lambda \exp(\beta_t)\, dt.$$

Therefore, since $B_0 = 0$, we obtain

$$1 - \exp(-Y_t) = -\lambda \int_0^t \exp(\beta_s)\, ds.$$

The rest of the proof is easy. $\square$

Lemma 3.2. *Let $\{X_t, t \geqq 0\}$ be a one-dimensional Brownian motion starting from 0. Then, if $\lambda > 0$, the stochastic process $\Lambda^\lambda = \{\Lambda_t^\lambda, t \geqq 0\}$ defined by*

$$\Lambda_t^\lambda = \exp\left(-\lambda \int_0^t \exp(X_s)\, dX_s - \frac{1}{2}\lambda^2 \int_0^t \exp(2X_s)\, ds\right) \tag{3.1}$$

is a (true) martingale with respect to the canonical filtration.

Proof. The local martingale property of Λ^λ is easily seen. Moreover, Itô's formula yields

$$\begin{aligned} &-\lambda \int_0^t \exp(X_s)\, dX_s - \frac{1}{2}\lambda^2 \int_0^t \exp(2X_s)\, ds \\ =& -\lambda \left(\exp(X_t) - 1 - \frac{1}{2}\int_0^t \exp(X_s)\, ds\right) - \frac{1}{2}\lambda^2 \int_0^t \exp(2X_s)\, ds \\ =& -\frac{1}{2}\lambda^2 \int_0^t \left(\exp(X_s) - \frac{1}{2\lambda}\right)^2 ds + \frac{1}{8}t + \lambda - \lambda \exp(X_t) \end{aligned}$$

and

$$\Lambda_t^\lambda \leqq \exp(\lambda + t/8 - \lambda \exp(X_t)) \leqq \exp(\lambda + t/8),$$

which shows that Λ^λ is bounded on any bounded time interval. $\square$

Remark 3.1. By using the result mentioned in McKean [14], Section 3.7, the martingale property of Λ^λ corresponds to the conservativeness of the diffusion process given as the unique solution of the stochastic differential equation $dX_t = dB_t - \lambda \exp(X_t)dt$ considered in Lemma 3.1. This gives another proof of Lemma 3.1 in this case and shows that the local martingale Λ^λ is not a martingale if $\lambda < 0$. For such "strict" local martingales, see [5], [16] and the references cited therein.

Now let $X = \{X_t, t \geqq 0\}$ be a one-dimensional Brownian motion starting from 0 defined on a probability space $(\Omega, \mathcal{F}, \tilde{P})$ and set $\mathcal{F}_t = \sigma\{X_s, s \leqq t\}$. Moreover, letting Λ_t^λ be the martingale given by (3.1), we define another probability measure P by

$$dP|_{\mathcal{F}_t} = \Lambda_t^\lambda \cdot d\tilde{P}|_{\mathcal{F}_t}.$$

Then Lemma 3.2 implies that the stochastic process $B = \{B_t, t \geqq 0\}$ given by

$$B_t = X_t + \lambda \int_0^t \exp(X_s)\, ds$$

is a Brownian motion under P by the Girsanov–Maruyama theorem.

For a non-negative Borel function g on $\mathbf{R}$, we set

$$J_t = E^P[g(B_t) \exp\left(-\lambda \exp(B_t) - \beta a_t - \frac{1}{2}\lambda^2 A_t\right)].$$

By Itô's formula, it is easy to show

$$J_t = e^{-\lambda} E^P[g(B_t) \exp\left(-\lambda \int_0^t \exp(B_s)\, dB_s - (\beta + \frac{\lambda}{2})\, a_t - \frac{1}{2}\lambda^2 A_t\right)].$$

Then we obtain

$$\begin{aligned} J_t &= e^{-\lambda} E^P[g(X_t) \exp\left(-(\beta + \frac{\lambda}{2}) \int_0^t \exp(X_s)\, ds\right)] \\ &= e^{-\lambda} E^P[(1 + 4\lambda A_{t/4})^{-(1/2+\beta/\lambda)} g(2B_{t/4} - \log(1 + 4\lambda A_{t/4}))], \end{aligned}$$

where we have used Lemma 3.1 and the self-similarity of Brownian motion for the second line.

Finally, setting

$$g(x) = \exp(\gamma x + \lambda \exp(x)),$$

we have proved the following:

Proposition 3.3. *For a Brownian motion $B = \{B_t, t \geqq 0\}$ and for any $t > 0, \gamma, \beta \in \mathbf{R}, \lambda > 0$, it holds that*

$$I_t(\gamma, \beta, \lambda) = e^{-\lambda} E[(1 + 4\lambda A_{t/4})^{-\gamma - 1/2 - \beta/\lambda} \exp\left(2\gamma B_{t/4} + \frac{\lambda \exp(2B_{t/4})}{1 + 4\lambda A_{t/4}}\right)]. \tag{3.2}$$

Now the proof of Theorem 2.1 is easy. We recall Yor's result (2.3). Then we have

$$\begin{aligned} I_t(\gamma, \beta, \lambda) = e^{-\lambda} \int_{\mathbf{R}} dy \int_0^\infty du\ (1 + 4\lambda u)^{-\gamma - 1/2 - \beta/\lambda} \exp\left(2\gamma y + \frac{\lambda e^{2y}}{1 + 4\lambda u}\right) \\ \times \frac{1}{u} \exp\left(-\frac{1 + e^{2y}}{2u}\right) \theta_{e^y/u}(t/4). \end{aligned}$$

Changing the variables (y, u) into (x, v) by

$$y = \frac{\lambda v + x}{2} \qquad \text{and} \qquad u = \frac{\exp(\lambda v) - 1}{4\lambda},$$

we obtain

$$I_t(\gamma,\beta,\lambda) = \int_{\mathbf{R}} e^{\gamma x}dx \int_0^\infty e^{-\beta v}dv \, \frac{\lambda}{4\sinh(\lambda v/2)} \times \exp(-\lambda(1+e^x)\coth(\lambda v/2))\theta_\phi(t/4)$$

and, consequently, (2.6) by the uniqueness of the Laplace transform.

In order to show the correspondence to an analytic proof of (2.6) given in the next section, we give another probabilistic proof by calculating the Laplace transform in time t of both hand sides of (3.2) instead of using Yor's result (2.3). The assertion of the following proposition is equivalent to that of Theorem 2.1 by virtue of the uniqueness of Laplace transforms.

We set

$$G_\nu(\gamma,\beta,\lambda) = \int_0^\infty \exp(-\frac{1}{2}\nu^2 t) I_t(\gamma,\beta,\lambda)\, dt, \qquad \nu > 0,$$

and

$$\Psi_\nu(y,v;\lambda) = \frac{\lambda}{\sinh(\lambda v/2)} \exp(-\lambda(1+e^y)\coth(\lambda v/2)) I_{2\nu}(\phi),$$

where ϕ is given by (2.5).

Proposition 3.4. *For any $\nu > 0$, it holds that*

$$G_\nu(\gamma,\beta,\lambda) = \int_{\mathbf{R}} e^{\gamma y}\, dy \int_0^\infty e^{-\beta v}\, \Psi_\nu(y,v;\lambda)\, dv. \tag{3.3}$$

Proof. We first recall the Lamperti relation (see [20], p.452): there exists a two-dimensional Bessel process $R = \{R_t, t \geqq 0\}$ starting from 1 such that

$$\exp(B_t) = R_{A_t}, \qquad t \geqq 0.$$

Then, by (3.2), we obtain

$$\begin{aligned} G_\nu(\gamma,\beta,\lambda) &= 4e^{-\lambda} \int_0^\infty \exp(-2\nu^2 t)\, dt \\ &\qquad \times E[(1+4\lambda A_t)^{-\gamma-1/2-\beta/\lambda} \exp\left(2\gamma B_t + \frac{\lambda \exp(2B_t)}{1+4\lambda A_t}\right)] \\ &= 4e^{-\lambda} \int_0^\infty (1+4\lambda s)^{-\gamma-1/2-\beta/\lambda}\, ds \\ &\qquad \times E[(R_s)^{2\gamma-2} \exp\left(-2\nu^2 C_s + \frac{\lambda (R_s)^2}{1+4\lambda s}\right)], \end{aligned}$$

where C_s is the inverse function of $s = A_t$ given by

$$C_s = \int_0^s \frac{d\tau}{(R_\tau)^2}.$$

Note that there exists a complex Brownian motion $Z = \{Z_t = Z_t^{(1)} + \sqrt{-1}Z_t^{(2)}, t \geqq 0\}$ with $Z_0 = 1$ and a continuous process $\Phi = \{\Phi_t, t \geqq 0\}$ with $\Phi_0 = 0$ such that

$$Z_t = R_t \exp(\sqrt{-1}\Phi_t), \qquad t \geqq 0.$$

Φ is the total winding of Z about 0. Moreover we recall the following formulae (cf. Pitman–Yor, [18], [19]): for $r, s, \alpha > 0$,

$$\begin{aligned} E[\exp(\sqrt{-1}\alpha\Phi_s); R_s \in dr] &= E[\exp(-\frac{1}{2}\alpha^2 C_s); R_s \in dr] \\ &= \exp(-\frac{1+r^2}{2s}) I_\alpha(\frac{r}{s}) \frac{rdr}{s}. \end{aligned}$$

Then we obtain

$$\begin{aligned} G_\nu(\gamma, \beta, \lambda) = 4e^{-\lambda} \int_0^\infty & (1+4\lambda s)^{-\gamma-1/2-\beta/\lambda} \frac{ds}{s} \\ & \times \int_0^\infty r^{2\gamma-1} \exp\left(\frac{\lambda r^2}{1+4\lambda s} - \frac{1+r^2}{2s} \right) I_{2\nu}(\frac{r}{s})\, dr. \end{aligned}$$

Finally, changing the variables from (s, r) into (y, v) by

$$r = (1+4\lambda s)^{1/2} e^{y/2} \qquad \text{and} \qquad s = \frac{\exp(\lambda v) - 1}{4\lambda},$$

we obtain (3.3). $\square$

4. Schrödinger Operators with Morse Potentials

For $\lambda > 0$ and $k \in \mathbf{R}$, we consider the Schrödinger operator $H^M_{\lambda,k}$ on $L^2(\mathbf{R})$ with the Morse potential $V^M_{\lambda,k}$ given by

$$H^M_{\lambda,k} = -\frac{1}{2}\frac{d^2}{dx^2} + V^M_{\lambda,k}, \qquad V^M_{\lambda,k}(x) = \frac{1}{2}\lambda^2 e^{2x} - \lambda k e^x.$$

For the motivation about the study of this operator and its close relation to the Maass Laplacians on the Poincaré upper half plane which we will study in the next section, see [11] and the references cited therein.

In this section we first show, by using our result (2.6), an integral representation (4.2) below of the heat kernel $q^M_{\lambda,k}(t, x, y)$, $t > 0, x, y \in \mathbf{R}$, with respect to the Lebesgue measure of the semigroup generated by $H^M_{\lambda,k}$. Next, after showing that (4.2) is also obtained by using (2.1) and some results in [11], we give another analytic proof of (2.6).

By the Feynman–Kac formula, we have

(4.1)

$$q^M_{\lambda,k}(t,x,y) = E[\exp\left(-\frac{1}{2}\lambda^2 e^{2x}A_t + \lambda k e^x a_t\right) | B_t = y-x] \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|y-x|^2}{2t}\right).$$

Using (2.6), we easily obtain the following proposition. It should be remarked that the right hand side of (4.1) may be regarded as a (double) Laplace transform of the joint distribution of (A_t, a_t, B_t) and (4.2) below is an equivalent assertion to (2.6).

Proposition 4.1. *Let $\lambda > 0$ and $k \in \mathbf{R}$. Then, for any $t > 0$, it holds that*

$$q^M_{\lambda,k}(t,x,y) = \int_0^\infty e^{2ku} \frac{1}{2\sinh(u)} \exp(-\lambda(e^x+e^y)\coth(u))\, \theta_{\bar\phi}(t/4)\, du, \tag{4.2}$$

where θ_r is the function defined by (2.2) and

$$\bar\phi = \frac{2\lambda\exp((x+y)/2)}{\sinh(u)}.$$

Remark 4.1. It is not difficult to show that $\theta_r(t) = O(r^N)$ holds as $r \downarrow 0$ for any $N > 0$ and, as a consequence, that the integral on the right hand side of (4.2) is convergent for any $\lambda > 0$ and $k \in \mathbf{R}$.

Remark 4.2. It is easy to show from (4.2) that

$$q^M_{\lambda,k}(t,x,y) = \int_0^\infty e^{-2ku} \frac{1}{2\sinh(u)} \exp(-|\lambda|(e^x+e^y)\coth(u))\, \theta_{\bar\phi'}(t/4)\, du, \tag{4.3}$$

holds with $\bar\phi' = 2|\lambda|\exp((x+y)/2)/\sinh(u)$ when $\lambda < 0$.

Next, starting from an explicit formula for the Green function $G^M_{\lambda,k}$ for $H^M_{\lambda,k}$ which has been obtained in [11] in two ways, by using the general theory of the Sturm–Liouville operators and by using the theory of the Bessel diffusions, we show (4.2) and then (2.6) conversely.

By Proposition 4.1 in [11], we have for $y > x$ and $\alpha > 0$

$$\begin{aligned} G^M_{\lambda,k}(x,y;\alpha^2/2) &\equiv \int_0^\infty \exp(-\frac{1}{2}\alpha^2 t) q^M_{\lambda,k}(t,x,y)\, dt \\ &= \frac{\Gamma(\alpha-k+1/2)}{\lambda\Gamma(1+2\alpha)} e^{-(x+y)/2} W_{k,\alpha}(2\lambda e^y) M_{k,\alpha}(2\lambda e^x), \end{aligned} \tag{4.4}$$

where it should be assumed that $\alpha > k - 1/2$ when $k > 0$. $W_{k,\alpha}$ and $M_{k,\alpha}$ are the Whittaker functions. Moreover, by the integral representation of the product of the

Whittaker functions (cf. [7], p.729), we also have

(4.5)

$$G^M_{\lambda,k}(x,y;\alpha^2/2) = 2\int_0^\infty \frac{\exp(2ku)}{\sinh(u)} \exp(-\lambda(e^x+e^y)\coth(u)) I_{2\alpha}\left(\frac{2\lambda e^{(x+y)/2}}{\sinh(u)}\right) du.$$

Now, recalling the characterization (2.1) of the Hartman–Watson distribution, we obtain (4.2).

Remark 4.3. In [11], the reference measure has not been mentioned and it has caused the difference of a constant factor "2" in the expressions of the Green functions. Formulae (4.4) and (4.5) give them with respect to the Lebesgue measure. Moreover the condition for α has been dropped in Proposition 4.1 in [11]. (4.4) and (4.5) do not hold for all α in general when $k > 0$ because $H^M_{\lambda,k}$ may have negative eigenvalues which are obtained as poles of the Gamma function on the right hand side of (4.4).

Our result (2.6) is obtained easily from (4.2). Setting $x = 0$ in (4.1) and (4.2), we have

$$\begin{aligned}
&\int_0^\infty e^{\lambda k u} E[\exp(-\frac{1}{2}\lambda^2 A_t)|a_t = u, B_t = y]\ \psi_t(u,y)\, du \\
&= \frac{2}{\lambda}\int_0^\infty e^{2ku} E[\exp(-\frac{1}{2}\lambda^2 A_t)|a_t = 2u/\lambda, B_t = y]\ \psi(2u/\lambda, y)\, du \\
&= \int_0^\infty \frac{\exp(2ku)}{2\sinh(u)} \exp(-\lambda(1+e^y)\coth(u))\ \theta_{\phi_1}(t/4)\, du
\end{aligned}$$

for all $k \in \mathbf{R}$, where

$$\phi_1 = \frac{2\lambda\exp(y/2)}{\sinh(u)}.$$

Therefore the uniqueness of the Laplace transform implies (2.6).

Remark 4.4. Since we have shown (2.6) without using Yor's result (2.3) or (2.4), we obtain another proof of it by letting λ tend to zero in (2.6). Moreover, in the case where $k = 0$, it is easy to show (cf. [11]) that

$$G^M_{\lambda,0}(x,y;\alpha^2/2) = 2I_\alpha(\lambda e^x)K_\alpha(\lambda e^y) \tag{4.6}$$

holds for $\alpha > 0$, $\lambda > 0$ and $x < y$. (4.6) is also obtained from (4.4) if we recall

$$M_{0,\alpha}(z) = 2^{2\alpha}\Gamma(\alpha+1)\sqrt{z} I_\alpha(z/2) \qquad \text{and} \qquad W_{0,\alpha}(z) = \sqrt{z/\pi} K_\alpha(z/2).$$

By using the integral representation

$$I_\alpha(a)K_\alpha(b) = \frac{1}{2}\int_0^\infty \exp(-\frac{1}{2}u - \frac{a^2+b^2}{2u}) I_\alpha(\frac{ab}{u})\, \frac{du}{u}, \qquad 0 < a < b,$$

of the product of the modified Bessel functions (cf. [7], p.725) and (2.1), we can show Yor's result (2.3) from (4.6) by the explicit inversion of the Laplace transform. It should be mentioned that Yor's result (2.3) is closely related to the study of the winding number of the two-dimensional Brownian motion. For details, see, e.g., Ito–McKean [12] Chapter 7, Pitman–Yor [19], Yor [21].

5. Maass Laplacian

In this section, by using our result (4.2) in the framework developed in [11], we give an explicit expression of the heat kernels of the semigroups generated by the Schrödinger operators with constant magnetic fields or the Maass Laplacians on the Poincaré upper half plane. Our result gives a simpler expression than Fay's original one (see [6],[11] and also [17]).

We begin with recalling some results in [11]. Letting $\mathbf{H}^2$ be the upper half plane with rectangular coordinates (x, y),

$$\mathbf{H}^2 = \{(x,y); x \in \mathbf{R}, y > 0\},$$

with the usual Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$, we consider the Schrödinger operator with magnetic field $H_k, k \in \mathbf{R}$, defined by

$$H_k = \frac{1}{2}y^2\left(\sqrt{-1}\frac{\partial}{\partial x} + \frac{k}{y}\right)^2 - \frac{1}{2}y^2\frac{\partial^2}{\partial y^2}.$$

A trivial modification $-2H_k + k^2$ gives the Maass Laplacian which plays important roles in several fields of mathematics. For details, see [11] and the references cited therein.

Let $q_k(t, z_1, z_2), t > 0, z_1, z_2 \in \mathbf{H}^2$, be the heat kernel with respect to the Riemannian volume $y^{-2}dxdy$ of the semigroup generated by H_k. Then it is known that there exists a function $g_t(\cdot)$ on $[0, \infty)$ such that

$$q_k(t, z_1, z_2) = \left(\frac{z_2 - \bar{z}_1}{z_1 - \bar{z}_2}\right)^k g_t(d(z_1, z_2)), \tag{5.1}$$

where $z_i = (x_i, y_i) \in \mathbf{H}^2$ is identified with $z_i = x_i + \sqrt{-1}y_i \in \mathbf{C}$ as usual, $d(z_1, z_2)$ is the hyperbolic distance between z_1 and z_2 and, for $w = |w|\exp(\sqrt{-1}\theta) \in \mathbf{C}$ with $-\pi < \theta \leqq \pi$, $w^k = |w|^k \exp(\sqrt{-1}k\theta)$.

Moreover it has been shown in [11] that $q_k(t, z_1, z_2)$ and $q^M_{\lambda,k}(t, \xi, \eta)$, studied in the previous section, are related through a one-dimensional Fourier transform in the following way:

$$q_k(t, z_1, z_2) = e^{-t/8-k^2t/2}\frac{\sqrt{y_1y_2}}{2\pi}\int_{-\infty}^{\infty} e^{-\sqrt{-1}(x_2-x_1)\lambda}\, q^M_{\lambda,k}(t, \log y_1, \log y_2)\, d\lambda. \tag{5.2}$$

With the help of harmonic analysis on $\mathbf{H}^2$, Fay [6] has shown an explicit form of the Green function for H_k, its spectral decomposition and, as a consequence, an explicit expression for $q_k(t, z_1, z_2)$. The Green function is also obtained by taking the Laplace transform of both hand sides of (5.2) and by using (4.4) or (4.5).

The following gives another simpler expression for $q_k(t, z_1, z_2)$, which, in fact, coincides with the expression of Fay, and completes the story developed in [11] about the Selberg trace formula on $\mathbf{H}^2$ in the framework of stochastic analysis .

Theorem 5.1. *Let $\varphi(b, r), 0 \leqq r \leqq b$, be a function defined by*

$$\varphi(b, r) = \cosh^{-1}\left(\frac{\cosh(b/2)}{\cosh(r/2)}\right).$$

Then the function g_t on the right hand side of (5.1) *is given by*

$$g_t(r) = \frac{\sqrt{2}\exp(-t/8 - k^2t/2)}{(2\pi t)^{3/2}}\int_r^{\infty} \frac{\cosh(2k\varphi(b, r))b\exp(-b^2/2t)}{(\cosh b - \cosh r)^{1/2}}\, db. \tag{5.3}$$

Before proceeding to the proof, we give some remarks on (5.3). By setting $k = 0$, we immediately obtain the well known formula for the heat kernel p^2 for H_0, the half of the Laplacian on $\mathbf{H}^2$. We refer to Davies [4] for the heat kernels of the semigroups generated by the Laplacians on the real hyperbolic spaces. Moreover, setting $r = 0$, we also easily obtain

$$q_k(t, z, z) = \frac{\exp(-t/8 - k^2t/2)}{(2\pi t)^{3/2}}\int_0^{\infty} \frac{\cosh(kb)b\exp(-b^2/2t)}{\sinh(b/2)}\, db,$$

which coincides with (3.5) in [11]. In general, we can show that our expression (5.3) for g_t coincides with that of Fay [6] (see [11] for details including some comments on Fay's original result). We omit the detailed proof since it only needs easy but lengthy calculations.

In the proof of (5.3) below, we shall use Gruet's calculation ([8]) of the heat kernel for the Laplacian on the three-dimensional hyperbolic space, which is available without any change. We recall his result for the reader's convenience. See also Gruet [9] for more general result.

Theorem 5.2 (Gruet). *Let $\Delta_{\mathbf{H}^n}$ be the Laplacian on the hyperbolic space $\mathbf{H}^n = \{z = (x,y); x \in \mathbf{R}^{n-1}, y > 0\}, d \geqq 2$, endowed with the Riemannian metric $y^{-2}(\sum_{i=1}^{n-1} dx_i^2 + dy^2)$. Then the heat kernel $p^n(t,z_1,z_2)$ for $\Delta_{\mathbf{H}^n}/2$ with respect to the Riemannian volume is given by*

$$
\begin{aligned}
p^n(t,z_1,z_2) = & \frac{\exp(-(n-1)^2t/8)}{\pi(2\pi)^{n/2}t^{1/2}}\Gamma\left(\frac{n+1}{2}\right) \\
& \times \int_0^\infty \frac{e^{(\pi^2-b^2)/2t}\sinh(b)\sin(\pi b/t)}{[\cosh(b)+\cosh(d_n(z_1,z_2))]^{(n+1)/2}}\,db,
\end{aligned} \tag{5.4}
$$

where $d_n(z_1,z_2)$ is the hyperbolic distance between z_1 and z_2 in $\mathbf{H}^n$.

Gruet has also shown that, from his expression (5.4), we can derive the classical formulae for $p^2 = q_0$ mentioned above and p^3, for which we know ([4])

$$
p^3(t,z_1,z_2) = \frac{e^{-t/2}}{(2\pi t)^{3/2}}\frac{d_3(z_1,z_2)}{\sinh(d_3(z_1,z_2))}\exp(-d_3(z_1,z_2)^2/2t). \tag{5.5}
$$

In particular, the right hand side of (5.5) is obtained from (5.4) by a simple residue calculus. Millson's formulae are also shown from (5.4) and the explicit forms of $p^n, n = 3,4,...$, are obtained inductively. It should be mentioned that Gruet's proof of (5.4) heavily depends on Yor's result (2.3).

Now we give a proof of (5.3).

Proof of Theorem 5.1. First of all we note that, by (5.1) and (5.2), we need only consider the case where $x_2 = x_1$. In this case the hyperbolic distance $r = d(z_1,z_2)$ is given by

$$
\cosh(d(z_1,z_2)) = \frac{y_1^2+y_2^2}{2y_1y_2}
$$

and we have

$$
q_k(t,z_1,z_2) = e^{-t/8-k^2t/2}\frac{\sqrt{y_1y_2}}{2\pi}\int_{-\infty}^\infty q^M_{\lambda,k}(t,\log y_1,\log y_2)\,d\lambda.
$$

Next we note that $q_k(t,z_1,z_2)$ and the function defined by the right hand side of (5.3) are, as functions in k, analytic on $\mathbf{C}$. The analyticity of $q_k(t,z_1,z_2)$ is seen from the probabilistic representation for it (see Section 2 of [11]). Therefore it is sufficient to show (5.3) when $|k| < 1/2$ by virtue of the uniqueness theorem for analytic functions.

Now we assume that $x_2 = x_1$ and $|k| < 1/2$. Then it holds that

$$
g_t(r) = \frac{\sqrt{y_1y_2}}{2\pi}e^{-t/8-k^2t/2}\int_0^\infty (q^M_{\lambda,k} + q^M_{-\lambda,k})(t,\log y_1,\log y_2)\,d\lambda.
$$

Moreover, setting $\phi_2 = 2\lambda\sqrt{y_1y_2}/\sinh(u)$, we have by (4.2) and (4.3)

$$(q^M_{\lambda,k} + q^M_{-\lambda,k})(t, \log y_1, \log y_2) = \int_0^\infty \frac{\cosh(2ku)}{\sinh(u)} \exp(-\lambda(y_1+y_2)\coth(u))\, \theta_{\phi_2}(t/4)\, du$$

and, therefore,

$$\begin{aligned} g_t(r) = {} & \frac{\sqrt{2}y_1y_2}{\pi^{5/2}t^{1/2}} e^{-t/8-k^2t/2+2\pi^2/t} \\ & \times \int_0^\infty \lambda\, d\lambda \int_0^\infty du \int_0^\infty d\xi\, \frac{\cosh(2ku)}{(\sinh(u))^2} \exp(-\lambda(y_1+y_2)\coth(u)) \\ & \times \exp\left(-\frac{2\xi^2}{t} - \frac{2\lambda\sqrt{y_1y_2}}{\sinh(u)}\cosh(\xi)\right) \sinh(\xi)\sin(4\pi\xi/t). \end{aligned}$$

Noting that the integral on the right hand side is absolutely convergent since $|k| < 1/2$, we first carry out the integral in λ. Then, after some calculations, we obtain

$$g_t(r) = \frac{1}{\pi(2\pi)^{3/2}t^{1/2}} e^{-t/8-k^2t/2} \int_0^\infty \cosh(2ku) F(u)\, du,$$

where

$$F(u) = \int_0^\infty \frac{\exp(2(\pi^2-\xi^2)/t)\sinh(\xi)\sin(4\pi\xi/t)}{(\cosh(r/2)\cosh(u)+\cosh(\xi))^2}\, d\xi.$$

Now we use (5.4) and (5.5). Then, setting

$$\tilde{\varphi}(r,u) = \cosh^{-1}(\cosh(r/2)\cosh(u)),$$

we obtain

$$F(u) = \frac{4\pi\tilde{\varphi}(r,u)}{t\sinh(\tilde{\varphi}(r,u))} \exp(-2\tilde{\varphi}(r,u)^2/t)$$

and

$$g_t(r) = \frac{4\exp(-t/8-k^2t/2)}{(2\pi t)^{3/2}} \int_0^\infty \frac{\cosh(2ku)\tilde{\varphi}(r,u)\exp(-2\tilde{\varphi}(r,u)^2/t)}{\sinh(\tilde{\varphi}(r,u))}\, du.$$

Finally, changing the variable from u into b by $2\tilde{\varphi}(r,u) = b$ or, equivalently, $u = \varphi(b,r)$, we obtain (5.3). □

6. Further Applications of Theorem 2.1

In this section we apply Theorem 2.1 to the computation of the semigroup of a hyperbolic drifted diffusion, which appeared naturally in Alili–Dufresne–Yor [2] in the context of a generalization of the Bougerol identity, and for the distribution of the asset process in a stochastic volatility model ([10]) in mathematical finance.

6.1. **A hyperbolic drifted diffusion.** Let P be the law of the original Brownian motion $B = \{B_t, t \geqq 0\}$ starting from 0 on the canonical path space $C([0,\infty);\mathbf{R})$ and denote $\mathcal{F}_t = \sigma\{B_s, s \leqq t\}$. We consider the stochastic differential equation

$$dY_t = dB_t + (\mu \tanh(Y_t) + \frac{\nu}{\cosh(Y_t)})\,dt, \qquad Y_0 = x,$$

for $\mu, \nu \in \mathbf{R}$ and denote by $P^{\mu,\nu}$ the probability law of the unique strong solution $Y^{\mu,\nu} = \{Y_t^{\mu,\nu}, t \geqq 0\}$. Setting

$$D_t^{\mu,\nu} = \exp\left(\int_0^t (\mu \tanh(B_s) + \frac{\nu}{\cosh(B_s)})\,dB_s - \frac{1}{2}\int_0^t (\mu \tanh(B_s) + \frac{\nu}{\cosh(B_s)})^2\,ds\right),$$

we have by the Girsanov–Maruyama theorem

$$dP^{\mu,\nu}|_{\mathcal{F}_t} = D_t^{\mu,\nu} \cdot dP|_{\mathcal{F}_t}.$$

The following theorem characterizes the law of $Y_t^{\mu,\nu}$ for a fixed time.

Proposition 6.1. *For any $t > 0, \lambda > 0$, it holds that*

$$\begin{aligned} &E_x[\exp(\sqrt{-1}\lambda \sinh(Y_t^{\mu,\nu}))] \\ &= e^{-\mu^2 t/2} \int_0^\infty dv \int_{-\infty}^\infty dy\ \exp(\sqrt{-1}\lambda \sinh(x) e^y + \mu y + \sqrt{-1}\lambda\nu v) \\ &\qquad\qquad \times \frac{\lambda}{4\sinh(\lambda v/2)} \exp\left(-\lambda(e^y+1)\coth(\lambda v/2)\right) \theta_\phi(t/4). \end{aligned}$$

Proof. We first borrow the following identity in law between the involved processes from [2];

$$\{\sinh(Y_t^{\mu,\nu}), t \geqq 0\} \overset{(\text{law})}{=} \{\exp(B_t^\mu)(\sinh(x) + \int_0^t \exp(-B_s^\mu)\,d\gamma_s^\nu), t \geqq 0\}, \tag{6.1}$$

where $B_s^{(\mu)} = B_s + \mu s$ and $\gamma_s^{(\nu)} = \gamma_s + \nu s$ for an auxiliary independent Brownian motion $\gamma = \{\gamma_s\}$. We recall that this result is obtained by applying Ito's formula and simplifying the martingale part of the right hand side. Reversing time in the integral on the right hand side of (6.1), we get the identity

$$\sinh(Y_t^{\mu,\nu}) \overset{(\text{law})}{=} \sinh(x)\exp(B_t^{(\mu)}) + \int_0^t \exp(B_s^{(\mu)})\,d\gamma_s^{(\nu)}. \tag{6.2}$$

for fixed $t > 0$. The theorem follows by taking the Fourier transform of both hand sides, using the independence of B and γ and applying Theorem 2.1. $\square$

Remark 6.1. Since B and γ are independent, Dambis-Dubins-Schwarz Theorem (see [20], p.181) allows us to conclude that, for fixed t,

$$(6.3)\qquad Y_t^{\mu,\nu} \overset{\text{(law)}}{=} \sinh^{-1}\left(\sinh(x)\exp(B_t^{(\mu)}) + \sqrt{A_t^{(\mu)}}N + \nu a_t^{(\mu)}\right),$$

where $A_t^{(\mu)}$,$a_t^{(\mu)}$ are defined as A_t,a_t with $\{B_t\}$ replaced by $\{B_t^{(\mu)}\}$ and N stands for a standard normal variable independent of B. The Cameron-Martin theorem shows that we need an explicit expression for the probability density of the triplet (A_t, a_t, B_t) in order to obtain that for the transition probability density $p_t^{\mu,\nu}(x,y)$ of $\{Y_t^{\mu,\nu}\}$. Indeed, a straightforward calculation shows that

$$p_t^{\mu,\nu}(x,y) = \cosh(y)E[e^{\mu B_t - \mu^2 t/2}\frac{1}{\sqrt{2\pi A_t}}\exp\left(-\frac{1}{2A_t}(\sinh(y) - \nu a_t - \sinh(x)e^{B_t})^2\right)]$$

Clearly the difficulty in expressing out $p_t^{\mu,\nu}(x,y)$ relies on the explicit inversion of the Laplace transform figuring in Theorem 2.1.

Remark 6.2. Now we clarify some connections of the present computations with those in a recent paper by Yor [23]. We assume that $\mu < 0$ and $x = 0$. In this case, by the time reversal, we see that the process $\{\sinh(Y_t^{\mu,\nu}), t \geqq 0\}$ converges in law to $\int_0^\infty \exp(B_s^{(\mu)})d\gamma_s^{(\nu)}$, which is called a subordinated perpetuity in [23]. The fact that the distribution of the later random variable is the unique invariant probability measure for the diffusion process $\{\sinh(Y_t^{\mu,\nu}), t \geqq 0\}$ has been extensively exploited to find out the explicit formula for the probability density of the subordinated perpetuity. In the spirit of the previous study (and only for $t = \infty$) the required object is the density of the couple $(\int_0^\infty \exp(B_s^{(\mu)})d\gamma_s, \int_0^\infty \exp(B_s^{(\mu)})ds)$, which seems to have some connections with generalized Lévy stochastic area formulae. This fact breaks down the hope to invert explicitly the Laplace transform figuring in Theorem 2.1.

6.2. **The Hull–White model.** In this paragraph we consider the Hull–White model in mathematical finance (see [10], [13]), which serves a model of an asset price process $\{S_t, t \geqq 0\}$ with a stochastic volatility $\{\sigma_t, t \geqq 0\}$. Our objective is to obtain an explicit form (via an elementary integral) of the distribution of S_t for fixed time t by applying Theorem 2.1.

The model is described by the following system of stochastic differential equations;

$$(6.4)\qquad \frac{dS_t}{S_t} = r\,dt + \sqrt{1-\rho^2}\sigma_t\,dw_t^{(1)} + \rho\sigma_t\,dw_t^{(2)}, \qquad \frac{d\sigma_t}{\sigma_t} = a\,dt + b\,dw_t^{(2)},$$

where a, b, r are real constants and $\{w_t^{(1)}, t \geqq 0\}$ and $\{w_t^{(2)}, t \geqq 0\}$ are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$ with a usual filtration $(\mathcal{F}_t)$. Moreover $\rho \in [0,1)$ is also a constant which describes the correlation between $\{S_t\}$ and $\{\sigma_t\}$.

We set

$$g = g(q) = (2(q^2 + \frac{\sigma_0^2}{8(1-\rho^2)b^2}))^{1/2}, \quad \tilde{\phi} = \frac{2g\exp(y/2)}{\sinh(gv/2)}, \quad \gamma = \frac{a}{b^2} - \frac{1}{2}$$

and

$$K \equiv K_t(x, y, v) = x - rt + \frac{\sigma_0 \rho}{b}(1 - e^y) + \frac{a\sigma_0\rho}{b^3}v.$$

Then we show the following:

Proposition 6.2. *Assume $b\sigma_0 < 0$. Then, for any fixed $t > 0$, the probability density $\varphi_t(x)$ of $\log(S_t/S_0)$ with respect to Lebesgue measure is given by*

$$(6.5) \quad \varphi_t(x) = \sqrt{\frac{b^2}{2\pi^2(1-\rho^2)\sigma_0^2}} \int_{\mathbf{R}} dq \int_{\mathbf{R}} dy \int_0^\infty dv \, \frac{g}{4\sinh(gv/2)} \, \theta_{\tilde{\phi}}(b^2t/4)$$

$$\times \exp\left(-g(e^y+1)\coth(gv/2)\right) \cos\left(\frac{\sqrt{2}bK}{\sigma_0\sqrt{1-\rho^2}}q\right) \exp\left(-\frac{K}{2(1-\rho^2)} + \gamma y - \frac{\gamma^2 b^2 t}{2}\right).$$

As a first argument towards a proof of Proposition 6.2, we remark that the volatility process $\{\sigma_t\}$ is a geometric Brownian motion given by

$$\sigma_t = \sigma_0 \exp(bw_t^{(2)} + b^2\gamma t)$$

and that

$$\log(S_t/S_0) = rt + \sqrt{1-\rho^2}\int_0^t \sigma_s \, dw_s^{(1)} + \rho \int_0^t \sigma_s \, dw_s^{(2)} - \frac{1}{2}\int_0^t \sigma_s^2 \, ds.$$

The following is easily obtained.

Lemma 6.3. *The conditional distribution of $\log(S_t/S_0)$ given $\mathcal{F}_t^{(2)} \equiv \sigma\{w_s^{(2)}, s \leqq t\} = \sigma\{\sigma_s, s \leqq t\}$ is the normal distribution with mean m_t and variance V_t, where*

$$m_t = rt + \rho\int_0^t \sigma_s \, dw_s^{(2)} - \frac{1}{2}\int_0^t \sigma_s^2 \, ds \qquad \textit{and} \qquad V_t = (1-\rho^2)\int_0^t \sigma_s^2 \, ds.$$

Proof of Proposition 6.2. By Lemma 6.3, we have

$$\varphi_t(x) = E[\frac{1}{\sqrt{2\pi V_t}}\exp(-\frac{(x-m_t)^2}{2V_t})].$$

Now, letting P^* be the probability measure on Ω defined by

$$dP^*|_{\mathcal{F}_t} = \exp\left(-b\gamma w_t^{(2)} - \frac{1}{2}b^2\gamma^2 t\right) \cdot dP|_{\mathcal{F}_t},$$

$w_t^* = w_t^{(2)} + b\gamma t$ is a Brownian motion under P^* by virtue of the Cameron–Martin theorem and we obtain

$$\varphi_t(x) = E^*[\frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_0^2 A_t^{(b)}}} \exp\left(-\frac{(G_1 + \sigma_0^2 A_t^{(b)}/2)^2}{2(1-\rho^2)\sigma_0^2 A_t^{(b)}}\right) \exp(b\gamma w_t^* - b^2\gamma^2 t/2)],$$

where E^* denotes the expectation with respect to P^*,

$$A_t^{(b)} = \int_0^t \exp(2bw_s^*)\, ds$$

and

$$G_1 = x - rt - \rho \int_0^t \sigma_s\, dw_s^* + \rho b\gamma \int_0^t \sigma_s\, ds.$$

Note that Itô's formula yields

$$\int_0^t \sigma_s\, dw_s^* = \frac{\sigma_0}{b}(\exp(bw_t^*) - 1 - \frac{b^2}{2}a_t^{(b)}), \qquad a_t^{(b)} = \int_0^t \exp(bw_s^*)\, ds.$$

Furthermore we define another Brownian motion $\{w_t, t \geqq 0\}$ under P^* by $w_t = bw^*_{b^{-2}t}$ and set

$$A_t = \int_0^t \exp(2w_s)\, ds \qquad a_t = \int_0^t \exp(w_s)\, ds.$$

Then it is easy to show

$$\varphi_t(x) = E^*[\sqrt{\frac{b^2}{2\pi(1-\rho^2)\sigma_0^2 A_{b^2t}}} \exp\left(-\frac{b^2(G_2 + \sigma_0^2 A_{b^2t}/2b^2)^2}{2(1-\rho^2)\sigma_0^2 A_{b^2t}}\right) \exp(\gamma w_{b^2t} - b^2\gamma^2 t/2)],$$

where

$$G_2 = x - rt + \frac{\sigma_0\rho}{b}(1 - \exp(w_{b^2t})) + \frac{a\sigma_0\rho}{b^3}a_{b^2t}.$$

Finally we recall the identity

$$\int_{\mathbf{R}} e^{-\beta q^2}\cos(\alpha q)\, dq = \sqrt{\frac{\pi}{\beta}}e^{-\alpha^2/4\beta}, \qquad \alpha \in \mathbf{R}, \beta > 0.$$

Then we obtain

$$\varphi_t(x) = \sqrt{\frac{b^2}{2\pi^2(1-\rho^2)\sigma_0^2}} \int_{\mathbf{R}} dq\, E^*[\exp\left\{-(q^2 + \frac{\sigma_0^2}{8(1-\rho^2)b^2})A_{b^2t}\right\}\cos\left(\frac{\sqrt{2}bG_2}{\sqrt{1-\rho^2}\sigma_0}q\right)$$
$$\times \exp\left(-\frac{G_2}{2(1-\rho^2)} + \gamma w_{b^2t} - \frac{b^2\gamma^2 t}{2}\right)].$$

Now, noting that the integral is absolutely convergent since $b\sigma_0 < 0$, we get (6.5) by using (2.6). □

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References

[1] L. Alili, Fonctionnelles exponentielles et certaines valeurs principales des temps locaux browniens, Thèse de doctorat de l'Univ. Paris VI, 1995.

[2] L. Alili, D. Dufresne and M. Yor, Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement brownien avec drift, in [24].

[3] A. Comtet, C. Monthus and M. Yor, Exponential functionals of Brownian motion and disordered systems, J. Appl. Prob., **35** (1998), 255–271.

[4] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, Cambridge, 1989.

[5] K.D. Elworthy, Xue-Mei Li and M. Yor, The importance of strict local martingales, applications to radial Ornstein–Uhlenbeck processes, Prob. Th. Rel. Fields, **115** (1999), 325–355.

[6] J. Fay, Fourier coefficients of the resolvent for a Fuchsian group, J. Reine Angew. Math., **293** (1977), 143–203.

[7] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, series and Products, Academic Press, New York, 1980.

[8] J.-C. Gruet, Semi-groupe du mouvement Brownien hyperbolique, Stochastics Stochastic Rep., **56** (1996), 53–61.

[9] J.-C. Gruet, Windings of hyperbolic Brownian motion, in [24].

[10] J. Hull and A. White, The pricing of options on assets with stochastic volatilities, J. Finance, **42** (1987), 281–300.

[11] N. Ikeda and H. Matsumoto, Brownian motion on the Hyperbolic plane and Selberg trace formula, J. Func. Anal., **163** (1999), 63–110.

[12] K. Itô and H.P. McKean, Jr., Diffusion Processes and Their Sample Paths, Springer-Verlag, Berlin, 1965.

[13] B. Leblanc, Une approche unifiee pour une forme exacte du prix d'une option dans les differents modeles a volatilite stochastique, Stochastic and Stochastic Reports, **57** (1996), 1–35.

[14] H.P. McKean, Jr., Stochastic Integrals, Academic Press, New York, 1969.

[15] H. Matsumoto and M. Yor, A version of Pitman's $2M - X$ theorem for geometric Brownian motions, C. R. Acad. Sc. Paris Série I, **328** (1999), 1067–1074.

[16] H. Matsumoto and M. Yor, Some changes of probabilities related to a geometric Brownian motion version of Pitman's $2M - X$ theorem, Elect. Comm. in Probab., **4** (1999), 15–23.

[17] S.J. Patterson, The Laplace operator on a Riemann surface, Compositio Math., **31** (1975), 83–107.

[18] J.W. Pitman and M. Yor, Bessel processes and infinitely divisible laws, Stochastic Integrals, ed. by D.Williams, Lecture Notes Math. **851**, 285–370, Springer-Verlag, Berlin, 1981.

[19] J.W. Pitman and M. Yor, Asymptotic laws of planar Brownian motion, Ann. Prob., **14** (1986), 733–779.

[20] D. Revuz and M. Yor, Continuous Martingales and Brownian Motion, 3rd. Ed., Springer-Verlag, Berlin, 1999.

[21] M. Yor, Loi de l'indice du lacet Brownien, et distribution de Hartman-Watson, Z.W., **53** (1980), 71–95.

[22] M. Yor, On some exponential functionals of Brownian motion, Adv. Appl. Prob., **24** (1992), 509-531.

[23] M. Yor, Interpretations in terms of Brownian and Bessel meanders of the distribution of a subordinated perpetuity, to appear in a Birkhäuser volume on Lévy processes edited by T.Mikosh.

[24] M. Yor (Ed.), Exponential Functionals and Principal Values related to Brownian Motion, A collection of research papers, Biblioteca de la Revista Matemática Iberoamericana, Madrid, 1997.

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