SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

NATHALIE EISENBAUM

On Ito's formula of Föllmer and Protter

Séminaire de probabilités (Strasbourg), tome 35 (2001), p. 390-395

<http://www.numdam.org/item?id=SPS_2001__35__390_0>

On Itô's formula of Föllmer and Protter

Nathalie Eisenbaum

Laboratoire de Probabilités et Modèles Aléatoires - Universités Paris VI et VII
4, Place Jussieu - case 188 - 75252 Paris cedex 05

Abstract : Föllmer and Protter have established an Itô formula for the d-dimensional Brownian motion and a function F in the Sobolev space $\mathcal{W}^{1,2}$. In this formula, the usual second order terms are replaced by quadratic covariations. We show here that these covariations are actually area integrals with respect to local times. We also extend their formula to the time-dependent case.

1 - Introduction and notations

Let X be a d-dimensional Brownian motion. We write : $X = (X^{(1)}, X^{(2)}, ... X^{(d)})$. Let G be an element of $\mathcal{L}^2(I\!R^d)$. Föllmer and Protter have established in [FP], that for all a in $I\!R^d$, except for some polar set, the quadratic covariation

$$[G(X), X^{(k)}]_t = \lim_n \sum_{i=1}^{n} (G(X_{t_{i+1}}) - (G(X_{t_i}))(X^{(k)}_{t_{i+1}} - X^{(k)}_{t_i})$$

(where $(t_i)_{1 \leq i \leq n}$ is a partition of $[0,1]$ depending on n, such that the mesh tends to 0 with n) exists as a limit in probability under $I\!P_a$ for each k in $\{1, ..., d\}$ and that :

$$[G(X), X^{(k)}]_t = -\int_0^t G(X_s) dX^{(k)}_s - \int_{1-t}^1 G(X_{1-s}) dX^{(k)}_{1-s} \qquad (1)$$

Denote by $\mathcal{W}^{1,2}$, the Sobolev space of functions in $\mathcal{L}^2(I\!R^d)$ such that the weak first partial derivatives belong to $\mathcal{L}^2(I\!R^d)$. Let F be an element of $\mathcal{W}^{1,2}_{\text{loc}}$. They established that $I\!P_a$-a.s. for all a in $I\!R^d$, except for some polar set :

$$F(X_t) = F(X_0) + \sum_{k=1}^{d} \int_0^t \frac{\partial F}{\partial x_k}(X_s) dX^{(k)}_s + \frac{1}{2} \sum_{k=1}^{d} [\frac{\partial F}{\partial x_k}(X), X^{(k)}]_t \qquad (2)$$

Clearly, the conditions of F are optimum. Indeed, for a fixed a, the existence of the process $(F(X_t) - F(X_0) - \sum_{k=1}^{d} \int_0^t \frac{\partial F}{\partial x_k}(X_s) dX^{(k)}_s, t \geq 0)$ requires at least that F belongs to $\mathcal{W}^{1,2}_{\text{loc}}$.
In a previous work, Föllmer, Protter and Shiryaev [FPS] proved this result for the special case $d = 1$ and $a = 0$. They also established the existence of the covariation of $(f(X^{(1)}_t, t), t \geq 0)$ and $(X^{(1)}_t, t \geq 0)$ and could write an Itô formula in the time-dependent case. A similar result has been established by Russo and Valois [RV]. Several authors have then extended that kind of result to other processes than Brownian motion, see Bardina and Jolis [BJ] and Moret and Nualart [MN].
In the case of a linear Brownian motion, we showed in [E], that the quadratic covariations appearing in [FPS] were particular examples of area integrals with respect to

the local time process of $X^{(1)}$. One of the interests of using local times is to make a clear connection beetween various Itô's formulas; namely : Bouleau and Yor's formula [BY], Föllmer, Protter and Shiryaev's formula [FPS], and Azéma, Jeulin, Knight and Yor's formula [AJKY]. In [E], an Itô formula is given, which summarizes and extends each of the above quoted formulas.

We keep here the point of view of the local times and first show the following proposition. We denote by $(L_s^x(X^{(k)}), x \in \mathbb{R}, s \geq 0)$ the local time process of $X^{(k)}$. We adopt the notation: $G(X_s^{(1)}, ..., X_s^{(k-1)}, x, X_s^{(k+1)}, ..., X_s^{(d)}) = G(X_s)\big|_{X_s^{(k)}=x}$.

Proposition 1.1 : *Let G be an element of $L^2(\mathbb{R}^d)$. Then for any a outside a polar set, we have $\mathbb{P}_a$-a.s.*

$$[G(X), X^{(k)}]_t = -\int_0^t \int_{\mathbb{R}} G(X_s)\big|_{X_s^{(k)}=x} dL_s^x(X^{(k)})$$

The precise meaning of the double integral with respect to $L(X^{(k)})$ will be recalled in the next section. Thanks to Proposition 1.1, (2) can be rewritten as follows :

$$F(X_t) = F(X_0) + \sum_{k=1}^d \int_0^t \frac{\partial F}{\partial x_k}(X_s) dX_s^{(k)} - \frac{1}{2} \sum_{k=1}^d \int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x_k}(X_s)\big|_{X_s^{(k)}=x} dL_s^x(X^{(k)})$$

We extend then this formula to the time-dependent case under the following form :

Theorem 1.2 : *Let F be a function defined on $\mathbb{R}^d \times \mathbb{R}^+$, such that F admits first order Radon-Nikodym derivatives with respect to each parameter. Moreover, we assume that these derivatives satisfy the following integrability conditions.*
For every compact K of $\mathbb{R}^d$, every $t > 0$ and every k in $\{1, ..., d\}$

$$\int_0^t \int_K |\frac{\partial F}{\partial t}(x, s))| dx \frac{ds}{s^{d/2}} < \infty$$
$$\int_0^t \int_K |\frac{\partial F}{\partial x_k}(x, s)| dx \frac{ds}{s^{(d+1)/2}} < \infty$$

Then, we have for every $a \in \mathbb{R}^d$, $\mathbb{P}_a$-a.s.

$$\begin{aligned} F(X_t, t) = F(X_0, 0) &+ \int_0^t \frac{\partial F}{\partial s}(X_s, s) ds + \sum_{k=1}^d \int_0^t \frac{\partial F}{\partial x_k}(X_s, s) dX_s^{(k)} \\ &- \frac{1}{2} \sum_{k=1}^d \int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x_k}(X_s, s)\big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}) \end{aligned}$$

In Section 2, we recall some facts on the stochastic integration with respect to local times. The proofs of Proposition 1.1 and Theorem 1.2 are given in Section 3.

2 - Preliminaries

We start by recalling a result on the stochastic integration of deterministic functions with respect to the local times. Without loss of generality, we restrict our attention to functions defined on $\mathbb{R} \times [0, 1]$.

Let f be a measurable function from $I\!\!R \times [0,1]$ to $I\!\!R$, B a real Brownian motion and a_1 a real number. We set :

$$||f||_{a_1} = I\!\!E_{a_1}[\int_0^1 f^2(B_s,s)ds]^{1/2} + I\!\!E_{a_1}[\int_0^1 |f(B_s,s)|\frac{|B_s - a_1|}{s}ds].$$

We know, thanks to [E], that stochastic integration with respect to $(L_s^x(B), 0 \le s \le 1, x \in I\!\!R)$ is well defined on the Banach space $\{f : ||f||_{a_1} < \infty\}$ in the following sense. Let f_Δ be an elementary function on $I\!\!R \times [0,1]$, meaning that

$$f_\Delta(x,t) = \sum_{(x_i,s_j)\in\Delta} f_{i,j} 1_{(x_i,x_{i+1}]}(x) 1_{(s_j,s_{j+1}]}(t),$$

where $\Delta = \{(x_i,s_j), 1 \le i \le n, 1 \le j \le m\}$ is a grid of $I\!\!R \times [0,1]$, and for every (i,j), $f_{i,j}$ is a real number. For such a function, integration with respect to $(L_s^x(B), 0 \le s \le 1, x \in I\!\!R)$ is defined by

$$\int_0^t \int_{I\!\!R} f_\Delta(x,s) dL_s^x(B) = \sum_{(x_i,s_j)\in\Delta} f_{i,j}(L_{s_{j+1}}^{x_{i+1}} - L_{s_j}^{x_{i+1}} - L_{s_{j+1}}^{x_i} + L_{s_j}^{x_i})$$

Let f be such that $||f||_{a_1} < \infty$. For any sequence of elementary functions $(f_{\Delta_k})_{k\in I\!\!N}$ converging to f for the norm $||.||_{a_1}$, the sequence $(\int_0^t \int_{I\!\!R} f_{\Delta_k}(x,s)dL_s^x(B))_{k\in I\!\!N}$ converges in L^1. The obtained limit does not depend on the choice of the sequence $(f_{\Delta_k})_{k\in I\!\!N}$ and represents the integral $\int_0^t \int_{I\!\!R} f(x,s)dL_s^x(B)$.
Moreover $I\!\!P_{a_1}$-a.s. for any $t \in [0,1]$:

$$\int_0^t \int_{I\!\!R} f(x,s)dL_s^x(B) = \int_0^t f(B_s,s)dB_s + \int_{1-t}^1 f(B_{1-s}, 1-s)dB_{1-s}$$

and

$$I\!\!E_{a_1}(|\int_0^t \int_{I\!\!R} f(x,s)dL_s^x(B)|) \le ||f||_{a_1}.$$

Consider now a measurable function F from $I\!\!R^d \times [0,1]$ to $I\!\!R$. Let $a = (a_1, a_2, ..., a_d)$ be an element of $I\!\!R^d$. Define :

$$||F||_{k,a} = I\!\!E_a[\int_0^1 F^2(X_s,s)ds]^{1/2} + I\!\!E_a[\int_0^1 |F(X_s,s)|\frac{|X_s^{(k)} - a_k|}{s}ds].$$

For any k, note that conditionally on $(X^{(i)}, 1 \le i \le d, i \ne k)$, $(F(X_s,s), 0 \le s \le 1)$ is a deterministic function of $((X_s^{(k)}, s), 0 \le s \le 1))$. Thanks to the above result, we know hence that as soon as :

$$||F||_{k,a} < \infty$$

then $I\!\!P_a$-a.s.,for any t, $\int_0^t \int_{I\!\!R} F(X_s,s)\big|_{X_s^{(k)}=x} dL_s^x(X^{(k)})$ is well defined and

$$\int_0^t \int_{I\!\!R} F(X_s,s)\big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}) = \int_0^t F(X_s,s)dX_s^{(k)} + \int_{1-t}^1 F(X_{1-s}, 1-s)dX_{1-s}^{(k)}.$$

Moreover :

$$I\!\!E_a[|\int_0^t \int_{I\!\!R} F(X_s,s)\big|_{X_s^{(k)}=x} dL_s^x(X^{(k)})|] \le ||F||_{k,a} \qquad (3)$$

Remark that Föllmer, Protter and Shiryaev did compute $[f(B_{\cdot},.), B_{\cdot}]$ for a deterministic f. But this computation requires some continuity property in the time variable for the function $f(x,t)$. Hence Föllmer and Protter could not use it to obtain (1) and had to refine the arguments of [FPS].

We will also need the following theorem, (established in [E],Section 5), concerning this time, integration of random processes.

Theorem 2.1: *Let $(A(x,t); x \in \mathbb{R}, 0 \leq t \leq 1)$ be a continuous random process taking values in $\mathbb{R}$, such that for any t in $[0,1]$ and almost every ω, $A(.,t)$ is absolutely continous with respect to dx. We note $\frac{\partial A}{\partial x}$ its derivative and ask $\frac{\partial A}{\partial x}$ to be continuous. For $\alpha < \beta$, let $(x_i)_{0\leq i\leq n}$ a subdivision of $[\alpha,\beta]$ and $(s_j)_{0\leq j\leq m}$ a subdivision of $[0,1]$. Note Δ the grid $\{(x_i,s_j), 1 \leq i \leq n, 1 \leq j \leq m\}$. Then, the expression*

$$\sum_{(x_i,s_j)\in\Delta} A(x_i,s_j)(L_{s_{j+1}}^{x_{i+1}} - L_{s_j}^{x_{i+1}} - L_{s_{j+1}}^{x_i} + L_{s_j}^{x_i})$$

admits a.s. a limit as $|\Delta|$ tends to 0. This limit is equal to :

$$-\int_0^t \frac{\partial A}{\partial x}(B_s,s)1_{[\alpha,\beta]}(B_s)ds + \int_0^t A(\beta,s)d_sL_s^{\beta} - \int_0^t A(\alpha,s)d_sL_s^{\alpha}.$$

3 - Proofs

Proof of Proposition 1.1

Let G be a measurable function from $\mathbb{R}^d$ to $\mathbb{R}$. Suppose that

$$\mathbb{E}_a[\int_0^1 G^2(X_s)ds]^{1/2} + \sum_{k=1}^{d} \mathbb{E}_a[\int_0^1 |G(X_s)|\frac{|X_s^{(k)} - a_k|}{s}ds] < \infty \tag{4}$$

then, thanks to the remarks of Section 2, we obtain $\mathbb{P}_a$-a.s.

$$\int_0^t \int_{\mathbb{R}} G(X_s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}) = \int_0^t G(X_s)dX_s^{(k)} + \int_{1-t}^1 G(X_{1-s})dX_{1-s}^{(k)}.$$

Looking closely to the proof of (1) in [FP], we see that Föllmer and Protter first establish (1) under the assumption (4). Hence we obtain $\mathbb{P}_a$-a.s.

$$\int_0^t \int_{\mathbb{R}} G(X_s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}) = -[G(X), X^{(k)}]_t$$

To lighten the integrability condition on G, we use the following result of Föllmer and Protter [FP]. They proved that if G is in $\mathcal{L}^2(\mathbb{R}^d)$ then for any a outside a polar set, (4) is verified. □

Proof of Theorem 1.2: Let F be a function satisfying the assumptions of Theorem 1.2. By a localization argument, we can assume that F has a compact support. We note then that for any k in $\{1,...,d\}$ and any $a \in R^d$, we have : $\|\frac{\partial F}{\partial x_k}\|_{k,a} < \infty$. Let g be

a $\mathcal{C}^\infty$ -function with compact support, from $I\!R$ to $I\!R+$, and such that : $\int_{I\!R} g(s)ds = 1$.
We set for $n \in I\!N^*$:

$$g_n(s) = ng(ns)$$

$$F_n(x,t) = \int_0^1 \int_{I\!R^d} F(y,s)g_n(t-s)g_n(x_1-y_1)g_n(x_2-y_2)...g_n(x_d-y_d)dsdy$$

Since the function F_n belongs to $\mathcal{C}^\infty(I\!R^d \times [0,1]$, we have, thanks to the usual Itô's formula, for every $\epsilon > 0$

$$F_n(X_t,t) = F_n(X_\epsilon,\epsilon) + \int_\epsilon^t \frac{\partial F_n}{\partial t}(X_s,s)ds + \sum_{k=1}^d \int_\epsilon^t \frac{\partial F_n}{\partial x_k}(X_s,s)dX_s^{(k)} + \frac{1}{2}\sum_{k=1}^d \int_\epsilon^t \frac{\partial^2 F_n}{\partial (x_k)^2}(X_s,s)ds$$

Using similar arguments to those developped in [AJKY], we note then that $F_n(X_t,t)$ (resp. $\int_\epsilon^t \frac{\partial F_n}{\partial t}(X_s,s)ds$; $\int_\epsilon^t \frac{\partial F_n}{\partial x_k}(X_s,s)dX_s^{(k)}$) converges in probability to $F(X_t,t)$ (resp. $\int_\epsilon^t \frac{\partial F}{\partial t}(X_s,s)ds$; $\int_\epsilon^t \frac{\partial F}{\partial x_k}(X_s,s)dX_s^{(k)}$) as n tends to ∞. Consequently $\frac{1}{2}\sum_{k=1}^d \int_\epsilon^t \frac{\partial^2 F_n}{\partial (x_k)^2}(X_s,s)ds$ converges in probability to :

$$F(X_t,t) - F(X_\epsilon,\epsilon) - \int_\epsilon^t \frac{\partial F}{\partial s}(X_s,s)ds - \sum_{k=1}^d \int_\epsilon^t \frac{\partial F}{\partial x_k}(X_s,s)dX_s^{(k)} \tag{5}$$

Besides, note that for each k, we have :

$$(\frac{\partial F_n}{\partial x_k}(x,s)1_{(\epsilon,t)}(s), x \in I\!R, s \in [0,1]) \underset{n\longrightarrow\infty}{\longrightarrow} (\frac{\partial F}{\partial x_k}(x,s)1_{(\epsilon,t)}(s), x \in I\!R, s \in [0,1])$$

for the norm $||.||_{k,a}$. Hence, we obtain

$$\int_\epsilon^t \int_{I\!R} \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}) \underset{n\longrightarrow\infty}{\overset{L^1}{\longrightarrow}} \int_\epsilon^t \int_{I\!R} \frac{\partial F}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}). \tag{6}$$

We compute now for a fixed k, $\int_\epsilon^t \int_{I\!R} \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)})$. Applying Theorem 2.1 to the process : $A(x,s) = \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=x}$, we obtain for any $\alpha < \beta$

$$\int_0^t \int_\alpha^\beta \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}) = -\int_0^t \frac{\partial^2 F_n}{\partial (x_k)^2}(X_s,s)1_{[\alpha,\beta]}(X_s^{(k)})ds + \int_0^t \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=\beta} d_sL_s^\beta(X^{(k)}) - \int_0^t \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=\alpha} d_sL_s^\alpha(X^{(k)})$$

Letting α and β tend respectively to $-\infty$ and $+\infty$, the left hand term of this equality converges in L^1 to $\int_0^t \int_a^b \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)})$, and the right hand term converges a.s. to $-\int_0^t \frac{\partial^2 F_n}{\partial (x_k)^2}(X_s,s)ds$. Consequently, we have :

$$\int_0^t \int_{I\!R} \frac{\partial F_n}{\partial x_k}(X_s,s)\Big|_{X_s^{(k)}=x} dL_s^x(X^{(k)}) = -\int_0^t \frac{\partial^2 F_n}{\partial (x_k)^2}(X_s,s)ds \tag{7}$$

Together (5), (6) and (7) give :

$$F(X_t,t) = F(X_\epsilon,\epsilon) + \int_\epsilon^t \frac{\partial F}{\partial t}(X_s,s)ds + \sum_{k=1}^d \int_\epsilon^t \frac{\partial F}{\partial x_k}(X_s,s)dX_s^{(k)}$$
$$-\frac{1}{2}\sum_{k=1}^d \int_\epsilon^t \int_{I\!R} \frac{\partial F}{\partial x_k}(X_s,s)\big|_{X_s^{(k)}=x} dL_s^x(X^{(k)})$$

By letting then ϵ tend to 0, Theorem 1.2 is established.□

References :

[AJKY] Azéma J., Jeulin T., Knight F. and Yor M. : Quelques calculs de compensateurs impliquant l'injectivité de certains processus croissants.*Séminaire de Probabilités XXXII, Lect. Notes in Maths. 1686, 316-327,Springer(1998).*

[BJ] Bardina X; and Jolis M. : An extension of Itô's formula for elliptic diffusion processes.*Stoch.Proc.Appl. 69,83-109 (1997).*

[BY] Bouleau N. and Yor M. Sur la variation quadratique des temps locaux de certaines semi-martingales. *C. R. Acad. Sc. Paris,V.292,491-494 (1981).*

[E] Eisenbaum N. : Integration with respect to local times.*To appear in Potential Analysis.*

[FP] Föllmer H. and Protter P. : On Itô's formula for multidimensional Brownian motion.*Probab. Theory Relat. Fields 116, 1-20 (2000).*

[FPS] Föllmer H., Protter P. and Shiryaev A.N. : Quadratic covariation and an extension of Itô's formula.*Bernoulli,1 (1/2),149-169(1995).*

[MN] Moret S. and Nualart D. : Quadratic covariation and Itô's formula for smooth nondegenerate martingales.*Journal of Theo. Probab. V.13,1,193-224 (2000)*

[RV] Russo F. and Vallois P. : Itô formula for C- functions of semi-martingales.*Prob. Theory Relat.Fields 104,27-42 (1996).*