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From Tanaka's Formula to Ito's Formula : Distributions, Tensor Products and Local Times

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Abstract : In this article we study the classical finite dimensional Ito formula from an infinite dimensional perspective. A finite dimensional semi-martingale is represented as a semi-martingale in a (countable) Hilbert space of tempered distributions. The classical Ito formula is obtained on action by a test function from the dual space. Finite dimensional stochastic differential equations with smooth coefficients are represented as an SDE in a Hilbert space. We obtain representations of the local time process, viewed as a distribution in the space variable, in terms of a Hilbert space valued process of finite variation. A basic feature of our representation, is the role of the tensor product.

Introduction

In [25], we had given a new proof of the Ito formula in dimension $d \geq 1$ for arbitrary semi-martingales, starting from Tanaka's formula. The idea was to use a multi-dimensional variant of the well known technique of proving Ito's formula for dimension $d = 1$ from the Tanaka's formula viz.

$$(X_t - a)^+ = (X_o - a)^+ + \int_0^t I_{\{X_s > a\}} \, dX_s + \frac{1}{2} L(t, \ a)$$

For fixed t, as a function of a, each term represents a tempered distribution, almost surely. It is natural to ask then if $(X_t - .)^+$, as an S' valued process, is an S'-valued semi-martingale and if so whether the above expansion for $(X_t - .)^+$ is true as a semi-martingale equation in S'. Here S' is the space of tempered distributions. In this paper, using a countable Hilbert space structure on S', we answer the above question in the affirmative. We prove in fact a stronger result in section 2. We show that if $\phi \in S'$, and (X_t) is a continuous d dimensional semi-martingale, then $\tau_{X_t}(\phi)$ is a Hilbert space valued semi-martingale where the Hilbert space comes from the countable Hilbertian structure of S'. Here $\tau_x : S' \to S'$, $x \in \mathbb{R}^d$ are the translation operators. We also give an explicit decomposition of this semi-martingale (see Theorem 2.3). Taking $\phi = \delta_o$ the dirac distribution concentrated at zero, we recover the finite dimensional Ito's formula from the equality, $f(X_t) = \ < f, \delta_{X_t} > \ = \ < f, \tau_{X_t}(\delta_o) >$, $f \in S$. Here $< ., . >$ denotes the duality between S and S'. We give two proofs of the fact that $(\tau_{X_t}(\phi))$ is a semi-martingale. One uses the Ito formula for Hilbert space valued semi-martingales as found in [18]. The other is centered around tensorial integration by parts of Hilbert space valued semi-martingales [18] and duality, and is a natural continuation of ideas used in [25]. This latter proof gives increased regularity (see Remark 2.4).

In section 3, we develop some of the implications of the results in section 2, for the theory of local times. The local time process (L_t) viz. the Hilbert space valued process of finite variation appearing in the semi-martingale decomposition of the process $(X_t - .)^+$, mentioned above, is shown to be identical with the Hilbert space valued process $\int_0^t \delta_{X_s}(.)\, d < X >_s$. Further for each t , L_t is given as a distribution by the locally integrable function $(x \to L(t,\, x))$, where $L(t,\, x)$ are the point local times of (X_t). In particular, this provides a mathematical justification for the physicist's interpretation of local times : $L_t^x = \int_0^t \delta_x(X_s)\, ds$ (see [28] for the case when (X_t) is conditioned Brownian motion).

How do the transformations on (X_t) , like $X_t \to X_t + Z_t$, or $X_t \to Y_t = \int_0^t h_s\, dX_s$ act at the level of local times? The results of section 3 are in this direction. Relatively simple formulae (involving tensor products of distributions) govern these transformations. These are consistent with similar formulae derived for point local times in [24], [26]. Here the fact that local times, as distributions, have compact support, plays a crucial role. Proposition [3.8] and the example that follows relate to intersection local times. These have been studied extensively via renormalisation and other techniques (see [15], [17], [31]). Our results show how the intersection local time can be described in terms of appropriate functionals of the local times of the underlying process. An interesting feature of the version of the generalised occupation density formula that we present (Lemma 3.1) is its connection with the Schwartz Kernel theorem. This approach also enables us to give a very natural interpretation of the local time as a tempered distribution realised as an integral along the path. Clearly such integrals exist for processes in more than one dimension - which is not true in general for point local times.

In recent times there has been a rapid growth in the literature on infinite dimensional stochastic calculus. For a representative (but far from exhaustive) sample see [6], [8], [10] , [11], [12], [13] , [16], [23], [27], [32], [33], [35], [36], [37], [38], [39], [40], and [41]. As random tempered distributions, local times have been studied by a number of authors. See [1], [2], [3], [4], [15], [20], [22], [34]. Our paper uses a countable Hilbertian framework to analyse classical finite dimensional semi-martingales. It embeds, so to say, the finite dimensional stochastic calculus in a countable Hilbertian, infinite dimensional framework. This is the natural framework for many important processes for example, stochastic partial differential equations. The infinite dimensional semi-martingales that we construct (see Theorem 2.3) out of say Brownian motion, are actually solutions of stochastic partial differential equations. Corresponding to the case of finite dimensional processes, satisfying Ito's stochastic differential equations, the infinite dimensional processes that we construct also satisfy a stochastic differential equation in S' and are Markov processes. This is analogous to the weak formulation of partial differential equations.

This paper is also related to another stream of results which extend Ito's formula - [5], [7], [14], [21], [24]. We present the Ito formula (corollary 2.5) for a continuous function (i.e., without the usual 'C^2' hypothesis) in an appropriate test function space. This formula is true for any continuous semi-martingales. But our main aim and focus in this paper is to bring out the duality inherent in the classical Ito's formula - somewhat analogous to reinterpreting partial differential equations in the weak or distributional form.

1. Preliminaries

On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying usual conditions, we are given a continuous $\mathcal{F}_t$-semi-martingale (X_t) with canonical decomposition $X_t = X_o + M_t + V_t$ and quadratic variation $< X >_t$. Recall the Tanaka formula : $\forall\ a \in I\!\!R$,

$$(X_t - a)^+ = (X_o - a)^+ + \int_0^t I_{\{X_s > a\}}\, dX_s + \frac{1}{2} L(t,\ a)$$

where $(L(t,\ a))_{t \geq 0}$ is the local time process at a. This is a continuous, non-decreasing adapted process satisfying : $(t,\ a,\ \omega) \to L(t,\ a,\ \omega)$ is $\mathcal{B}(0,\ \infty) \times \mathcal{B}(I\!\!R) \times \mathcal{F}$ measurable and for each a, $a.s.$ $L(t,\ a) = \int_0^t I_{\{X_s = a\}} L(ds,\ a)\ \forall\, t \geq 0$. We also have the occupation density formula : $\forall\, f \geq 0$ measurable, a.s.,

$$\int_0^t f(X_s)\, d < X >_s = \int f(a) L(t,a)\, da \quad \forall\, t \geq 0$$

We refer to [26] for these and other facts relating to the local times of semi-martingales.

Countable Hilbert Spaces : Recall the Schwartz space $S(I\!\!R^d)$ (also denoted by S_d or simply $\mathcal{S}$ when the dimension d is understood) defined by

$$S(I\!\!R^d) = \left\{ f \in C^\infty(I\!\!R^d) : \forall\, p \geq 1,\ \max_{0 \leq j_1 + \cdots + j_d \leq p} \sup_x (1 + \mid x \mid^p) \mid \partial_1^{j_1} \cdots \partial_j^{j_d} f(x) \mid < \infty \right\}$$

Let $g_k(x) = (-1) e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$ $k = 0,\ 1,\ 2,\ \cdots$ be the k th - Hermite polynomial and $h_k(x) = \frac{1}{\sqrt{\sqrt{\pi} 2^k k!}} \times e^{-x^2/2} g_k(x)$, $k = 0,\ 1,\ 2,\ \cdots$. For $j = (j_1 \cdots j_d)$ $j_i \in Z_+$, let $h_j(x) = h_{j_1}(x_1) \cdots h_{j_d}(x_d)$ where $x = (x_1 \cdots x_d)$. The space $S_{d,\ p}$ (or simply S_p) is the completion of S w.r.t. the norm $\|.\|_p$ defined by

$$\|\phi\|_p^2 = \sum_{|j|=0}^{\infty} (2 \mid j \mid + d)^{2p} (\phi,\ h_j)^2, \qquad p \in I\!\!R$$

where $\mid j \mid = j_1 + \cdots + j_d$ and $(.,\ .)$ denotes the inner product in $L^2(I\!\!R^d)$. If $p_1 < p_2$, $\|.\|_{p_1} < \|.\|_{p_2}$. The usual topology on S is given by the family of semi-norms

$$\|\phi\|_p^* = \max_{0 \leq j_1 + \cdots + j_d \leq p} \sup_{x \in I\!\!R^d} (1 + \mid x \mid^2)^p \mid \partial_{x_1}^{j_1} \cdots \partial_{x_d}^{j_d} \phi(x) \mid$$

where $p \in Z_+$. The following proposition shows that this topology is also the same as those given by the semi-norms $\|.\|_p$, $p \geq 1$.

Proposition 1.1. Let $\phi \in S$.

a) $\forall\, n \geq 0$ $\exists\, m > n$ and a constant $C = C(n)$ such that $\|\phi\|_n^* \leq C \|\phi\|_m$

b) $\forall\, n \geq 0\ \exists l > n$ and a constant $C' = C'(n)$ such that $\|\phi\|_n \leq C'\ \|\phi\|_l^*$.

Proof. a) If $\psi \in S$ then $\psi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \partial_1 \cdots \partial_d \psi(y)\, dy$

$$\Rightarrow \quad |\, \psi(x) \,| \leq C \left(\int_{I\!R^d} |\, (1+ |\, y \,|^2) \partial_1 \cdots \partial_d \psi(y) \,|^2 \;\, dy \right)^{1/2}$$

Let $n \geq 0$ and $\phi \in S$. Let $\psi(x) = (1+ |\, x \,|^2)^n D^k \phi(x)$ where

$$D^k = \partial_1^{k_1} \cdots \partial_d^{k_d} \qquad k = (k_1 \cdots k_d), \qquad |\, k \,| \leq n$$

Let $\eta(x) = (1+ |\, x \,|^2) \partial_1 \cdots \partial_d \psi(x)$. Then we have $\|\phi\|_n^* \leq C \sup_{|k| \leq n} \|\eta\|_o$, where $\|\eta\|_o$ is the $L^2(I\!R^d)$ norm. Using the recurrence relations (see [10], appendix)

$$h_j'(x) = \sqrt{\frac{j}{2}}\; h_{j-1}(x) - \sqrt{\frac{j+1}{2}}\; h_{j+1}(x)$$

$$x h_j(x) = \sqrt{\frac{j}{2}}\; h_{j-1}(x) + \sqrt{\frac{j+1}{2}}\; h_{j+1}(x)$$

and the definition of $\|.\|_p$, the existence of m and C are easily established.

b) Let $A = (x_1^2 + \cdots + x_d^2) - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_d^2}$. Then it is easy to see that $\|\phi\|_p = \|A^p \phi\|_o$. The existence of l and C' follow easily, using the above recurrence relations. □

Remark 1.2. It can be shown that $m = 3n + 2 + d$ and $l = 2n + 1$.

The space $(S_p,\ \|.\|_p)$ is a Hilbert space. We denote its dual by S_p' with the dual norm $\|.\|_p'$ defined by

$$\|\phi\|_p' = \sup\{|< f,\ \phi >|, \quad f \in S, \quad \|f\|_p \leq 1\}$$

It is well known (see [9]) that $S = \cap_{p \geq 0} S_p$ and $S' = \cup_{p \geq 0} S_p'$ where S' denotes the dual of S viz. the space of tempered distributions.

Stochastic Integration : We shall use the theory of stochastic integration on Hilbert spaces developed in [18], [19]. If H and K are Hilbert spaces, we will denote by $L(H,\ K)$ the set of bounded linear operators $T : H \to K$. Then if (X_t) is a continuous H valued semi-martingale and (Y_t) is a $L(H,\ K)$ valued locally bounded predictable process the stochastic integral $\int_0^t Y_s\, dX_s$ is a continuous K valued semi-martingale. Note that when (Y_t) is a (locally bounded predictable) K valued process, then (Y_t) can be considered as an $L(H,\ H \otimes_{hs} K)$ valued process, where $H \otimes_{hs} K$ is Hilbert Schmidt tensor product of H and K. In this case $\int_0^t Y_s\, dX_s$ is a $H \otimes_{hs} K$ valued process. We also denote this process by $\int_0^t Y_s \otimes\ dX_s$. In proving our results, we shall need the following facts.

Proposition 1.3. Let $H = I\!R$ and (X_s) a continuous real semi-martingale.

a) Let K and K' be Hilbert spaces and $T : K \to K'$ be a continuous linear functional. Let (Y_s) be an $L(H,\ K)$ valued locally bounded predictable process.

Then, $\forall\, t$, a.s. $T(\int_0^t Y_s\, dX_s) = \int_0^t TY_s\, dX_s$. In particular if $K = S_p'$ for some $p > 0$ and (Y_s) is K valued and $f \in S$, then $\forall\, t$, a.s.

$$< f,\ \int_0^t Y_s\, dX_s > = \int_0^t < f,\ Y_s > \ dX_s$$

b) Suppose (Y_s) is an S_p' valued locally bounded previsible process such that, $\forall\, t$, $\cup_{s\leq t}$ supp $(Y_s) \subset G$ a.s., where $G \subset I\!R^d$ is a compact set and supp (Y_s) is the support of the distribution $Y_s \in S'$. Then a.s., supp $(\int_0^t Y_s\, dX_s) \subset G$ and if $\phi \in S'$ then

$$\phi * \int_0^t Y_s\, dX_s \ = \int_0^t \phi\ *\ Y_s\, dX_s$$

c) Given $p > 0$ and $q > 2p$, $\exists$ a linear homeomorphism $T_{p,\,q} : \otimes_{1,\ hs}^d S_p'(I\!R) \rightarrow S_q'(I\!R^d)$ such that $T_{p,\,q}(\otimes_{hs}\phi_i) = \otimes_d \phi_i$ where $\otimes_{hs}$ is the Hilbert Schmidt tensor product and $\otimes_d$ denotes the tensor product of distributions.

Proof. a) The proof of a) is immediate if we approximate (Y_s) by simple processes and use the continuity of T.

b) Suppose $f \in S$, supp $f \cap G = \phi$. Let $t > 0$. Then a.s.,

$$< f,\ \int_0^t Y_s\, dX_s > = \int_0^t < f,\ Y_s > \ dX_s = 0$$

The set $N = \{f \in S,\ \text{supp } f \cap G = \phi\}$ has a countable dense subset. Hence a.s.,

$$< f,\ \int_0^t Y_s\, dX_s > = \ 0 \qquad \forall\, f \in N$$

$$\Rightarrow \text{supp } (\int_0^t Y_s\, dX_s) \subset G$$

Now let $\phi \in S'$. If $f \in S$ then from the definition of convolution (see [30]),

$$< f,\ \phi * \int_0^t Y_s\, dX_s > = < f * \tilde{\phi},\ \ \int_0^t Y_s\, dX_s >$$

where $\tilde{\phi}(x) = \phi(-x)$. Let ψ_n be C^∞ functions, $\psi_n(x) \equiv 1$ $\|x\| \leq n$, supp $\psi_n \subset \{x : \|x\| \leq n+1\}$. Then, $(f * \tilde{\phi})\psi_n \in S$ and we have, a.s.,

$$\begin{aligned}
< f,\ \phi * \int_0^t Y_s\, dX_s > \ &= \ < f * \tilde{\phi},\ \ \int_0^t Y_s\, dX_s > \\
&= \ \lim_{n\to\infty} < (f * \tilde{\phi})\psi_n,\ \ \int_0^t Y_s\, dX_s > \\
&= \ \lim_{n\to\infty} \int_0^t < (f * \tilde{\phi})\psi_n,\ Y_s > \ dX_s \\
&= \ \int_0^t < f * \tilde{\phi},\ Y_s > \ dX_s = \int_0^t < f,\ \phi * Y_s > \ dX_s \\
&= \ < f,\ \int_0^t \phi * Y_s\, dX_s >
\end{aligned}$$

c) we give the proof for $d = 2$. Since S_p' and S_{-p} are isomorphic as Hilbert spaces (see [11]), it is sufficient to construct the map T from $S_{-p}(I\!R) \otimes_{hs} S_{-p}(I\!R)$ to

$S_{-q}(\mathbb{R}^2)$. Let $q > 2p$ and $(h_i^{-p} \otimes h_j^{-p})$ and $(h_{(i,\, j)}^{-q})$ be respectively the ONB in $S_{-p}(\mathbb{R}) \otimes S_{-p}(\mathbb{R})$ and $S_{-q}(\mathbb{R}^2)$. Recall the notation $h_{(i,\, j)}$ and note that

$$h_i^{-p} = (2i+1)^p h_i$$

$$h_i^{-p} \otimes h_j^{-p} = (2i+1)^p (2j+1)^p h_i \otimes h_j$$

$$h_{(i,\, j)}^{-q} = (2(i+j)+2)^q h_{(i,\, j)}$$

Now define $T : S_{-p}(\mathbb{R}) \otimes_{hs} S_{-p}(\mathbb{R}) \to S_{-q}(\mathbb{R}^2)$ as follows :

$$T(h_i^{-p} \otimes h_j^{-p}) = \frac{((2i+1)(2j+1))^p}{((2(i+j)+2)^q} h_{(i,\, j)}^{-q}$$

If $x \in S_{-p}(\mathbb{R}) \otimes_{hs} S_{-p}(\mathbb{R})$ then

$$x = \sum_{i,\, j} c_{i,\, j} h_i^{-p} \otimes_{hs} h_j^{-p}$$

Extend T by linearity :

$$Tx = \sum_{i,\, j} c_{i,\, j} \frac{((2i+1)(2j+1))^p}{(2(i+j)+2)^q} h_{(i,\, j)}^{-q}$$

It is easy to see that T is a linear homeomorphism. If $f,\, g \in S_{-p}(\mathbb{R})$, and

$$f \otimes_{hs} g = \sum_{i,\, j} < f,\, h_i^{-p} >_{-p} < g,\, h_j^{-p} >_{-p} h_i^{-p} \otimes_{hs} h_j^{-p}$$

then

$$\begin{aligned} T(f \otimes_{hs} g) &= \sum_{i,\, j} < f,\, h_i^{-p} >_{-p} < g,\, h_j^{-p} >_{-p} \frac{((2i+1)(2j+1))^p}{(2(i+j)+2)^q} h_{(i,\, j)}^{-q} \\ &= \sum_{i,\, j} < f,\, h_i >< g,\, h_j > (2(i+j)+2)^{-q} h_{(i,\, j)}^{-q} \\ &= \sum_{i,\, j} < f \otimes g,\, h_{(i,\, j)}^{-q} >_{-q} h_{(i,\, j)}^{-q} \\ &= f \otimes_d g \end{aligned}$$

where $< f, g >$ denotes the inner product in $L^2(\mathbb{R})$ between f and g . Note that the last equality holds in S_{-q}. □

For $a \in \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}$ define $\tau_a f : \mathbb{R}^d \to \mathbb{R}$ by $\tau_a f(x) = f(x-a)$. Then $\tau_a : S' \to S'$ is defined in the usual way : $< f,\, \tau_a \phi > \, = \, < \tau_{-a} f,\, \phi >,\, f \in S,\, \phi \in S'$. Then we have the following proposition.

Proposition 1.4. Let $\phi \in S'$ let $g(x) = \tau_x \phi$, $x \in \mathbb{R}^d$. Then there exists an integer n such that $g : \mathbb{R}^d \to S'_n$, is twice continuously differentiable. Further,

$$\begin{aligned} g'(x)h &= -\sum_{i=1}^{d} (\partial_i \tau_x \phi)\, h_i; \quad h \in \mathbb{R}^d \\ g''(x)(h \otimes h) &= \frac{1}{2} \sum_{i,\, j=1}^{d} (\partial_{ij} \tau_x \phi)\, h_i h_j \quad h \in \mathbb{R}^d \end{aligned}$$

and $g'' : \mathbb{R}^d \to L(\mathbb{R}^d \otimes \mathbb{R}^d,\ S'_n)$ is uniformly continuous on bounded sets.

Proof. $\phi \in S' \Rightarrow \phi \in S'_m$ for some m. We will show that $\exists n > m$ such that the following inequalities hold $\forall\ f \in S : \forall\ x,\ \ h \in \mathbb{R}^d$

$$\|\tau_{x+h} f - \tau_x f - \sum_{i=1}^{d} (\partial_i \tau_x f)\ h_i\|_m \le C(m,\ x,\ d)\|f\|_n \|h\|^2 \qquad ...(1)$$

$$\|\tau_{x+h} f - \tau_x f - \sum_{i=1}^{d} (\partial_i \tau_x f)\ h_i - \frac{1}{2} \sum_{i,\ j=1}^{d} (\partial_i \partial_j \tau_x f)\ h_i h_j\|_m \le C(m,\ x,\ d)\|f\|_n \|h\|^2 \ ...(2)$$

Now the proof is completed using duality :

$$\begin{aligned} \|g(x+h) - g(x) \quad &+ \quad \sum_{i=1}^{d} (\partial_i \tau_x \phi)\ h_i\|'_n \\ &= \sup_{f \in S,\ \|f\|_n \le 1} |< f,\ \tau_{x+h}\phi - \tau_x \phi + \sum_{i=1}^{d} (\partial_i \tau_x \phi)\ h_i >| \\ &\le \sup_{f \in S,\ \|f\|_n \le 1} \left\{ \|\phi\|'_m \|\tau_{-x-h} f - \tau_{-x} f - \sum_{i=1}^{d} (\partial_i \tau_{-x} f)\ h_i\|_m \right\} \\ &\le \sup_{f \in S,\ \|f\|_n \le l} \left\{ \|\phi\|'_m C(n,\ x,\ d)\|f\|_n \|h\|^2 \right\} \\ &= C(n,\ x,\ d)\|\phi\|'_m \|h\|^2. \end{aligned}$$

This proves the first equality. The second is proved similarly. The uniform continuity of g'' follows from that of τ_x.

To prove inequalities (1) and (2) it is sufficient to show that given $m\ \ \exists\ n > m$ such that (1) and (2) hold with norms $\|.\|^*_m,\ \ \|.\|^*_n$ in place of $\|.\|_m,\ \ \|.\|_n$ respectively. Using Proposition 1.1 the required inequalities for $\|.\|_m$ and $\|.\|_n$ follow for an appropriate choice of n. Inequality (1) for the norms $\|.\|^*_m$ and $\|.\|^*_n$ are obtained by applying Taylor's formula to functions of the form

$$B(t) = \partial_1^{j_1} \cdots \partial_d^{j_d} \left(f(z + x + th) - f(z + x) - \sum_{i=1}^{d} \partial_i f(z + x) h_i\ t \right)$$

where $0 \le t \le 1$ and $z,\ x,\ h \in \mathbb{R}^d$, $f \in S$, and using the elementary inequalities

$$(1 + |\ x\ |^2) \le 4(1 + |\ z\ |^2)(1 + |\ x + z\ |^2)$$

The constant $C(m,\ x,\ d) = C(m,\ d)(1 + |\ x\ |^2)^n$. Inequality (2) is proved similarly. $\square$

2. From Tanaka Formula to Ito Formula

In this section, we prove the Ito formula viz. the semi-martingale expansion of δ_{X_t} in the space S'_n , for a suitable n. As mentioned in the Introduction, we give two proofs : one which starts from the Tanaka formula and uses the tensorial integration by parts formula as in [18], and the other using the generalised Ito formula for Hilbert space valued processes X ([18]).

Observe that the locally integrable function $x \to I_{(0,\,\infty)}(x)$ is in S'. From the properties of the translation operator τ_x (see Proposition 1.4) it follows that $I_{(0,\,\infty)}(X_s - .)$ is a locally bounded previsible S'_m valued process, for a previsible process (X_t) for some m. Here the notation $\phi(X_s - .)$ for $\phi \in S' \cap L^1_{\rm loc}$ means the random distribution $\tau_{X_s}\tilde{\phi}$ where $\tilde{\phi}(x) = \phi(-x)$. By the remarks made earlier in Section 1, it follows that when (X_t) is a continuous real semi-martingale, the stochastic integral $\int_0^t I_{(0,\,\infty)}(X_s - .)\, dX_s$ is a S'_m valued continuous semi-martingale.

Lemma 2.1. Let (X_t) be a continuous semi-martingale and $(L(t,\, x))_{t\geq 0,\, x\in I\!R}$, its local time process. There exists an integer $n > 0$ and an S'_n- valued, continuous, adapted process of finite variation, denoted by $L(t)$ such that $\forall\, t$, a.s., $L(t) = (x \to L(t,\, x))$ in S'. Moreover this process is identical with the S'_n- valued process of finite variation $(\int_0^t \delta_{X_s}\ d < X >_s)_{t\geq 0}$. Finally $((X_t - .)^+)_{t\geq 0}$ is an S'_n-valued continuous semi-martingale and we have the Tanaka formula :

$$\text{a.s. } (X_t - .)^+ = (X_o - .)^+ + \int_0^t I_{(0,\,\infty)}(X_s - .)\, dX_s + \frac{1}{2}L(t) \qquad t \geq 0 \qquad ...(3)$$

Proof. Fix $n > 0$ such that the distributions $x \to I_{(0,\,\infty)}(x)$, $x \to (x)^+$ and δ_x (the dirac distribution at x) all belong to S'_n. Note that $(X_t - .)^+$ and $\int_0^t I_{(0,\,\infty)}(X_s - .)\, dX_s$ are S'_n valued continuous adapted processes and the latter, following remarks above, is an S'_n valued semi-martingale.

Define

$$L(t) \equiv 2\Big\{(X_t - .)^+ - (X_o - .)^+ - \int_0^t I_{(0,\,\infty)}(X_s - .)\, dX_s\Big\}$$

Then $L(t)$ is an S'_n-valued continuous adapted process. Hence from proposition 1.3 a) and Fubini's theorem for stochastic integrals. We have for $f \in S$,

$$\begin{aligned}
< f,\, L_t > &= 2\Big\{< f,\, (X_t - .)^+ - (X_o - .)^+ > - \int_0^t < f,\, I_{(0,\,\infty)}(X_s - .) > \ dX_s\Big\} \\
&= 2\int_{I\!R} f(x)\Big\{(X_t - x)^+ - (X_o - x)^+ - \int_0^t I_{(0,\,\infty)}(X_s - x)\ dX_s\Big\}\, dx \\
&= \int_{I\!R} f(x)L(t,\, x)\, dx \qquad \forall\, t, \quad \text{a.s.} \\
&= \int_0^t f(X_s)\, d < X >_s \\
&= < f,\, \int_0^t \delta_{X_s}\, d < X >_s >
\end{aligned}$$

Note that $\int_0^t \delta_{X_s}\ d < X >_s = \int_0^t \tau_{X_s}\delta_o\ d < X >_s$ is an S'_n-valued continuous adapted process of finite variation. □

Remark 2.2. Let $p > 0$ be such that $\|f\|_o^* \leq C_p\|f\|_p \quad \forall\ f \in S$ (Proposition 1.1). Let $0 \leq t_o < t_1 < \cdots < t_n = t$

$$|< f,\, L_{t_j} - L_{t_{j-1}} >| \;=\; |\int f(x)(L(t_j,\, x) - L(t_{j-1},\, x)\, dx\,|$$

$$\leq \|f\|_o^* \int (L(t_j,\ x) - L(t_{j-1},\ x))\, dx$$
$$\leq C_p \|f\|_p (< X >_{t_j} - < X >_{t_{j-1}})$$

$$\|L_{t_j} - L_{t_j-1}\|'_p = \sup_{f \in S,\ \|f\|_p \leq 1} |< f,\ L_{t_j} - L_{t_{j-1}} >|$$
$$\leq C_p (< X >_{t_j} - < X >_{t_{j-1}})$$

$$\Rightarrow \text{ a. s. },\ \sum_{j=1}^{n} \|L_{t_j} - L_{t_j-1}\|'_p \leq C_p < X >_t$$

$\Rightarrow (L_t)$ is an S'_p valued process of finite variation, with the total variation bounded by $< X >_t \ \forall\, t$, a.s.

Theorem 2.3. Let $\phi \in S' = S'(I\!R^d)$, $d \geq 1$ and let (X_t) be a continuous $I\!R^d$-valued semi-martingale. Then $\exists\ n > 0$ such that the process $(\tau_{X_t}\phi)_{t \geq 0}$ is an S'_n-valued continuous semi-martingale and we have

$$\tau_{X_t}\phi = \tau_{X_o}\phi - \sum_{i=1}^{d} \int_0^t \partial_{x_i}(\tau_{X_s}\phi)\, dX_s^i + \frac{1}{2} \sum_{i,\ j=1}^{d} \int_0^t \partial_{x_i}\partial_{x_j}(\tau_{X_s}\phi)\, d < X^i,\ X^j >_s \quad ...(4)$$

1st proof. We consider the case $d = 2$. Let $(X_t) = (X_t^1,\ X_t^2)$. By Lemma 2.1, $\exists$ an $n_o > 0$ such that $(X_t^1 - .)^+$ and $(X_t^2 - .)^+$ are S'_{n_o} valued semi-martingales. Since the operators $\partial_{x_i} : S^1_{n_o}(I\!R^2) \to S^1_{n_o+n_1}(I\!R^2)$, $i = 1,\ 2$, are linear and continuous for some $n_1 > 0$, we can apply the operator $\partial^2_{x_i}$ to $(X_t^i - .)^+$ and using the (Tanaka) formula for the latter and Proposition 1.3 a), we see that the following equation holds in $S^1_{n_o+n_1}(I\!R)$, almost surely, for $i = 1,\ 2$:

$$\delta_{X_t^i} = \delta_{X_0^i} - \int_0^t \partial_{x_i}(\delta_{X_s^i})\, dX_s^i + \frac{1}{2} \int_0^t \partial^2_{x_i}(\delta_{X_s^i})\, d < X^i >_s .$$

Here we used Lemma 2.1 to identify L_t^1 with $\int_0^t \delta_{X_s^1}\, d < X^1 >_s$ etc. Applying the tensorial integration by parts formula (see Métivier and Pellaumail(1969)), we get

$$\begin{aligned}\delta_{X_t^1} \otimes_{hs} \delta_{X_t^2} = {} & \delta_{X_0^1} \otimes_{hs} \delta_{X_0^2} - \int_0^t (\partial_{x_1}\delta_{X_s^1}) \otimes_{hs} \delta_{X_s^2}\, dX_s^1 - \int_0^t \delta_{X_s^1} \otimes_{hs} (\partial_{x_2}\delta_{X_s^2})\, dX_s^2 \\ & + \frac{1}{2} \int_0^t (\partial^2_{x_1}\delta_{X_s^1}) \otimes_{hs} \delta_{X_s^2}\, d < X^1 >_s + \frac{1}{2} \int_0^t \delta_{X_s^1} \otimes_{hs} (\partial^2_{x_2}\delta_{X_s^2}) d < X^2 >_s \\ & + \int_0^t (\partial_{x_1}\delta_{X_s^1}) \otimes_{hs} (\partial_{x_2}\delta_{X_s^2})\, d < X^1,\ X^2 >_s .\end{aligned}$$

Applying the operator $T_{p,\ q} : S'_p(I\!R) \otimes_{hs} S'_p(I\!R) \to S'_q(I\!R^2)$ of Proposition 1.3 c) with $p = n_0 + n_1$ and $q = 2(n_0 + n_1) + \epsilon$, where $\epsilon > 0$, to the above equation we get that the following equation holds in $S'_q(I\!R^2)$, where q is as above, $a.s,\ \forall t > 0$:

$$\begin{aligned}\delta_{(X_t^1,\ X_t^2)} = {} & \delta_{(X_0^1,\ X_0^2)} - \int_0^t \partial_{x_1}(\delta_{(X_s^1,\ X_s^2)})\, dX_s^1 - \int_0^t \partial_{x_2}(\delta_{(X_s^1,\ X_s^2)})\, dX_s^2 \\ & + \frac{1}{2} \int_0^t \partial^2_{x_1}(\delta_{(X_s^1,\ X_s^2)})\, d < X^1 >_s + \frac{1}{2} \int_0^t \partial^2_{x_2}(\delta_{(X_s^1,\ X_s^2)})\, d < X^2 >_s \\ & + \int_0^t \partial^2_{x_1 x_2}(\delta_{(X_s^1,\ X_s^2)})\, d < X^1,\ X^2 >_s .\end{aligned}$$

We now observe that by Proposition 1.3(b), the distributions $\int_0^t \partial_{x_1}\delta_{X_s}\, dX_s^1$, $\int_0^t \partial_{x_2}\delta_{X_s}\, dX_s^2$, almost surely have compact support and hence in the above equation we can convolve each term with $\phi \in S'(I\!R^2)$. Using the fact that $\tau_x\phi = \phi * \delta_x$ and the fact that $\phi * \delta_x \in S'_{q+r}(I\!R^2)$, for some $r > 0$, if $\delta_x \in S'_q(I\!R^2)$, we get the required result with $n = 2(n_0 + n_1) + 1 + r$.

2nd Proof : Given $\phi \in S'$ and $x \in I\!R^d$, let $T(x) = \tau_x\phi \in S'$. By proposition 1.3 $\exists n > 0$ such that $\tau_x\phi \in S'_n$ $\forall\, x \in I\!R^d$ and the map $T : I\!R^d \to S'_n$ is twice continuously Fréchet differentiable with $T'(x) = -(\partial_1\tau_x\phi, \cdots, \partial_n\tau_x\phi)$ and $T''(x) = ((\partial_{ij}(\tau_x\phi)))_{1\le i,\ j\le d}$ where the derivatives are in the sense of distributions. (X_t) being an $I\!R^d$- valued process we can apply the Ito formula of Métevier and Pellaumail [18], and the result follows immediately. □

Remark 2.4 We remark that in the special case of the process (δ_{X_t}) , the first proof gives a value of n such that $\delta_{X_t} \in S'_n$, which does not depend on the dimension d. In the case of the second proof however, the value of n depends on the estimate obtained in Proposition 1.1 , which depends on the dimension.

Corollary 2.5 Let n be such that the process (δ_{X_t}) is an S'_n valued semi-martingale and equation (4) holds. If $f \in S_n$ is a continuous function then $a.s.$, $\forall t \ge 0$ we have,

$$\begin{aligned} f(X_t) &= f(X_o) - \sum_{i=1}^d \int_0^t < f,\ \partial_{x_i}(\delta_{X_s}) > \ dX_s^i \\ &+ \frac{1}{2}\sum_{i,\ j=1}^d \int_0^t < f,\ \partial_{x_i}\partial_{x_j}(\delta_{X_s}) > \ d < X^i,\ X^j >_s . \end{aligned}$$

Remark 2.6 For $p > d$, $f \in S_p$ is a continuous function : From the estimates for h_j given in [29] , Lemma 1.5.1, one can easily show that for $p > d$, the partial sums

$$S_N f(x) = \sum_{|j|=1}^N (2 \mid j \mid + d)^p (f,\ h_j) h_j(x)$$

converge uniformly on compact sets to f.

Stochastic Differential Equations We now consider the case when the finite dimensional process (X_t) is a diffusion driven by a stochastic differential equation of the type

$$\begin{aligned} dX_t &= \sigma(X_t)\, dB_t + b(X_t)\, dt \\ X_0 &= x \qquad \qquad ...(5) \end{aligned}$$

where $\sigma : I\!R^d \to I\!R^{d^2}$ and $b : I\!R^d \to I\!R^d$ are measurable functions and $(B_t) = (B_t^1, ..., B_t^d)$ is a d-dimensional standard Brownian motion. To state the next proposition, we fix some notation. We define operators $A : S' \to L(I\!R^d \to S')$ and $L : S' \to S'$ given by :

$$A\psi(h) = -\sum_j \sum_i < \sigma_{ij},\ \psi > (\partial_i\psi) h_j$$

where $h \in I\!R^d$ and $\psi \in S'$;

$$L\psi = \frac{1}{2}\sum_{i,\, j} < (\sigma\sigma^t)_{ij},\, \psi > \partial_{ij}^2\psi - \sum_i < b_i,\, \psi > \partial_i\psi$$

Of course these operators are well defined only if σ_{ij} and b_i are in S. In the following proposition we assume this is the case. Below we denote by $\mathcal{B}_\omega$ the Borel sigma field on S' generated by the weak topology.

Proposition 2.7 Let $\sigma_{ij},\, b_i,\, i,\, j = 1, \ldots d$ belong to S. Let (X_t) be the unique solution of eqn.(5). Then $(\tau_{X_t}\delta_0)$ satisfies the S' valued stochastic differential equation

$$\begin{aligned} dY_t &= A(Y_t)\, dB_t + L(Y_t)\, dt \\ Y_0 &= \delta_x \end{aligned} \qquad \ldots.(6)$$

Further, $(\tau_{X_t}\delta_0)$ is a $(S'(I\!R^d),\, \mathcal{B}_\omega)$ valued Markov process.

Proof It follows easily from eqn (4) that $\tau_{X_t}(\delta_0)$ satisfies the SDE (6). To prove the Markov property of $\tau_{X_t}(\delta_0)$, we consider the subset E of S' defined by $E = \{\delta_x; x \in I\!R^d\}$. We consider the Borel sigma field on E generated by the weak topology. Let $i : E \to I\!R^d$ be the natural Borel isomorphism $i(\delta_x) = x$. For an element A of the Borel sigma field of S' generated by the weak topology , let A' be the Borel subset of $I\!R^d$ given by $A' = i(A \cap E)$. Then the Markov property of $\tau_{X_t}(\delta_0)$ follows from the Markov property of (X_t) and the following equalities :

$$\begin{aligned} \{\tau_{X_t}(\delta_0) \in A\} &= \{\tau_{X_t}(\delta_0) \in A \cap E\} \\ &= \{X_t \in A'\} \end{aligned} \qquad \square$$

3. Local times and the occupation density formula

Let (X_t) be a continuous semi-martingale, $(L(t,\, x))_{t \geq 0}$ its local time process at $x \in I\!R$. $(L_t)_{t\geq 0}$ will denote the S'-valued induced local time process obtained in Lemma 2.1. In this section, we show how various transformations on the process (X_t) can be expressed at the level of local times, by simple formulae using the occupation density formula, rephrased in the language of tensor products of distributions.

Lemma 3.1. Let n be as in lemma 2.1, so that $(L_t)_{t\geq 0}$ is an $S_n'(I\!R)$-valued process. Suppose $(h_s)_{s\geq 0}$ is an $S_n'(I\!R^d)$ valued locally bounded (measurable) process. Then $\int_0^t L(ds) \otimes_{hs} h_s$ is an $S_n'(I\!R) \otimes_{hs} S_n'(I\!R^d)$ valued continuous adapted process of finite variation. Further if $\epsilon > 0$ and $T_{n,2n+\epsilon} : S_n'(I\!R) \otimes_{hs} S_n'(I\!R^d) \to S_{2n+\epsilon}'(I\!R^{d+1})$, then the distribution $T_{n,2n+\epsilon}(\int_0^t L(ds) \otimes_{hs} h_s)$ is given $\forall\, t,\ a.s.$ by

$$< f,\, T_{n,2n+\epsilon}\Big(\int_0^t L(ds) \otimes_{hs} h_s\Big) > = \int da \int_0^t < f_a,\, h_s > L(ds,\, a) \qquad \ldots(7)$$

$\forall\, f \in S(\mathbb{R}^{d+1})$ and where $f_a(x) = f(a,\ x)$ $a \in \mathbb{R}$, $x \in \mathbb{R}^d$. In particular, if (Y_s) is an $\mathbb{R}^d$ valued measurable process then, $\forall\, t a.s$,

$$T_{n,2n+\epsilon}\left(\int_0^t L(ds) \otimes_{hs} \delta_{Y_s}\right) = \int_0^t \delta_{(X_s,\ Y_s)}\ d < X >_s \qquad \forall\, t \geq 0 \qquad ...(8)$$

Proof : The process $\int_0^t L(ds) \otimes_{hs} h_s$ is well defined (see remarks on stochastic integration in Section 1) and is an $S_n'(\mathbb{R}) \otimes_{hs} S_n'(\mathbb{R}^d)$ valued continuous process of finite variation. In the case when (h_s) is left continuous and bounded, we have,

$$T_{n,2n+\epsilon}\left(\int_0^t L(ds) \otimes_{hs} h_s\right) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(L_{t_{k+1}^n \wedge t} - L_{t_k^n \wedge t}\right) \otimes_d h_{t_k^n}$$

where for each n, $0 = t_o^n < t_1^n < \cdots < t_k^n \uparrow \infty$, and $\sup_k \mid t_k^n - t_{k-1}^n \mid \to 0$ as $n \to \infty$.

We recall that $S(\mathbb{R}^{d+1}) = S(\mathbb{R})\hat{\otimes}S(\mathbb{R}^d)$ where $\wedge$ denotes the completion of the algebraic tensor product in the π - topology (see [30]). Let $f = f_1 \otimes f_2$, $f_1 \in S(\mathbb{R})$ and $f_2 \in S(\mathbb{R}^d)$. Using the occupation density formula eqn. (7) is verified first for f as above and (h_s) simple; and then for (h_s) left continuous and finally for general (h_s) by an application of the monotone class theorem. Since finite linear combinations of functions of the form $f_1 \otimes f_2$ are dense, the result follows for $f \in S(\mathbb{R}^{d+1})$. Eqn. (8) follows from (7) and the generalised occupation density formula by taking $h_s = \delta_{Y_s}$. □

Remark 3.2. Lemma 3.1 can be viewed as a stochastic analogue of the Schwartz Kernel theorem, which states that the space $S'(\mathbb{R}^{d+1})$ is isomorphic to the space $L(S(\mathbb{R}^d) \to S'(\mathbb{R}))$, of continuous linear maps from $S(\mathbb{R}^d)$ to $S'(\mathbb{R})$ see [30]. In effect, the element of $S'(\mathbb{R}^{d+1})$ viz. $T_{n,2n+\epsilon}(\int_0^t L(ds) \otimes_{hs} h_s)$ is identified with the element of $L(S(\mathbb{R}^d) \to S'(\mathbb{R}))$ given by

$$f \in S(\mathbb{R}^d) \longrightarrow \int_0^t < f,\ h_s > L(ds,\ .) \in S'(\mathbb{R}).$$

Consider now the relationship between the local times $L_t^x(X)$ of a continuous semimartingale $X_t = X_o + M_t + V_t$ and the local times $L_t^x(M)$ of its martingale part (M_t). From the results in [24], [26] (p.218), taking $X_o \equiv 0$,

$$L_t^x(X) = \lim_{n\to\infty} \sum_{t_i \in \Delta_n} \left(L_{t_{i+1}\wedge t}^{x-V_{t_i}}(M) - L_{t_i \wedge t}^{x-V_{t_i}}(M)\right)$$

where $\{\Delta_n\}$ is a sequence of partitions of $[0,\ t]$ and the limit is taken in probability.

Let $(L_t(X))_{t\geq 0}$ and $(L_t(M))_{t\geq 0}$ denote the S_n' - valued processes of finite variation given by Lemma 2.1 applied to (X_t) and (M_t) respectively. Let $\epsilon > 0$ and let $T_{n,2n+\epsilon} : S_n'(\mathbb{R}) \otimes_{hs} S_n'(\mathbb{R}) \to S_{2n+\epsilon}'(\mathbb{R}^2)$. Proposition 3.3 below gives us the relationship between these processes. To state it, we need some facts from the theory of distributions. Let $\mathcal{E}(\mathbb{R}^d)$ denote the linear space of C^∞ - functions with the topology of uniform convergence of functions and their derivatives on compact sets. Let $\mathcal{E}'(\mathbb{R}^d)$ denote its dual. It is well known (see [30]) that $\mathcal{E}'(\mathbb{R}^d)$ consists of the tempered distributions with compact support. Let $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be the map

$\sigma(x,\ y) = x + y$. This induces a map, again denoted by $\sigma : \mathcal{E}(I\!R) \to \mathcal{E}(I\!R^2)$ given by $\sigma\ f(x,\ y) = f \circ \sigma(x,\ y) = f(x+y)$. Clearly the map σ is continuous. Let $\sigma' : \mathcal{E}'(I\!R^2) \to \mathcal{E}'(I\!R^1)$ denote its transpose. Recall that $\sigma'(S \otimes T) = S * T$ where $S,\ T \in \mathcal{E}'(I\!R)$ and $*$ denotes convolution.

Proposition 3.3. Let (X_t), (M_t), $(L_t(X))$ and $(L_t(M))$ be as above. Then

$$\forall\, t,\ \ L_t(X) = \sigma' \circ\ T_{n,2n+\epsilon}\left(\int_0^t L_{ds}(M) \otimes_{hs} \delta_{X_o+V_s}\right) \qquad \text{a. s.}$$

where the equality holds in $S'(I\!R)$.

Proof. It is easy to see using Lemma 3.1, eqn. (7) that $T_{n,2n+\epsilon}(\int_0^t L_{ds}(M) \otimes_{hs} \delta_{X_o+V_s})$ has compact support for each t, almost surely. Hence if $f \in \mathcal{E}(I\!R)$ then

$$\begin{aligned}
< f,\ \sigma' \circ\ T_{n,2n+\epsilon}\left(\int_0^t L_{ds}(M) \otimes_{hs} \delta_{X_o+V_s}\right) > &= < \sigma f,\ \int_0^t \delta_{(M_s,\ X_o+V_s)}\ d < M >_s> \\
&= \int_0^t < \sigma f,\ \delta_{(M_s,\ X_o+V_s)} >\ d < M >_s \\
&= \int_o^t \sigma f(M_s,\ X_o + V_s)\ d < M >_s \\
&= \int_0^t f(X_o + M_s + V_s)\ d < M >_s \\
&= \int_0^t f(X_s)\ d < X >_s \\
&= < f,\ \int_0^t \delta_{X_s}\ d < X >_s> \\
&= < f,\ L_t(X) >
\end{aligned}$$

□

Remark 3.4. Let $\{\Delta_n\}$ be a sequence of partitions of $[0,\ t]$ with $\|\Delta_n\| = \sup_i \mid t_i^n - t_{i-1}^n \mid\to 0$ as $n \to \infty$. The following computations show that the above result is consistent with those obtained in [24] and [26] for point local times :

$$\begin{aligned}
L_t(X) &= \sigma' \circ\ T_{n,2n+\epsilon}\left(\lim_{n\to\infty} \sum_{t_i \in \Delta_n} (L_{t_{i+1}\wedge t}(M) - L_{t_i \wedge t}(M)) \otimes_{hs} \delta_{X_o+V_{t_i}}\right) \\
&= \lim_{n\to\infty} \sum_{t_i \in \Delta_n} \left(L_{t_{i+1}\wedge t}(M) - L_{t_i \wedge t}(M)\right) * \delta_{X_o+V_{t_i}} \\
&= \lim_{n\to\infty} \sum_{t_i \in \Delta_n} \tau_{X_o+V_{t_i}} \left(L_{t_{i+1}\wedge t}(M) - L_{t_i \wedge t}(M)\right)
\end{aligned}$$

Note that for $x \in I\!R$, the distribution $\tau_{-x} L_t(M)$ is given by the locally integrable function $y \to L_t^{x+y}(M)$ where $L_t^x(M)$ are the point local times of $(M_t)_{t\geq 0}$ (See comments preceding proposition 3.3).

Proposition 3.5. Suppose $X_t = M_t + N_t$ where (M_t) and (N_t) are continuous orthogonal martingales. Then $\forall\ t,\ \ L_t(X) = \sigma' \circ\ T_{n,2n+\epsilon}(\int_0^t L_{ds}(M) \otimes \delta_{N_s} + \int_0^t L_{ds}(N) \otimes \delta_{M_s})$ a.s.

Proof. The proof of this proposition is the same as that of Proposition 3.3, using the fact that $< X >_t = < M >_t + < N >_t$. □

Remark 3.6. If $M_t = M_t^c + M_t^d$ is the decomposition of a local martingale (M_t) into the continuous and purely discontinuous parts, then we can show, as above, that

$$\forall\, t, \quad L_t(M) = \sigma' \circ T_{n,2n+\epsilon}\Big(\int_0^t L_{ds}(M^c) \otimes_{hs} \delta_{M_s^d}\Big).$$

We now consider the relationship between $(L_t(X))$ and $(L_t(Y))$ where (X_t) is a continuous semi-martingale and (Y_t) is given by $Y_t = \int_0^t h_s \, dX_s$, where (h_s) is locally bounded previsible process. Let n be such that $L_t(X) \in S_n'$ and let $T_{n,2n+\epsilon} : S_n'(\mathbb{R}) \otimes_{hs} S_n'(\mathbb{R}) \to S_n'(\mathbb{R}^2)$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the map $\pi(x,\, y) = y$. This induces a map, again denoted by $\pi : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R}^2)$ by $\pi f(x,\, y) = f \circ \pi(x,\, y) = f(y)$, $f \in \mathcal{E}(\mathbb{R})$. Let $\pi' : \mathcal{E}'(\mathbb{R}^2) \to \mathcal{E}'(\mathbb{R})$ denote the transpose of π.

Proposition 3.7. Let (X_t) , (h_s) and (Y_t) be as above. Then $\forall\, t$, $L_t(Y) = \pi' \circ T_{n,2n+\epsilon}\left(\int_0^t L_{ds}(X) \otimes_{hs} h_s^2 \delta_{Y_s}\right)$ a.s.

where $h_s^2 \delta_{Y_s}$ is the product of the distribution $\delta_{Y_s(\omega)}$ with the scalar $h_s^2(\omega)$.

Proof. Let $t > 0$ and $f \in \mathcal{E}(\mathbb{R})$. Then a.s.,

$$\begin{aligned}
< f,\, L_t(Y) > &= \int_0^t f(Y_s)\, d < Y >_s \\
&= \int_0^t h_s^2 f(Y_s)\, d < X >_s \\
&= \int_0^t h_s^2 \pi f(X_s,\, Y_s)\, d < X >_s \\
&= \int_0^t h_s^2 < \pi f,\, \delta_{X_s} \otimes_d \delta_{Y_s} > \, d < X >_s \\
&= \int_0^t < \pi f,\, \delta_{X_s} \otimes_d h_s^2 \delta_{Y_s} > \, d < X >_s \\
&= < \pi f,\, T_{n,2n+\epsilon}\Big(\int_0^t L_{ds}(X) \otimes_{hs} h_s^2 \delta_{Y_s}\Big) > \\
&= < f,\, \pi' \circ T_{n,2n+\epsilon}\Big(\int_0^t L_{ds}(X) \otimes_{hs} h_s^2 \delta_{Y_s}\Big) > \qquad \Box
\end{aligned}$$

We next take up intersection local times. As mentioned in the introduction, this has been studied by a number of authors. Using proposition 3.8, we show in the example that follows that at the level of distributions, the double intersection local time of two dimensional Brownian motion is an explicit functional (involving tensor products) of the local times of the marginal processes.

In the following we will write an element of $\mathbb{R}^4$ as $(x,\, x',\, y,\, y')$. Let $\alpha : \mathbb{R}^4 \to \mathbb{R}^2$ be the map $\alpha(x,\, x',\, y,\, y') = (x - y,\, x' - y')$. Then α induces a map denoted again by α, $\alpha : \mathcal{E}(\mathbb{R}^2) \to \mathcal{E}(\mathbb{R}^4)$ where for $f \in \mathcal{E}(\mathbb{R}^2)$,

$$(\alpha f)(x,\, x',\, y,\, y') = f \circ \alpha(x,\, x',\, y,\, y') = f(x - y,\, x' - y').$$

Let $\alpha' : \mathcal{E}'(\mathbb{R}^4) \to \mathcal{E}'(\mathbb{R}^2)$ denote its transpose. Let (X_t) and (Y_t) be two continuous semi-martingales and $(L_t(X))$ and $(L_t(Y))$ their local time processes given by Lemma 2.1, belonging to $S_n'(\mathbb{R})$ for some n. We shall denote by $T_{n,2n+\epsilon}$ the operator $T_{n,2n+\epsilon} : S_n'(\mathbb{R}) \otimes_{hs} S_n'(\mathbb{R}) \to S_{2n+\epsilon}'(\mathbb{R}^2)$ as well as the operator $T_{n,2n+\epsilon}$: $\otimes^4_{1,\ hs} S_n'(\mathbb{R}) \to S_{2n+\epsilon}'(\mathbb{R}^4)$ given by lemma 1.3(c).

Proposition 3.8.

a) $\forall\, t$, $\int_0^t \int_0^t \delta_{(X_s,\ Y_u)}\, d < X >_s d < Y >_u = T_{n,2n+\epsilon}(L_t(X) \otimes_{hs} L_t(Y))$ a.s. where equality holds in $\mathcal{E}'(\mathbb{R}^2)$.

b) If (X_s'), (Y_s') are arbitrary locally bounded processes then, in $\mathcal{E}'(\mathbb{R}^4)$, $\forall\, t > 0$,

$$\int_0^t \int_0^t \delta_{(X_s,\ X_s',\ Y_u,\ Y_u')}\, d < X >_s d < Y >_s$$
$$= T_{n,2n+\epsilon}((\int_0^t L_{ds}(X) \otimes_{hs} \delta_{X_s'}) \otimes_{hs} (\int_0^t L_{ds}(Y) \otimes_{hs} \delta_{Y_s'}))\ \text{a.s.}$$

c) If (X_s'), (Y_s') are as in b) then $\forall\, t$,

$$\int_0^t \int_0^t \delta_{(X_s - Y_u,\ X_s' - Y_u')}\, d < X >_s d < Y >_u$$
$$= \alpha' \circ T_{n,2n+\epsilon}\{(\int_0^t L_{ds}(X) \otimes_{hs} \delta_{X_s'}) \otimes_{hs} (\int_0^t L_{du}(Y) \otimes_{hs} \delta_{Y_u'})\} \quad \text{a.s.}$$

where the equality holds in $\mathcal{E}'(\mathbb{R}^2)$.

Proof. a) This follows from the occupation density formula. If $f \in \mathcal{E}(\mathbb{R}^2)$ then

$$\begin{aligned}
\int_0^t \int_0^t f(X_s,\ Y_u)\, d < X >_s d < Y >_u &= \int_0^t < f(.,\ Y_u),\ L_t(X) > d < Y >_u \\
&= \int dy L_t^y(Y) < f(.,\ y),\ L_t(X) > \\
&= \int \int dy\ dx L_t^y(Y) L_t^x(X) f(x,\ y) \\
&= < f,\ L_t(X) \otimes_d L_t(Y) > \\
&= < f,\ T_{n,2n+\epsilon}(L_t(X) \otimes_{hs} L_t(Y)) >
\end{aligned}$$

b) Let $f = f_1 \otimes f_2$ where $f_i \in \mathcal{E}(\mathbb{R}^2)$ $\qquad i = 1,\ 2$

$$\int_0^t \int_0^t f(X_s,\ X_s',\ Y_u,\ Y_u')\, d < X >_s d < Y >_u$$
$$= < f_1,\ T_{n,2n+\epsilon}(\int_0^t L_{ds}(X) \otimes_{hs} \delta_{X_s'}) > \ \times\ < f_2,\ T_{n,2n+\epsilon}(\int_0^t L_{ds}(Y) \otimes_{hs} \delta_{Y_s'}) >$$
$$= < f_1 \otimes f_2,\ T_{n,2n+\epsilon}((\int_0^t L_{ds}(X) \otimes_{hs} \delta_{X_s'}) \otimes_{hs} (\int_0^t L_{ds}(Y) \otimes_{hs} \delta_{Y_s'})) >$$

Since finite linear combinations of functions of the form $f_1 \otimes f_2$ are dense in $\mathcal{E}(\mathbb{R}^4)$, the result follows for all $f \in \mathcal{E}(\mathbb{R}^4)$.

c) Let $f \in \mathcal{E}(\mathbb{R}^2)$. Using b) we have,

$$< f,\ \alpha' \circ T_{n,2n+\epsilon}\left\{\left(\int_0^t L_{ds}(X) \otimes_{hs} \delta_{X_s'}\right) \otimes_{hs} \left(\int_0^t L_{ds}(Y) \otimes_{hs} \delta_{Y_s'}\right)\right\} >$$
$$= \int_0^t \int_0^t \alpha f(X_s,\ X_s',\ Y_u,\ Y_u')\, d < X >_s d < Y >_u$$
$$= \int_0^t \int_0^t f(X_s - Y_u,\ X_s' - Y_u')\, d < X >_s d < Y >_u$$
$$= < f\ ,\quad \int_0^t \int_0^t \delta_{(X_s - Y_u,\ X_s' - Y_u')}\, d < X >_s d < Y >_u > . \qquad \square$$

Example 3.9. We take $\boldsymbol{X}(s) = (X_s^1,\ X_s^2)$ a 2-dimensional Brownian motion, $\boldsymbol{X}(0) \equiv 0$. In [31], [15], it is shown that there exists an $L^2(I\!R^2)$ valued random variable $\tilde{L}(x)$ (the intersection local time) which is almost surely, continuous in x if $x \neq 0$, such that for $f \in \mathcal{E}(I\!R^2)$,

$$\int_0^1 \int_0^1 f(\boldsymbol{X}_s - \boldsymbol{X}_u)\, ds\, du = \int_{I\!R^2} f(x) \tilde{L}(x)\, dx \qquad \text{a.s.}$$

Hence in the sense of distributions a.s.

$$\int_0^1 \int_0^1 \delta_{\boldsymbol{X}_s - \boldsymbol{X}_u}\, ds\, du = \tilde{L}(.)$$

From Proposition 3.8c, with $(X_s,\ X_s') \equiv (Y_s,\ Y_s') \equiv \boldsymbol{X}(s)$, it follows that, $\tilde{L}(.)$ is determined by the local times of the independent Brownian motions (X_s^1) and (X_s^2) by the formula,

$$\tilde{L}(.) = (\alpha' \circ T_{n,2n+\epsilon}) \left\{ \int_0^1 L_{ds}(X^1) \otimes_{hs} \delta_{X_s^2} \otimes_{hs} \int_0^1 L_{du}(X^1) \otimes_{hs} \delta_{X_u^2} \right\} \qquad \square$$

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Note from the Rédaction

A referee has pointed out the following reference, containing some independent work, that was suggested by M. Motoo to T. Nakajima, on the same subject:

Nakajima, T.: A certain classs of distribution-valued additive functionals I - for the case of Brownian motion. J. Math. Kyoto Univ., to appear.