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Quantum stochastic calculus for the uniform measure and Boolean convolution

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Abstract

We study a subspace of the Fock space, called Boolean Fock space, and its associated non-commutative processes obtained by combinations of annihilators and creators. These processes include the Boolean Brownian and Poisson processes obtained by replacing the classical convolution by its Boolean counterpart, and a family of Bernoulli processes. Using a quantum stochastic calculus constructed by time changes, we complete the existing non-commutative relations between basic probability laws. In particular the uniform distribution has the role played by the exponential law in the classical setting of tensor independence.

Key words: Quantum stochastic calculus, Boolean independence, uniform measure.
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1 Introduction

The Brownian and Poisson processes can be realized as operator processes on the symmetric Fock space, the classical notion of independence of increments being expressed in Fock space using tensor products. In non-commutative probability, two other definitions of independence and convolution are available, namely the free and Boolean independence, cf. [2], [3], [13], [15], [16]. Each definition yields another notion of Brownian motion and Poisson process, which can be realized on different forms of the Fock space, namely the full Fock space in the case of free convolution, cf. [14]. The interest in the Boolean convolution is to provide a simple model to illustrate the free case, and the Boolean analogs of Brownian motion and the Poisson processes can be used to approximate their classical counterparts. The aim of this paper is twofold.
(i) We realize the Boolean Brownian motion and Poisson process on a subspace of the symmetric Fock space, which will be called Boolean Fock space. Such processes have no classical versions, however we show that the Boolean Fock space can be identified to the L^2 space of a classical Bernoulli process, obtained itself by combinations of creation and annihilation operators.
(ii) Poisson random variables can be constructed non-commutatively by addition of the conservation (or number) operator to Gaussian random variables. On the other hand, it has been shown in [10] that the geometric law can be obtained in a similar

way from the exponential law, using a construction of quantum stochastic calculus based on time changes. We show that in the Boolean setting, the uniform density plays the role of exponential density, i.e. the Gaussian, exponential and uniform laws can be respectively linked to the Poisson, geometric and Bernoulli laws in a unified non-commutative framework.

We proceed with a more detailed description of the main results. Let ρ be one of the probability densities

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad \rho(x) = e^{-x} 1_{[0,\infty[}(x), \quad \rho(x) = \frac{1}{2} 1_{[-1,1]}(x),$$

$x \in \mathbb{R}$. The Gram-Schmidt orthogonalization procedure defines three families of orthogonal polynomials, respectively the Hermite, Laguerre and Legendre polynomials, which satisfy the differential equation

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0, \quad \lambda \in \mathbb{N}, \tag{1.1}$$

with respectively $(\sigma(x), \tau(x)) = (1, -x)$, $(\sigma(x), \tau(x)) = (x, 1-x)$, $(\sigma(x), \tau(x)) = (1-x^2, -2x)$, cf. [8].

For each choice of the probability density ρ, we can form a Banach space of sequences $B = \mathbb{R}^\infty$ with a measure P denoted formally by $dP = d\rho^{\otimes\infty}$ which is the completion of a measure defined on cylinder sets. Denote by $\theta_k : B \to \mathbb{R}$, $k \in \mathbb{N}$, the coordinate functionals, which are independent random variables distributed according to $d\rho$, and by $D : L^2(B) \longrightarrow L^2(B) \otimes l^2(\mathbb{N})$ the densely defined and closable gradient operator defined as

$$Df(\theta_0, \ldots, \theta_n) = (\partial_k f(\theta_0, \ldots, \theta_n))_{k \in \mathbb{N}}, \quad n \in \mathbb{N}.$$

For each density function ρ, a gradient operator $\tilde{D} : L^2(B) \to L^2(B) \otimes L^2(\mathbb{R}_+)$ is defined by composition of D with a random injection $i : L^2(B) \otimes l^2(\mathbb{N}) \longrightarrow L^2(B) \otimes L^2(\mathbb{R}_+)$, see Relation (5.2) below. This operator is closable and admits a closable adjoint $\tilde{\delta}$, cf. [10], [11]. A family $\{\tilde{a}_u^-, \tilde{a}_u^\circ, \tilde{a}_u^+\}$ of unbounded operators on $L^2(B)$ is defined as

$$\tilde{a}_u^- F = (\tilde{D}F, u)_2, \quad \tilde{a}_u^+ F = \tilde{\delta}(uF), \quad \tilde{a}_u^\circ F = \tilde{\delta}(u\tilde{D}F),$$

$u \in L^2(B) \otimes L^2(\mathbb{R}_+; \mathbb{C})$, for F in a dense domain. These operators complement the usual triple $\{a_h^-, a_h^\circ, a_h^+\}$, $h \in L^2(\mathbb{R}_+; \mathbb{C})$, of annihilation, creation and number (or conservation) operators on the symmetric Fock space, cf. [7], [9]. We recall below the interpretation of these operators in the tensor case, this paper being concerned with the second part, cf. Sect. 3 and 4, i.e. with the Boolean case which will be shown to correspond to ρ uniform on $[-1, 1]$.

1. Tensor independence. In this case the symmetric Fock space has at least two probabilistic interpretations.

- *Wiener interpretation.* This corresponds to the choice $\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. In this case, $a_u^- = \tilde{a}_u^-$, and $a_u^+ = \tilde{a}_u^+$, and

$$a^-_{1_{[0,t]}} + a^+_{1_{[0,t]}} = \tilde{a}^-_{1_{[0,t]}} + \tilde{a}^+_{1_{[0,t]}}$$

is identified to the classical Brownian motion, and

- *Poisson interpretation.* The classical Poisson process is constructed as $t + a^-_{1_{[0,t]}} + a^+_{1_{[0,t]}} + a^\circ_{1_{[0,t]}}$. Here, ρ is the exponential density $\rho(x) = e^{-x} 1_{[0,\infty[}(x)$ and

$$a_u^- + a_u^+ + a_u^\circ = \tilde{a}_u^- + \tilde{a}_u^+,$$

hence the Poisson process is also given by $\tilde{a}^-_{1_{[0,t]}} + \tilde{a}^+_{1_{[0,t]}}$.

2. Boolean independence. In this case we will use a strict subspace $\Gamma_\gamma(L^2(\mathbb{R}_+))$ of the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$. The Boolean Brownian and Poisson processes are still given by $a^-_{1_{[0,t]}} + a^+_{1_{[0,t]}}$ and $t\varphi + a^-_{1_{[0,t]}} + a^+_{1_{[0,t]}} + a^\circ_{1_{[0,t]}}$, where φ is the vacuum state, but they have no classical interpretation, cf. Prop. 3.3 in Sect. 3. However, with ρ the uniform density $\rho(x) = \frac{1}{2} 1_{[-1,1]}(x)$,

$$a^-_{(1_{[0,t]}-t)} + a^+_{(1_{[0,t]}-t)} + a^\circ_{(1_{[0,t]}-t)}, \quad t \in \mathbb{R}_+,$$

can be identified to a classical Bernoulli process, cf. Prop. 4.1 of Sect. 4.

The following properties 1-4 hold for ρ Gaussian, and from [10] for ρ exponential. Their proof in the uniform case is the other goal of this paper, cf. Sects. 4 and 5.

1. The sum $\left(\tilde{a}^+_{1_{[0,t]}} + \tilde{a}^-_{1_{[0,t]}}\right)_{t\in\mathbb{R}_+}$ can be identified to the classical process (Brownian, compensated Poisson or Bernoulli) associated to the sequence $(\tau_k)_{k\in\mathbb{N}}$, see Cor. 5.1. In the uniform case we obtain in particular the identity

$$a^-_{(1_{[0,t]}-t)} + a^+_{(1_{[0,t]}-t)} + a^\circ_{(1_{[0,t]}-t)} = \tilde{a}^-_{1_{[0,t]}} + \tilde{a}^+_{1_{[0,t]}}, \quad t \in \mathbb{R}_+.$$

2. The sum $\tilde{a}^+_{i(e_k)} + \tilde{a}^-_{i(e_k)}$ equals the classical random variable $\tau(\theta_k)$ of Eq. 1.1, which has respectively a Gaussian, exponential or uniform distribution, cf. Relation (5.4).

3. Let $\mathrm{i} = \sqrt{-1}$. The operator $\tilde{a}^\circ_{i(e_k)} + \mathrm{i}s\tilde{a}^+_{i(e_k)} - \mathrm{i}s\tilde{a}^-_{i(e_k)} + s^2\sigma(\theta_k)$, $s \in \mathbb{R}\setminus\{0\}$, has a discrete probability law μ, namely a Poisson or geometric law, respectively for ρ Gaussian and exponential. If ρ is the uniform density, we show in Sect. 5 that this distribution μ is given as

$$\mu\left(\{n(n+1)\}\right) = \frac{2n+1}{2s}\pi(\mathcal{J}_{n+1/2}(s))^2, \quad n \in \mathbb{N}, \tag{1.2}$$

$\mathcal{J}_\nu$ being the Bessel function of the first kind, $\nu \in \mathbb{R}_+$, cf. Prop. 5.2 of Sect. 5.

4. The commutator $[\tilde{a}^+_{i(e_k)}, \tilde{a}^-_{i(e_k)}]$ equals (the multiplication operator by) the random variable $\sigma(\theta_k)$ of (1.1), cf. Lemma 5.2.

In Sect. 2 we recall the definitions of Boolean independence and convolution according to [3], [13], [15]. In Sect. 3 we construct the Boolean Brownian motion and Poisson process. In Sect. 4 we show that a classical Bernoulli process can be also constructed by combining the annihilation and creation operators. In this interpretation, the Boolean Fock space is identified to the L^2 space of a countable product of copies of the uniform density. In Sect. 5 we introduce the operators $\tilde{a}^-_h$, $\tilde{a}^\circ_h$, $\tilde{a}^+_h$, defined by infinitesimal perturbations of jump times, and we link the uniform density to the discrete distribution of Relation (1.2). In Sect. 6, we study the corresponding continuous time construction of quantum stochastic calculus, in which iterated integrals of adapted integrands turn out to be anticipating.

2 Boolean independence and convolution

In this section we recall the basic definitions of Fock space and Boolean independence. Let $L^2(\mathbb{R}_+) = L^2(\mathbb{R}_+;\mathbb{C})$, let $(\cdot,\cdot)_2$ and $|\cdot|_2$ denote the Hermitian product and norm on $L^2(\mathbb{R}_+)$, while $|z|$ denote the modulus of $z \in \mathbb{C}$. Let $\Gamma(L^2(\mathbb{R}_+))$ denote the symmetric Fock space over $L^2(\mathbb{R}_+)$, with its gradient and divergence operators $\nabla^- : \Gamma(L^2(\mathbb{R}_+)) \longrightarrow \Gamma(L^2(\mathbb{R}_+)) \otimes L^2(\mathbb{R}_+)$ and

$$\nabla^+ : \Gamma(L^2(\mathbb{R}_+)) \otimes L^2(\mathbb{R}_+) \longrightarrow \Gamma(L^2(\mathbb{R}_+))$$

defined by linearity and polarization and density as

$$\nabla^-(h_1 \circ \cdots \circ h_n) = \sum_{k=1}^{k=n} \left(h_1 \circ \cdots \circ \hat{h}_k \circ \cdots \circ h_n\right) \otimes h_k,$$

where "$\hat{h}_k$" denotes the omission of h_k in the product, and

$$\nabla^+(f_1 \circ \cdots \circ f_n \otimes g) = f_1 \circ \cdots \circ f_n \circ g,$$

$f_1, \ldots, f_n, g \in L^2(\mathbb{R}_+)$.

Definition 2.1 *Let $\mathcal{S}$ denote the linear space, dense in $\Gamma(L^2(\mathbb{R}_+))$, generated by vectors of the form $h_1 \circ \cdots \circ h_n$, $h_1, \ldots, h_n \in L^2(\mathbb{R}_+)$, $n \in \mathbb{N}$.*

The annihilation, creation and conservation operators a^-_u, a°_u and a^+_u, $u \in L^2(\mathbb{R}_+)$, on $\Gamma(L^2(\mathbb{R}_+))$ are defined as

$$a^-_u F = (\nabla^- F, u)_2, \quad a^+_u F = \nabla^+(F \otimes u), \quad a^\circ_u F = \nabla^+(u \nabla^- F), \qquad F \in \mathcal{S}.$$

Definition 2.2 *Let $\mathcal{A}$ denote the set of closable operators X that leave $\mathcal{S}$ invariant, and admit an adjoint denoted by X^* on $\mathcal{S}$.*

Let $\langle\cdot,\cdot\rangle$ denote the Hermitian product on $\Gamma(L^2(\mathbb{R}_+))$, and let Ω denotes the unit vector in $\Gamma(L^2(\mathbb{R}_+))$. We consider the non-commutative probability space $(\mathcal{A},\varphi)$, where $\mathcal{A}$ is the algebra of operators on $\Gamma(L^2(\mathbb{R}_+))$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is the linear functional defined as

$$\varphi(X) = \langle X\Omega, \Omega\rangle, \qquad X \in \mathcal{A}.$$

Self-adjoint elements of $\mathcal{A}$ are called non-commutative random variables. We recall the following definition, cf. [15].

Definition 2.3 *Two non-commutative random variables X, Y are said to be Boolean independent if*

$$\varphi(X^{k_1}Y^{k_2}X^{k_3}Y^{k_4}\cdots) = \varphi(X^{k_1})\varphi(Y^{k_2})\varphi(X^{k_3})\varphi(Y^{k_4})\cdots,$$

and

$$\varphi(Y^{k_1}X^{k_2}Y^{k_3}X^{k_4}\cdots) = \varphi(Y^{k_1})\varphi(X^{k_2})\varphi(Y^{k_3})\varphi(X^{k_4})\cdots,$$

for any $k_1 \geq 1,\ k_2 \geq 1,\ k_3 \geq 1,\ k_4 \geq 1, \ldots$.

The distribution μ_X of $X \in \mathcal{A}$ is the linear functional $P \mapsto \varphi(P(X))$ defined on the algebra $\mathbb{C}[X]$ of complex polynomials in one variable.

Definition 2.4 *Let X and Y be Boolean independent, of distributions μ_X and μ_Y. The Boolean convolution of μ_X and μ_Y is defined to be the distribution of $X+Y$, and is denoted as $\mu_X \uplus \mu_Y$.*

The Boolean Gauss law with variance σ^2 and the Boolean Poisson distribution with intensity $\lambda > 0$ are the probability measures

$$\frac{1}{2}\delta_{-\sigma} + \frac{1}{2}\delta_{\sigma} \quad \text{and} \quad \frac{1}{\lambda+1}\left(\delta_0 + \lambda\delta_{\lambda+1}\right),$$

cf. [15].

3 Boolean Fock space, Brownian motion and Poisson process

We now introduce a Boolean Fock space $\Gamma_\gamma(L^2(\mathbb{R}_+))$ with parameter $\gamma > 0$ as a subspace of the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$. To this end we define a Boolean symmetric tensor product. Let

$$Z_n^\gamma = \left\{(t_1,\ldots,t_n) \in \mathbb{R}_+^n \ : \ \left[\frac{t_i}{\gamma}\right] \neq \left[\frac{t_j}{\gamma}\right],\ i \neq j\right\},$$

where $[x]$ denotes the integral part of $x \in \mathbb{R}_+$.

Definition 3.1 *For $f_1, \dots, f_n \in L^2(\mathbb{R}_+)$, let*

$$(f_1 \diamond^\gamma \cdots \diamond^\gamma f_n)(t_1, \dots, t_n) = 1_{Z_n^\gamma}(t_1, \dots, t_n) f_1 \circ \cdots \circ f_n(t_1, \dots, t_n).$$

We denote by $L^2(\mathbb{R}_+)^{\diamond^\gamma n}$ the subspace of $L^2(\mathbb{R}_+)^{\otimes n}$ which is the completion of the vector space generated by

$$\left\{ f_1 \diamond^\gamma \cdots \diamond^\gamma f_n \ : \ f_1, \dots, f_n \in L^2(\mathbb{R}_+) \right\},$$

with respect to the norm

$$\| \cdot \|^2_{L^2(\mathbb{R}_+)^{\diamond^\gamma n}} = n! \ \| \cdot \|^2_{L^2(\mathbb{R})^{\otimes n}}, \quad n \in \mathbb{N},$$

and denote by $\Gamma_\gamma(L^2(\mathbb{R}_+))$ the Boolean Fock space defined as

$$\Gamma_\gamma(L^2(\mathbb{R}_+)) = \bigoplus_{n \in \mathbb{N}} L^2(\mathbb{R}_+)^{\diamond^\gamma n}.$$

For $u \in L^2(\mathbb{R}_+)$, the exponential vector $\xi^\gamma(u)$ is defined as

$$\xi^\gamma(u) = \sum_{n \in \mathbb{N}} \frac{1}{n!} u^{\diamond^\gamma n}.$$

Let $\mathcal{S}_\gamma = \Gamma_\gamma(L^2(\mathbb{R}_+)) \cap \mathcal{S}$, and let $\mathcal{U}$ denote the set of processes of the form

$$u = \sum_{k=1}^{k=n} F_k \otimes h_k, \quad F_1, \dots, F_n \in \mathcal{S}, \quad h_1, \dots, h_k \in L^2(\mathbb{R}_+).$$

Let $\pi^\gamma : \Gamma(L^2(\mathbb{R}_+)) \longrightarrow \Gamma_\gamma(L^2(\mathbb{R}_+))$ denote the orthogonal projection on $\Gamma_\gamma(L^2(\mathbb{R}_+))$, which can be viewed as a conditional expectation.

Definition 3.2 *We define the operators $\nabla^{\gamma-}$, $a_u^{\gamma-}$, $a_u^{\gamma+}$, $a_u^{\gamma\circ}$, resp. $\nabla^{\gamma+}$ on $\mathcal{S}$, resp. $\mathcal{U}$, as*

$$\nabla^{\gamma-} = \nabla^- \circ \pi^\gamma, \qquad \nabla^{\gamma+} = \pi^\gamma \circ \nabla^+,$$

and

$$a_u^{\gamma-} = a_u^- \circ \pi^\gamma, \quad a_u^{\gamma+} = \pi^\gamma \circ a_u^+, \quad a_u^{\gamma\circ} = \pi^\gamma \circ a_u^\circ \circ \pi^\gamma.$$

The operator $\nabla^{\gamma+} : \Gamma(L^2(\mathbb{R}_+)) \otimes L^2(\mathbb{R}_+) \to \Gamma(L^2(\mathbb{R}_+))$ is closable and adjoint of $\nabla^{\gamma-} : \Gamma(L^2(\mathbb{R}_+)) \to \Gamma(L^2(\mathbb{R}_+)) \otimes L^2(\mathbb{R}_+)$:

$$\langle \nabla^{\gamma-} F, u \rangle_{\Gamma(L^2(\mathbb{R}_+)) \otimes L^2(\mathbb{R}_+)} = \langle F, \nabla^{\gamma+}(u) \rangle_{\Gamma(L^2(\mathbb{R}_+))}, \quad F \in \mathcal{S}, \quad u \in \mathcal{U}.$$

The operators $\nabla^{\gamma-}$ and $\nabla^{\gamma+}$ satisfy

$$\nabla^{\gamma-}(h_1 \diamond^\gamma \cdots \diamond^\gamma h_n) = \sum_{k=1}^{k=n} \left(h_1 \diamond^\gamma \cdots \diamond^\gamma \hat{h}_k \diamond^\gamma \cdots \diamond^\gamma h_n \right) \diamond^\gamma h_k,$$

$$\nabla^{\gamma+}(f_1 \diamond^\gamma \cdots \diamond^\gamma f_n \otimes g) = f_1 \diamond^\gamma \cdots \diamond^\gamma f_n \diamond^\gamma g,$$

$f_1, \ldots, f_n, g \in L^2(\mathbb{R}_+)$, and $a_u^{\gamma-}$, $a_u^{\gamma\circ}$ and $a_u^{\gamma+}$, $u \in L^2(\mathbb{R}_+)$ satisfy

$$a_u^{\gamma-}F = \langle \nabla^{\gamma-}F, u\rangle_{\Gamma(L^2(\mathbb{R}_+))\otimes L^2(\mathbb{R}_+)}, \quad a_u^{\gamma+}F = \nabla^{\gamma+}(F \otimes u), \quad a_u^{\gamma\circ}F = \nabla^{\gamma+}(u\nabla^{\gamma-}F),$$

$F \in \mathcal{S}_\gamma$. The next proposition shows in particular that $a_u^{\gamma-} + a_u^{\gamma+}$ has the Boolean Gaussian distribution $\frac{1}{2}\delta_{-\|u\|} + \frac{1}{2}\delta_{\|u\|}$.

Proposition 3.1 *Let $h, u \in L^2([0,\gamma])$ and $\alpha \in \mathbb{C}$ with*

$$|u|_2 = 1 \quad \textit{and} \quad |\alpha|^2 + |h|_2^2 = 1.$$

The law of $a_u^{\gamma-} + a_u^{\gamma+}$ in the state $\alpha\Omega + h$ has support $\{-1, 0, 1\}$, with respective probabilities

$$\frac{1}{2}|\alpha - (u,h)_2|^2, \; |h|_2^2 - |(u,h)_2|^2, \; \frac{1}{2}|\alpha + (u,h)_2|^2.$$

Proof. We determine the action of the Weyl operator $\exp(zi(a_u^{\gamma+} + a_u^{\gamma-}))$, by showing that

$$\begin{aligned} &\exp(zi(a_u^{\gamma+} + a_u^{\gamma-}))(\alpha\Omega + h) \\ &= \; h - u(u,h)_2 + (\alpha\Omega + u(u,h)_2)\cos(z) + \mathrm{i}\,(\alpha u + (u,h)_2\Omega)\sin(z), \quad z \in \mathbb{R}. \end{aligned}$$

For this we compute by induction:

$$\left(a_u^{\gamma+} + a_u^{\gamma-}\right)^n h = \begin{cases} h, & n = 0, \\ u(u,h)_2, & n = 2k > 0, \\ (u,h)_2\Omega, & n = 2k+1 \geq 1, \end{cases}$$

and

$$\left(a_u^{\gamma+} + a_u^{\gamma-}\right)^n \Omega = \begin{cases} \Omega, & n = 2k \geq 0, \\ u, & n = 2k+1 \geq 1, \quad k \in \mathbb{N}. \end{cases}$$

Hence the Fourier transform of $a_u^{\gamma+} + a_u^{\gamma-}$ in the pure state $\alpha\Omega + h$ is given by

$$\begin{aligned} &\langle \exp(\mathrm{i}z(a_u^{\gamma+} + a_u^{\gamma-}))(\alpha\Omega + h), \alpha\Omega + h\rangle \\ &\quad = |h|_2^2 - |(u,h)_2|^2 + \left(|\alpha|^2 + |(u,h)_2|^2\right)\cos(z) + \mathrm{i}((h,u)_2\alpha + (u,h)_2\bar{\alpha})\sin(z), \end{aligned}$$

$z \in \mathbb{R}$. □

The operators π^γ, $a_u^{\gamma-}$, $a_u^{\gamma+}$, $a_u^{\gamma\circ}$, acting on the two-dimensional space $\mathrm{span}(\Omega, u)$ can be respectively represented by the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $a_u^{\gamma+} + a_u^{\gamma-}$, $\mathrm{i}(a_u^{\gamma+} - a_u^{\gamma-})$ and $[a_u^{\gamma-}, a_u^{\gamma+}]$ give a representation of the Pauli matrices σ_x, σ_y, σ_z.

Proposition 3.2 *Let $u \in L^2([0,\gamma])$ be the indicator function of a Borel set of Lebesgue measure $\alpha > 0$. Then*

$$\alpha\pi^\gamma + a_u^{\gamma-} + a_u^{\gamma+} + a_u^{\gamma\circ}$$

has the Boolean Poisson distribution with parameter α, i.e. $\frac{1}{\alpha+1}(\delta_0 + \alpha\delta_{\alpha+1})$.

Proof. Let $X_u = \alpha\pi^\gamma + a_u^{\gamma-} + a_u^{\gamma+} + a_u^{\gamma\circ}$. We have $X_u\Omega = u + \alpha\Omega$ and $X_u h = hu + \alpha(h,u)_2\Omega$, hence $(X_u)^k\Omega = (\alpha+1)^{k-1}(u+\alpha\Omega)$, $(X_u)^k u = (\alpha+1)^{k-1}(u+\alpha\Omega)$, $k \geq 1$, which implies

$$\varphi(e^{ziX_u}) = \sum_{k=0}^{\infty} \frac{(z1)^k}{k!}\varphi(X_u^k) = \alpha \sum_{k=0}^{\infty} \frac{(z1)^k}{k!}(\alpha+1)^{k-1} = \frac{\alpha}{\alpha+1}e^{iz(\alpha+1)},$$

which is the characteristic function of $\frac{1}{\alpha+1}(\delta_0 + \alpha\delta_{\alpha+1})$. □

We define the processes $(a_t^{\gamma-})_{t\in\mathbb{R}_+}$, $(a_t^{\gamma+})_{t\in\mathbb{R}_+}$, $(a_t^{\gamma\circ})_{t\in\mathbb{R}_+}$ by

$$a_t^{\gamma-} = a_{1_{[0,t]}}^{\gamma-}, \quad a_t^{\gamma+} = a_{1_{[0,t]}}^{\gamma+}, \quad a_t^{\gamma\circ} = a_{1_{[0,t]}}^{\gamma\circ}, \quad t \in \mathbb{R}_+.$$

The following result, combined to Props. 3.1 and 3.2, shows that $(a_t^{\gamma-} + a_t^{\gamma+})_{t\in[0,\gamma]}$ is the Boolean analog of Brownian motion, and that $(t\pi^\gamma + a_t^{\gamma-} + a_t^{\gamma+} + a_t^{\gamma\circ})_{t\in[0,\gamma]}$ is a realization of the Boolean Poisson process.

Proposition 3.3 *let $u, v \in L^2([0,\gamma])$.*

i) If u, v are orthogonal, then $a_u^{\gamma-} + a_u^{\gamma+}$ is Boolean independent of $a_v^{\gamma-} + a_v^{\gamma+}$.

ii) If u, v are indicator functions with disjoint supports, then $\alpha\pi^\gamma + a_u^{\gamma-} + a_u^{\gamma+} + a_u^{\gamma\circ}$ and $\alpha\pi^\gamma + a_v^{\gamma-} + a_v^{\gamma+} + a_v^{\gamma\circ}$, with $\alpha = \int_0^\infty u(s)ds$ and $\beta = \int_0^\infty v(s)ds$, are Boolean independent.

Proof. i) This property follows from the facts that

$$(a_u^{\gamma-} + a_u^{\gamma+})^k (a_v^{\gamma-} + a_v^{\gamma+})^l \Omega = \begin{cases} \Omega, & k,l \text{ even} \\ u & k \text{ odd}, \ l \text{ even} \\ (u,v)_2 u & k \text{ even}, \ l \text{ odd} \\ (u,v)_2\Omega & k \text{ odd}, \ l \text{ odd}, \end{cases}$$

and

$$(a_u^{\gamma-} + a_u^{\gamma+})^k (a_v^{\gamma-} + a_v^{\gamma+})^l h = \begin{cases} (v,h)_2(u,v)_2 u, & k,l \text{ even} \\ (v,h)_2(u,v)_2\Omega & k \text{ odd}, \ l \text{ even} \\ (v,h)_2\Omega & k \text{ even}, \ l \text{ odd} \\ (v,h)_2 u & k \text{ odd}, \ l \text{ odd}, \end{cases}$$

which imply that

$$\varphi((a_u^{\gamma-} + a_u^{\gamma+})^{k_1}(a_v^{\gamma-} + a_v^{\gamma+})^{k_2}(a_u^{\gamma-} + a_u^{\gamma+})^{k_3}(a_v^{\gamma-} + a_v^{\gamma+})^{k_4}\cdots) = \begin{cases} 1 & k_1, k_2, \ldots \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$ □

ii) The relation $uv = 0$ implies in the notation of the proof of Prop. 3.2:

$$\begin{aligned} \varphi(X_u^{k_1} X_v^{k_2} X_u^{k_3} X_v^{k_4}\cdots) &= \alpha(\alpha+1)^{k_1-1}\beta(\beta+1)^{k_2-1}\alpha(\alpha+1)^{k_3-1}\beta(\beta+1)^{k_4-1}\cdots, \\ &= \varphi(X_u^{k_1})\varphi(X_v^{k_2})\varphi(X_u^{k_3})\varphi(X_v^{k_4})\cdots, \qquad k_1 \geq 1, k_2 \geq 1, \ldots, \end{aligned}$$

hence the Boolean independence of X_u and X_v.

□

Remark 1 *The sequence* $\left(a_t^{\frac{1}{n}-} + a_t^{\frac{1}{n}+}\right)_{n\geq 1}$ *converges to* $a_t^- + a_t^+$ *pointwise on* $\mathcal{S}_\gamma$*, hence in distribution, as* n *goes to* ∞*. Similarly, the sequence*

$$\left(t\pi^{\frac{1}{n}} + a_t^{\frac{1}{n}-} + a_t^{\frac{1}{n}+} + a_t^{\frac{1}{n}\circ}\right)_{n\geq 1}$$

converges to $tI_d + a_t^- + a_t^+ + a_t^\circ$*, pointwise on* $\mathcal{S}_\gamma$ *as* n *goes to* ∞*. Hence the Brownian motion and Poisson process are limits of their Boolean counterparts in the sense of pointwise convergence on* $\mathcal{S}_\gamma$*.*

Due to the non-commutativity of the Boolean independence property, the Boolean Brownian Poisson processes obtained in this way do not have classical realizations. Nevertheless, we show in the next section (Prop. 4.1) that $a_u^{\gamma\circ} + a_u^{\gamma+} + a_u^{\gamma-}$ can be identified to a multiplication operator by a classical random variable.

4 Probabilistic interpretation of $\Gamma_1(L^2(\mathbb{R}_+))$

In the remaining of this paper we set $\gamma = 1$ and write "$\diamond$" instead of "$\diamond^1$". In this section we construct a probabilistic interpretation for the Boolean subspace $\Gamma_1(L^2(\mathbb{R}_+))$ of $\Gamma(L^2(\mathbb{R}_+))$. We show that in this interpretation, a classical Bernoulli process can be constructed from $a_u^{1-} + a_u^{1+} + a_u^{1\circ}$. Consider the space $B = \mathbb{R}^{\mathbb{N}}$ with the metric

$$d(x,y) = \sup_{n\in\mathbb{N}} |x_n - y_n|,$$

and the probability measure defined on cylinder sets as

$$P(\{x \ : \ (x_{k_1},\ldots,x_{k_d}) \in E\}) = \frac{1}{2^d}\int_{E\cap[-1,1]^d} dt_1\cdots dt_d, \quad k_1 \neq \cdots \neq k_d, \ d \in \mathbb{N}.$$

The coordinate functionals

$$\theta_k : B \to \mathbb{R}, \quad k \in \mathbb{N},$$

are independent, uniformly distributed random variables on $[-1,1]$. Let

$$T_k = k + (1+\theta_k)/2, \quad k \in \mathbb{N},$$

be the kth jump time of the point process $(Y(t))_{t\in\mathbb{R}_+}$ defined as

$$Y(t) = \sum_{k\in\mathbb{N}} 1_{[T_k,\infty[}(t), \quad t \in \mathbb{R}_+. \tag{4.1}$$

For bounded $A \in \mathcal{B}(\mathbb{R}_+)$, let

$$\mathcal{F}_A = \sigma\left(\sum_{k=1}^{\infty} 1_O(T_k) \ : \ O \subset A, \ O \in \mathcal{B}(\mathbb{R}_+)\right),$$

and $\mathcal{F}_t = \mathcal{F}_{[0,t]}$, $t \in \mathbb{R}_+$. We define the filtration $(\tilde{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$ as $\tilde{\mathcal{F}}_t = \mathcal{F}_{[0,[t]]}$, $t \in \mathbb{R}_+$, where $[t]$ denotes the integral part of $t \in \mathbb{R}_+$. The compensator $(\nu_t)_{t\in\mathbb{R}_+}$ of $(Y(t))_{t\in\mathbb{R}_+}$ with respect to its natural filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is

$$\sum_{k\geq 0} \frac{1}{k+1-t} 1_{[k,T_k[}(t)dt,$$

cf. [6], and $(Y(t)-t)_{t\geq 0}$ is not a $(\mathcal{F}_t)$-martingale. For $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, denote by $\tilde{I}_n(f_n)$ the $(\tilde{\mathcal{F}}_t)$-adapted iterated stochastic integral with respect to the compensated process $(Y(t) - t)_{t\in\mathbb{R}_+}$:

$$\tilde{I}_n(f_n) = n! \int_0^\infty \int_0^{[t_n]} \cdots \int_0^{[t_2]} f_n(t_1,\ldots,t_n) d(Y_{t_1} - t_1) \cdots d(Y_{t_n} - t_n).$$

Let

$$K = \left\{ f \in L^2(\mathbb{R}_+) \ : \ \int_k^{k+1} f(t)dt = 0, \quad k \in \mathbb{N} \right\},$$

let $K^{\circ n} = L^2(\mathbb{R}_+)^{\circ n} \cap K^{\otimes n}$ equipped with the $L^2(\mathbb{R}_+)^{\circ n}$ norm, and let $\Phi(K)$ be the subspace of $\Gamma_1(L^2(\mathbb{R}_+))$ defined as

$$\Phi(K) = \bigoplus_{n\geq 0} K^{\circ n}.$$

For $f_n \in K^{\circ n}$ we have

$$\tilde{I}_n(f_n) = \sum_{k_1 \neq \cdots \neq k_n} f_n(T_{k_1},\ldots,T_{k_n}) = n! \sum_{k_1 < \cdots < k_n} f_n(T_{k_1},\ldots,T_{k_n}),$$

and

$$E\left[\tilde{I}_n(f_n)\tilde{I}_m(g_m)\right] = 1_{\{n=m\}} n! (f_n, g_m)_{L^2(\mathbb{R}_+)^{\otimes n}}, \quad f_n \in K^{\circ n}, \ g_m \in K^{\circ m}.$$

Consequently, the mapping

$$\begin{aligned} \Psi : \Phi(K) &\longrightarrow L^2(B) \\ f_n &\mapsto \tilde{I}_n(f_n) \end{aligned}$$

is bijective since the set of multiple stochastic integrals is total in $L^2(B)$. The exponential vector $\xi^1(u)$, $u \in K$, is here identified to

$$\xi^1(u) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \tilde{I}_n(u^{\circ n}) = 1 + \sum_{n\geq 1} \sum_{k_1<\cdots<k_n} u(T_{k_1})\cdots u(T_{k_n}) = \prod_{n\in\mathbb{N}} (1 + u(T_n)).$$

Under this identification, any square-integrable $(\tilde{\mathcal{F}}_t)$-adapted process $u \in L^2(B) \otimes L^2(\mathbb{R}_+)$ belongs to $\mathrm{Dom}(\nabla^{1+})$, and

$$\nabla^{1+}(u) = \int_0^\infty u(t)d(Y(t) - t), \tag{4.2}$$

cf. Corollary 1 of [11]. Let $\pi_K : L^2(\mathbb{R}_+) \longrightarrow K$ denote the orthogonal projection on K. The following proposition shows that the process $(a^{1-}_{\pi_K 1_{[0,t]}} + a^{1+}_{\pi_K 1_{[0,t]}} + a^{1\circ}_{\pi_K 1_{[0,t]}})_{t\in\mathbb{R}_+}$ is identified to the classical compensated process $(Y(t)-t)_{t\in\mathbb{R}_+}$. This result corresponds to the fact that the linear combination $x\sigma_x + y\sigma_y + z\sigma_z + t$ can yield all Bernoulli probability laws for $x, y, z, t \in \mathbb{R}_+$, when $a^{1+}_u + a^{1-}_u$, $\mathrm{i}(a^{1+}_u - a^{1-}_u)$ and $[a^{1-}_u, a^{1+}_u]$ are identified with the Pauli matrices σ_x, σ_y, σ_z, acting on $\mathrm{span}(\Omega, u)$.

Proposition 4.1 *Let $u \in K$. The operator $a^{1-}_u + a^{1+}_u + a^{1\circ}_u$ is identified to the multiplication operator on $\Phi(K)\cap\mathcal{S}_1$ by the single stochastic integral $\tilde{I}_1(u)$.*

Proof. The proof follows by application of the following Lemma. □

Lemma 4.1 *Let $u, f \in K$ such that $uf \in L^2(\mathbb{R}_+)$. The multiplication formula for the multiple stochastic integral $\tilde{I}_n(f^{\circ n})$ and $\tilde{I}(u)$ can be stated for $n \geq 1$ as*

$$\tilde{I}_n(f^{\circ n})\tilde{I}_1(u) = \tilde{I}_{n+1}(f^{\circ n}\circ u) + n\tilde{I}_n((uf)\circ f^{\circ(n-1)}) + n\tilde{I}_{n-1}((f^{\circ n}(*,\cdot), u(\cdot))_2).$$

Proof. We have

$$\begin{aligned}
\tilde{I}_n(f^{\circ n})\tilde{I}_1(u) &= \sum_{k_1\neq\cdots\neq k_n, k_{n+1}} f(T_{k_1})\cdots f(T_{k_n})u(T_{k_{n+1}}) \\
&= \sum_{k_1\neq\cdots\neq k_{n+1}} f(T_{k_1})\cdots f(T_{k_n})u(T_{k_{n+1}}) \\
&\quad + n\sum_{k_1\neq\cdots\neq k_n} f(T_{k_1})\cdots f(T_{k_{n-1}})\left(f(T_{k_n})u(T_{k_n}) - \int_{k_n}^{k_n+1} f(t)u(t)dt\right) \\
&\quad + n\sum_{k_1\neq\cdots\neq k_n} f(T_{k_1})\cdots f(T_{k_{n-1}})\int_{k_n}^{k_n+1} f(t)u(t)dt. \quad \square
\end{aligned}$$

5 Quantum stochastic processes in discrete time

In this section we link the uniform distribution to a discrete law (Prop. 5.2) by addition of a number operator defined via a discrete-time quantum stochastic calculus. We start by considering a different approach to non-commutative stochastic calculus, allowing to write the multiplication operator $\int_0^\infty u(t)d(Y(t)-t)$ as a sum of a gradient operator and its adjoint.

Definition 5.1 *Let $\mathcal{P}$ be the set of functionals of the form $f(\theta_0,\ldots,\theta_{n-1})$, f polynomial, $n\geq 1$, and let $\mathcal{V}$ be the set of processes u of the form*

$$u = \sum_{k=1}^{k=n} F_k\otimes u_k, \quad F_1,\ldots,F_n\in\mathcal{P}, \quad u_1,\ldots,u_n\in L^2(\mathbb{R}_+), \quad n\in\mathbb{N}.$$

The sets $\mathcal{P}$ and $\mathcal{V}$ are respectively dense in $L^2(B)$ and in $L^2(B)\otimes L^2(\mathbb{R}_+)$. We now define a gradient operator by perturbation of the jump times of $(Y(t))_{t\in\mathbb{R}_+}$, cf. (4.1), i.e. by differentiation with respect to the coordinate functionals $(\theta_k)_{k\in\mathbb{N}}$. Define

$$\tilde{D} : L^2(B) \longrightarrow L^2(B)\otimes L^2(\mathbb{R}_+)$$

with

$$\tilde{D}f(\theta_0,\dots,\theta_n) = \sum_{k=1}^{n}\left((\theta_k-1)1_{]k,T_k]}(t) + (\theta_k+1)1_{]T_k,k+1]}(t)\right)\partial_k f(\theta_0,\dots,\theta_n),\quad t\in\mathbb{R}_+,$$

cf. Def. 2 and Def. 3 of [11]. The operator $\tilde{D}$ is closable and admits an adjoint $\tilde{\delta} : L^2(B)\otimes L^2(\mathbb{R}_+)\longrightarrow L^2(B)$.

Proposition 5.1 *We have the identity*

$$\tilde{\delta}(v) = \int_0^\infty v(s)d(Y(s)-s) - \int_0^\infty \tilde{D}_s v(s)ds,\qquad v\in\mathcal{V}. \tag{5.1}$$

Proof. cf. Prop. 5 of [11]. □

Consequently, if $v\in L^2(B)\otimes L^2(\mathbb{R}_+)$ is $(\tilde{\mathcal{F}}_t)$-adapted, then $v\in \mathrm{Dom}(\tilde{\delta})\cap\mathrm{Dom}(\nabla^{1+})$ and $\tilde{\delta}(v)$, $\nabla^{1+}(v)$ both coincide with the stochastic integral of v with respect to $(Y(t))_{t\in\mathbb{R}_+}$, compensated with dt:

$$\tilde{\delta}(v) = \nabla^{1+}(v) = \int_0^\infty v(t)d(Y(t)-t).$$

We can now state the definition of the three basic operators.

Definition 5.2 *For $h\in L^2(\mathbb{R}_+)$, define the closable operators $\tilde{a}_h^-$, $\tilde{a}_h^\circ$, $\tilde{a}_h^+$ on $\mathcal{P}$ as*

$$\tilde{a}_h^- F = (\tilde{D}F, h)_2,\quad \tilde{a}_h^\circ F = \tilde{\delta}\left(h(\cdot)\tilde{D}_\cdot F\right),\quad \tilde{a}_h^+ F = \tilde{\delta}(h\otimes F),\qquad F\in\mathcal{P}.$$

The operator $\tilde{a}_h^+$ is adjoint of $\tilde{a}_h^-$ on $\mathcal{P}$ and $\tilde{a}_h^\circ$ is self-adjoint on $\mathcal{P}$. Let $\tilde{a}_t^\varepsilon = \tilde{a}^\varepsilon_{1_{[0,t]}}$, $t\in\mathbb{R}_+$, $\varepsilon = -,\circ,+$.

Corollary 5.1 *The operator $\tilde{a}_t^- + \tilde{a}_t^+$ is the multiplication operator by $Y(t)-t$, $t\in\mathbb{R}_+$, and $\left(\tilde{a}_t^- + \tilde{a}_t^+\right)_{t\in\mathbb{R}_+}$ is identified to the classical process $(Y(t)-t)_{t\in\mathbb{R}_+}$.*

Proof. This follows from Prop. 5.1. □

We define the mapping $i : L^2(B)\otimes l^2(\mathbb{N})\longrightarrow L^2(B)\otimes L^2(\mathbb{R}_+)$ as

$$i_t(u) = \sum_{k=1}^{\infty} u_k\left((\theta_k-1)1_{]k,T_k]}(t) + (\theta_k+1)1_{]T_k,k+1]}(t)\right),\quad t\in\mathbb{R}_+. \tag{5.2}$$

With this notation we have $\tilde{D} = i \circ D$, where $D : L^2(B) \longrightarrow L^2(B) \otimes L^2(\mathbb{N})$ is the discrete-time gradient densely defined as

$$Df(\theta_0, \ldots, \theta_n) = (D_k f(\theta_0, \ldots, \theta_n))_{k\in\mathbb{N}} = (\partial_k f(\theta_0, \ldots, \theta_n))_{k\in\mathbb{N}}, \quad t \in \mathbb{R}_+.$$

The definitions of $\tilde{a}_h^+$, $\tilde{a}_h^-$, $\tilde{a}_h^\circ$ can be extended by letting h equal the random process

$$i(e_k) := \left((\theta_k - 1)1_{]k,T_k]} + (\theta_k + 1)1_{]T_k,k+1]}\right), \quad k \in \mathbb{N},$$

where $(e_k)_{k\in\mathbb{N}}$ denotes the canonical basis of $l^2(\mathbb{N})$.

Proposition 5.2 *Let $s \in \mathbb{R} \setminus \{0\}$. The non-commutative random variable*

$$\tilde{a}^\circ_{i(e_k)} + \mathrm{i}s\tilde{a}^+_{i(e_k)} - \mathrm{i}s\tilde{a}^-_{i(e_k)} + s^2(1 - \theta_k^2)$$

has a discrete distribution μ carried by $\{n(n+1) \ : \ n \in \mathbb{N}\}$ and given by

$$\mu(\{n(n+1)\}) = \pi \frac{n + 1/2}{s} (\mathcal{J}_{n+1/2}(s))^2, \quad n \in \mathbb{N}. \tag{5.3}$$

Here $\mathcal{J}_p$, $p > 0$, denotes the Bessel function of the first kind, defined as

$$\mathcal{J}_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!\Gamma(p+k+1)}, \qquad x \in \mathbb{R}.$$

For the proof of Prop. 5.2 we will need the following Lemmas.

Lemma 5.1 *The operators $\tilde{a}^-_{i(e_k)}$, $\tilde{a}^\circ_{i(e_k)}$, $\tilde{a}^+_{i(e_k)}$ satisfy*

$$\tilde{a}^-_{i(e_k)} f(\theta_k) = (1 - \theta_k^2)\partial f(\theta_k), \quad \tilde{a}^\circ_{i(e_k)} = -(1 - \theta_k^2)\partial^2 f(\theta_k) + 2\theta_k \partial f(\theta_k),$$

and

$$\tilde{a}^+_{i(e_k)} = -(1 - \theta_k^2)\partial f(\theta_k) + 2\theta_k f(\theta_k).$$

Proof. The relation $\tilde{a}^-_{i(e_k)} f(\theta_k) = (1 - \theta_k^2)\partial f(\theta_k)$ follows easily from the definition of $\tilde{D}$ as $\tilde{D} = i \circ D$. Using the duality between $\tilde{D}$ and $\tilde{\delta}$, a one-dimensional integration by parts on $[-1, 1]$ gives $\tilde{a}^\circ_{i(e_k)} = -(1 - \theta_k^2)\partial^2 f(\theta_k) + 2\theta_k \partial f(\theta_k)$. The last relation is obtained from $\tilde{a}^\circ_{i(e_k)} = \tilde{a}^+_{i(e_k)} D_k$, $k \geq 0$. □

Consequently, $\tilde{a}^+_{i(e_k)} + \tilde{a}^-_{i(e_k)}$ is identified to a multiplication operator:

$$\tilde{a}^+_{i(e_k)} + \tilde{a}^-_{i(e_k)} = 2\theta_k, \quad k \in \mathbb{N}, \tag{5.4}$$

and $\tilde{a}^+_{i(e_k)} + \tilde{a}^-_{i(e_k)}$ has a uniform distribution on $[-2, 2]$. Defining the Hermitian operators $Q_h = \tilde{a}_h^+ + \tilde{a}_h^-$, $P_h = \mathrm{i}(\tilde{a}_h^- - \tilde{a}_h^+)$, $P_t = P_{1_{[0,t]}}$, $Q_t = Q_{1_{[0,t]}}$, $t \in \mathbb{R}_+$, we have

$$Q_{i(e_k)} f(\theta_k) = 2\theta_k f(\theta_k), \quad P_{i(e_k)} f(\theta_k) = -\mathrm{i}(-2(1 - \theta_k^2)\partial f(\theta_k) + 2\theta f(\theta_k)).$$

Lemma 5.2 *For $s \in \mathbb{R}$,*

$$\exp\left(\mathrm{i}\frac{s}{2}Q_{i(e_k)}\right)\tilde{a}^{\circ}_{i(e_k)}\exp\left(-\mathrm{i}\frac{s}{2}Q_{i(e_k)}\right) = \tilde{a}^{\circ}_{i(e_k)} - \mathrm{i}s\tilde{a}^{+}_{i(e_k)} + \mathrm{i}s\tilde{a}^{-}_{i(e_k)} + s^2(1-\theta_k^2),$$

and the following commutation relations hold:

$$\left[\tilde{a}^{-}_{i(e_k)}, \tilde{a}^{+}_{i(e_l)}\right] = -2(1-\theta_k^2)1_{\{k=l\}}, \tag{5.5}$$

$$\left[P_{i(e_k)}, Q_{i(e_k)}\right] = 2\mathrm{i}(1-\theta_k^2), \tag{5.6}$$

$$\left[\tilde{a}^{\circ}_{i(e_k)}, Q_{i(e_k)}\right] = \mathrm{i}P_{i(e_k)}, \qquad k, l \in \mathbb{N}. \tag{5.7}$$

Proof. We omit the index k and use Lemma 5.1. We have

$$\begin{aligned}
&\tilde{a}^{\circ}\exp(-\mathrm{i}s\theta)f(\theta) = \left(-(1-\theta^2)\partial_\theta^2 + 2\theta\partial_\theta\right)(f(\theta)\exp(-\mathrm{i}s\theta)) \\
&\quad = \ -(1-\theta^2)(-2\mathrm{i}sf'(\theta)\exp(-\mathrm{i}s\theta) + f''(\theta)\exp(-\mathrm{i}s\theta) - s^2 f(\theta)\exp(-\mathrm{i}s\theta)) \\
&\qquad +2(-\mathrm{i}s\theta f(\theta) + \theta f'(\theta))\exp(-\mathrm{i}s\theta),
\end{aligned}$$

hence

$$\begin{aligned}
\exp(\mathrm{i}s\theta)\tilde{a}^{\circ}\exp(-\mathrm{i}s\theta) &= (-(1-\theta^2)\partial_\theta^2 + 2\theta\partial_\theta)f(\theta) + \mathrm{i}s((1-\theta^2)\partial_\theta - 2\theta)f(\theta) \\
&\quad +\mathrm{i}s((1-\theta^2)\partial_\theta)f(\theta) + s^2(1-\theta^2)f(\theta) \\
&= \tilde{a}^{\circ}f(\theta) - \mathrm{i}s\tilde{a}^{+}f(\theta) + \mathrm{i}s\tilde{a}^{-}f(\theta) + s^2(1-\theta^2)f(\theta).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\tilde{a}^{-}, \tilde{a}^{+}] &= (1-\theta^2)\partial(-(1-\theta^2)\partial) + 2\theta(1-\theta^2)\partial + (1-\theta^2)\partial((1-\theta^2)\partial - 2\theta) \\
&= -(1-\theta^2)^2\partial^2 + 2\theta(1-\theta^2)\partial + 2\theta(1-\theta^2)\partial + (1-\theta^2)^2\partial^2 \\
&\quad -2\theta(1-\theta^2)\partial - 2\theta(1-\theta^2)\partial - 2(1-\theta^2) = -2(1-\theta^2),
\end{aligned}$$

hence (5.5) and (5.6). Concerning (5.7) we have

$$\begin{aligned}
&(-(1-\theta^2)\partial_\theta^2 + 2\theta\partial_\theta)(\theta f(\theta)) - \theta(-(1-\theta^2)\partial_\theta^2 + 2\theta\partial_\theta)f(\theta) \\
&\quad = \ -(1-\theta^2)(2f'(\theta) + \theta f''(\theta)) + 2\theta(f(\theta) + \theta f''(\theta)) + \theta((1-\theta^2)\partial_\theta^2 - 2\theta\partial_\theta)f(\theta) \\
&\quad = \ -2(1-\theta^2)f'(\theta) + 2\theta f(\theta) = \mathrm{i}Pf(\theta). \quad \square
\end{aligned}$$

Proof of Prop. 5.2. Let R_n, $n \geq 0$, be the Legendre polynomial of degree n, which satisfies the differential equation

$$(1-x^2)R_n''(x) - 2xR_n'(x) + n(n+1)R_n(x) = 0, \tag{5.8}$$

and the orthogonality relation

$$\int_{-1}^{1} R_n(x)R_m(x)dx/2 = \frac{1}{2n+1}1_{\{n=m\}}, \quad n, m \in \mathbb{N}.$$

We have

$$\tilde{a}^{\circ}_{i(e_k)} R_n(\theta_k) = \tilde{\delta}\tilde{D}R_n(\theta_k) = n(n+1)R_n(\theta_k), \quad k,n \in \mathbb{N}. \tag{5.9}$$

From Lemma 5.2, the law of $\tilde{a}^{\circ}_{i(e_k)} + \mathrm{i}s\tilde{a}^{+}_{i(e_k)} - \mathrm{i}s\tilde{a}^{-}_{i(e_k)} + s^2(1-\theta_k^2)$ in the vacuum state Ω is the same as the law of $\tilde{a}^{\circ}_{i(e_k)}$ in the state $\exp\left(\mathrm{i}\frac{s}{2}Q_{i(e_k)}\right)\Omega$, cf. [1]. From (5.9), the spectrum of $\tilde{a}^{\circ}$ is $\{n(n+1) \ : \ n \in \mathbb{N}\}$ and the Legendre polynomial R_n is eigenvector for $\tilde{a}^{\circ}$ of even eigenvalue $n(n+1) \in \mathbb{N}$. In order to determine the law of $\tilde{a}^{\circ}$ in the state $\exp(\mathrm{i}sx)$, it is sufficient to decompose $\exp(\mathrm{i}sx)$ into a series of Legendre polynomials. From [12], p. 194, we have

$$\int_{-1}^{1} x^m R_n(x)\frac{dx}{2} = \frac{m!}{(m-n)!!(m+n+1)!!},$$

if $m-n$ is even and $m \geq n$, with

$$p!! = \prod_{0\leq 2k\leq p} (2k), \ p \text{ even, and } p!! = \prod_{0\leq 2k+1\leq p} (2k+1), \ p \text{ odd.}$$

For other values of m, n, the integral is equal to zero. Using Legendre's duplication formula (cf. [4], p. 64):

$$\frac{\Gamma(a)\Gamma(a+1/2)}{\Gamma(2a)} = \frac{\sqrt{\pi}}{2^{2a-1}}, \quad a \in \mathbb{R}_+,$$

where Γ is the Gamma function, it follows:

$$\begin{aligned}
\int_{-1}^{1} e^{\mathrm{i}sy} R_n(y)\frac{dy}{2} &= (\mathrm{i}s)^n \sum_{k\geq 0} \frac{(\mathrm{i}s)^{2k}}{(2k)!!(2k+2n+1)!!} = (\mathrm{i}2s)^n \sum_{k=0}^{\infty} \frac{(\mathrm{i}s)^{2k}(k+n)!}{k!(2n+2k+1)!} \\
&= \sqrt{\pi}(\mathrm{i}2s)^n \sum_{k=0}^{\infty} \frac{(\mathrm{i}s)^{2k}}{2^{2n+2k+1}k!\Gamma(n+k+3/2)!} \\
&= \mathrm{i}^n \sqrt{\frac{\pi}{2s}} \left(\frac{s}{2}\right)^{n+1/2} \sum_{k=0}^{\infty} \frac{(-s^2/4)^k}{k!\Gamma(n+k+3/2)!} = (\mathrm{i})^n \sqrt{\frac{\pi}{2s}} \mathcal{J}_{n+1/2}(s).
\end{aligned}$$

The expansion

$$e^{\mathrm{i}sx} = \sum_{n=0}^{\infty} \left(\sqrt{2n+1}\int_{-1}^{1} e^{\mathrm{i}sy} R_n(y)\frac{dy}{2}\right)\sqrt{2n+1}R_n(x)$$

gives (5.3), since

$$\mu\left(\{n(n+1)\}\right) = \left|\sqrt{2n+1}\int_{-1}^{1} e^{\mathrm{i}sy} R_n(y)\frac{dy}{2}\right|^2. \quad \square$$

6 Quantum stochastic calculus by time changes

In this section, $\Phi(K) = \bigoplus_{n\in\mathbb{N}} K^{\circ n}$ is identified to $L^2(B)$ and we use the decomposition

$$\tilde{a}_t^- + \tilde{a}_t^+ = a^\circ_{\pi_K 1_{[0,t]}} + a^-_{\pi_K 1_{[0,t]}} + a^+_{\pi_K 1_{[0,t]}} = (Y(t) - t)_{t\in\mathbb{R}_+}$$

of $(Y(t) - t)_{t\in\mathbb{R}_+}$ in annihilation parts to construct a non-commutative Itô calculus. For $f \in l^2(\mathbb{N})$ with finite support, define the exponential functional

$$\zeta^1(f) = \exp\left(\sum_{k\in\mathbb{N}} f_k\theta_k\right),$$

and denote by Ξ the vector space generated by such random variables, which is dense in $L^2(B,P)$. Let $A \in \mathcal{B}(\mathbb{R})$. Denote by Ψ_A the set of operators in $\mathcal{A}$ with $\mathcal{P}\bigcup\Xi \subset \mathrm{Dom}(X)$, that can be written as $X \otimes I_d$ on $\Gamma(L^2(A)) \otimes \Gamma(L^2(A^c))$.

Definition 6.1 *A process $(X(t))_{t\in\mathbb{R}_+}$ of operators is said to be $(\tilde{\mathcal{F}}_t)$-adapted if $X(t) \in \Psi_{[0,[t]]}$, $t \in \mathbb{R}_+$.*

We start by defining quantum stochastic integrals of simple adapted processes.

Definition 6.2 *If $(X(t))_{t\in\mathbb{R}_+}$ is a simple adapted process of operators of the form*

$$X(t) = \sum_{i=0}^{i=n} X_i 1_{[i,i+1[}(t), \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N},$$

where $X_i \in \Psi_{[0,i]}$, $i = 0,\ldots,n$, let

$$\int_0^t X(s)d\tilde{a}_s^\varepsilon = \sum_{i=0}^{i=n} X_i \tilde{a}^\varepsilon_{1_{[i\wedge t,(i+1)\wedge t[}}, \quad \varepsilon = -,\circ,+. \tag{6.1}$$

The following proposition extends this definition to non-adapted processes, provided smoothness conditions are satisfied, see also [5].

Proposition 6.1 *If $(X(t))_{t\in\mathbb{R}_+}$ is a process of operators in $\mathcal{A}$, let*

$$\int_0^\infty X(s)d\tilde{a}_s^- F = \int_0^\infty X(s)\tilde{D}_s F ds, \quad \int_0^\infty X(s)d\tilde{a}_s^\circ F = \tilde{\delta}(X(\cdot)\tilde{D}_\cdot F), \tag{6.2}$$

$$\int_0^\infty d\tilde{a}_s^+ X(s)F = \tilde{\delta}(X(\cdot)F), \quad \int_0^\infty d\tilde{a}_s^- X(s)F = \int_0^\infty \tilde{D}_s X(s)F ds, \tag{6.3}$$

provided $XF = (X(t)F)_{t\in\mathbb{R}_+}$ satisfies respectively $X\tilde{D}F \in L^2(B)\otimes L^2(\mathbb{R}_+)$, $X\tilde{D}F \in \mathrm{Dom}(\tilde{\delta})$, $XF = (X(t)F)_{t\in\mathbb{R}_+} \in \mathrm{Dom}(\tilde{\delta})$, $\tilde{D}XF \in L^2(B)\otimes L^2(\mathbb{R}_+)$, $F \in L^2(B)$. These definitions coincide with Def. 6.2 on simple adapted processes.

Proof. We have $D_i = D_i \otimes I$ on $\Gamma(L^2([0,i[)) \otimes \Gamma(L^2([i,\infty[))$, hence $X(s)$ commutes with D_i for $s \in [i, i+1[$, $i \in \mathbb{N}$, and

$$X(s)\tilde{D}_s F = i_s(X(s)DF) = \tilde{D}_s X(s)F, \ a.s., \quad F \in \Xi, \ s \in \mathbb{R}_+, \tag{6.4}$$

hence $X\tilde{D}F$ satisfies the conditions of Def. 6.1, and (6.1) is equivalent to (6.2) and (6.3) on simple $(\tilde{\mathcal{F}}_t)$-adapted processes, from Def. 5.2. □

The integral $\int_0^\infty X(t)d\tilde{a}_t^+$ is defined by duality from $\int_0^\infty d\tilde{a}_t^- X(x)^*$. Conditions for the existence of the stochastic integral of adapted operator processes as an unbounded operator on the vector space Ξ of exponential vectors can be obtained from the next proposition:

Proposition 6.2 *Let $(X(t))_{t\in\mathbb{R}_+}$ be a simple adapted process in $\mathcal{A}$. We have*

$$\langle \zeta(g), \int_0^\infty X(s)d\tilde{a}_s^\varepsilon \zeta(f)\rangle = \langle \zeta(g), \int_0^\infty h_\varepsilon(s)X(s)\zeta(f)ds\rangle,$$

$\zeta(f), \zeta(g) \in \Xi$, $\varepsilon = -, \circ, +$, *with*

$$h_+ = i(\bar{g}), \quad h_- = i(f), \quad h_\circ = i(f\bar{g}). \tag{6.5}$$

This shows that if $(X(t))_{t\in\mathbb{R}_+}$ is an $(\tilde{\mathcal{F}}_t)$-adapted process of operators such that

$$(X(t)\zeta(f))_{t\in\mathbb{R}_+} \in L^2(B) \otimes L^2(\mathbb{R}_+), \quad \forall\, \zeta(f) \in \Xi,$$

then $\int_0^\infty X(s)d\tilde{a}_s^-$ is uniquely densely defined. We have

$$\langle \zeta(g), \int_0^\infty X(s)d\tilde{a}_s^\varepsilon \zeta(f)\rangle = \langle \int_0^\infty X(s)^* d\tilde{a}_s^{*\varepsilon}\zeta(g), \zeta(f)\rangle, \quad \zeta(f),\ \zeta(g) \in \Xi,$$

if $(X(t))_{t\in\mathbb{R}_+}$ and its adjoint $(X(t)^*)_{t\in\mathbb{R}_+}$ are simple adapted processes that satisfy the above conditions, with $*\varepsilon = +, \circ, -$ respectively if $\varepsilon = -, \circ, +$.

Proof. The proof is an application of Relation (6.4) and the fact that $\tilde{D}\zeta(f) = i(f)\zeta(f)$. The last relation is a consequence of the duality relations between $\tilde{a}_u^+$ and $\tilde{a}_u^-$, and of the self-adjointness of $\tilde{a}_u^\circ$, cf. [1], [10] for the analog statements for ρ respectively Gaussian and exponential. □

Proposition 6.3 *Let X, Z be simple $(\tilde{\mathcal{F}}_t)$-adapted processes in $\mathcal{A}$ such that $\Xi \subset$ $\mathrm{Dom}(X(s)Z(s))$, $s \in \mathbb{R}_+$. We have the equality*

$$\begin{aligned}\int_0^t X(s)d\tilde{a}_s^\varepsilon \int_0^t Z(s)d\tilde{a}_s^\eta &= \int_0^t d\tilde{a}_s^\varepsilon X(s)\left(\int_0^s Z(u)d\tilde{a}_u^\eta\right) + \int_0^t \left(\int_0^s X_u d\tilde{a}_u^\varepsilon\right) Z(s)d\tilde{a}_s^\eta \\ &\quad + \int_0^t X(s)Z(s)d\tilde{a}_s^\varepsilon \cdot d\tilde{a}_s^\eta, \end{aligned} \tag{6.6}$$

where the composition of operators holds in the weak sense and the product $d\tilde{a}_s^\varepsilon \cdot d\tilde{a}_s^\eta$ is given by the multiplication table

$\cdot$	$d\tilde{a}_t^+$	$d\tilde{a}_t^-$
$d\tilde{a}_t^+$	0	0
$d\tilde{a}_t^-$	$dY(t)$	0

Proof. The statement of (6.6) in the weak sense means the following identities, which will be proved using the duality between $\tilde{\delta}$ and $\tilde{D}$:

$$\begin{aligned}
&\langle \int_0^t Z(s)d\tilde{a}_s^+G, \int_0^t X(s)^* d\tilde{a}_s^- F\rangle \\
&= \int_0^t \langle \int_0^s Z(u)d\tilde{a}_u^+G, X(s)^*\tilde{D}_sF\rangle ds + \int_0^t \langle Z(s)G, \tilde{D}_s \int_0^s X(u)^* d\tilde{a}_u^- F\rangle ds, \\
&\langle \int_0^t Z(s)d\tilde{a}_s^-G, \int_0^t X(s)^* d\tilde{a}_s^+ F\rangle \\
&= \int_0^t \langle \tilde{D}_sG, \int_0^s Z(u)^* d\tilde{a}_u^+ X(s)^* F\rangle ds + \int_0^t \langle \tilde{D}_s \int_0^s X_u d\tilde{a}_u^- Z(s)G, F\rangle ds, \\
&\langle \int_0^t Z(s)d\tilde{a}_s^+G, \int_0^t X(s)^* d\tilde{a}_s^+ F\rangle \\
&= \int_0^t \langle \tilde{D}_sX(s) \int_0^s Z(u)d\tilde{a}_u^+G, F\rangle ds + \int_0^t \langle Z(s)G, \int_0^s \tilde{D}_sX(u)^* d\tilde{a}_u^+ F\rangle ds, \\
&\quad + \langle \int_0^t X(s)Z(s)GdX(s), F\rangle, \\
&\langle \int_0^t Z(s)d\tilde{a}_s^-G, \int_0^t X(s)^* d\tilde{a}_s^- F\rangle \\
&= \int_0^t \langle \int_0^s Z(u)d\tilde{a}_u^-G, X(s)^*\tilde{D}_sF\rangle ds + \int_0^t \langle Z(s)\tilde{D}_sG, \int_0^s X(u)^* d\tilde{a}_u^- F\rangle ds,
\end{aligned}$$

for $F, G \in \mathcal{P}$. By linearity and adaptedness of X, Z it suffices to prove these relations for $X = Z = 1_{[0,t]}$. We have

$$\begin{aligned}
\langle \tilde{a}_t^- G, \tilde{a}_t^- F\rangle &= \langle \int_0^t \tilde{D}_u G du, \int_0^t \tilde{D}_s F ds\rangle \\
&= \int_0^t \int_0^s \langle \tilde{D}_u G, \tilde{D}_s F\rangle du ds + \int_0^t \int_0^u \langle \tilde{D}_u G, \tilde{D}_s F\rangle du ds \\
&= \langle \tilde{\delta}\left(1_{[0,t]}(\cdot) \int_0^\cdot \tilde{D}_u G du\right), F\rangle + \langle G, \tilde{\delta}\left(1_{[0,t]}(\cdot) \int_0^\cdot \tilde{D}_u F du\right)\rangle \\
&= \langle \int_0^t d\tilde{a}_s^+ \tilde{a}_s^- G, F\rangle + \langle G, \int_0^t d\tilde{a}_s^+ \tilde{a}_s^- F\rangle \\
&= \langle G, \left(\int_0^t \tilde{a}_s^+ d\tilde{a}_s^- + \int_0^t d\tilde{a}_s^+ \tilde{a}_s^-\right) F\rangle, \quad F, G \in \mathcal{P},
\end{aligned}$$

and

$$\begin{aligned}
\langle \tilde{a}_t^+ F, \tilde{a}_t^- G\rangle &= \int_0^t \langle \tilde{a}_s^+ F, \tilde{D}_s G\rangle ds + \langle F, \int_0^t \tilde{a}_s^- G d(Y(s) - s)\rangle - \int_0^t \langle \tilde{D}_s F, \tilde{a}_s^- G\rangle ds \\
&= \int_0^t \langle \tilde{a}_s^+ F, \tilde{D}_s G\rangle ds + \langle F, \int_0^t d\tilde{a}_s^+ \tilde{a}_s^- G\rangle
\end{aligned}$$

$$
\begin{aligned}
& +\langle F, \int_0^t d\tilde{a}_s^- \tilde{a}_s^- G\rangle - \int_0^t \langle \tilde{D}_s F, \tilde{a}_s^- G\rangle ds \\
= & \int_0^t \langle \tilde{a}_s^+ F, \tilde{D}_s G\rangle ds + \langle F, \int_0^t d\tilde{a}_s^- \tilde{a}_s^- G\rangle \\
= & \langle F, \int_0^t \tilde{a}_s^- d\tilde{a}_s^- G\rangle + \langle F, \int_0^t d\tilde{a}_s^- \tilde{a}_s^- G\rangle, \quad F, G \in \mathcal{P}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\langle \tilde{a}_t^+ F, \tilde{a}_t^+ G\rangle = & \int_0^t \langle \tilde{a}_s^+ F d(Y(s) - s), G\rangle - \int_0^t \langle \tilde{a}_s^+ F, \tilde{D}_s G\rangle ds \\
& +\langle F, \int_0^t \tilde{a}_s^+ G d(Y(s) - s)\rangle - \int_0^t \langle \tilde{D}_s F, \tilde{a}_s^+ G\rangle ds + \langle Y(t) F, G\rangle \\
= & \langle \int_0^t d\tilde{a}_s^+ \tilde{a}_s^+ F, G\rangle + \langle \int_0^t d\tilde{a}_s^- \tilde{a}_s^+ F, G\rangle - \int_0^t \langle \tilde{a}_s^+ F, \tilde{D}_s G\rangle ds \\
& +\langle F, \int_0^t d\tilde{a}_s^+ \tilde{a}_s^+ G\rangle + \langle F, \int_0^t d\tilde{a}_s^- \tilde{a}_s^+ G\rangle - \int_0^t \langle \tilde{D}_s F, \tilde{a}_s^+ G\rangle ds + \langle Y(t) F, G\rangle \\
= & \langle \int_0^t d\tilde{a}_s^- \tilde{a}_s^+ F, G\rangle + \langle F, \int_0^t d\tilde{a}_s^- \tilde{a}_s^+ G\rangle + \langle Y(t) F, G\rangle \\
= & \langle \int_0^t d\tilde{a}_s^- \tilde{a}_s^+ F, G\rangle + \langle \int_0^t \tilde{a}_s^- d\tilde{a}_s^+ F, G\rangle + \langle Y(t) F, G\rangle, \quad F, G \in \mathcal{P}. \quad \square
\end{aligned}
$$

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