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GENERICITY IN DETERMINISTIC AND STOCHASTIC DIFFERENTIAL EQUATIONS

J.J. Alibert* and K. Bahlali**

ABSTRACT. We prove that the convergence of the approximation with time delay, as well as pathwise uniqueness, are generic properties in ordinary differential equations as well as in stochastic differential equations. This is done in the case where the coefficients are neither bounded nor time continuous. The approximation with time delay is used to obtain existence of weak solutions for SDE. We also prove L^2-convergence of this approximation when only pathwise uniqueness is assumed.

KEY WORDS. Approximation with time delay, generic property, pathwise uniqness, strong and weak solution.

Introduction.

Let (E, d) be a complete metric space and $F \subset E$. The subset F is said to be meager (or of first category of Baire), if it is contained in a countable union of closed nowhere dense subsets of E. The complement of a meager set is called a residual set (or a set of second category of Baire). Let $(\mathcal{P})$ be a property which is satisfied by some elements of E. $(\mathcal{P})$ is said to be generic if the set $F := \{x \in E : x \text{ satisfies } (\mathcal{P})\}$ is residual. In this case property $(\mathcal{P})$ is said to hold almost surely in the Baire sense. For more details about the categories of sets see e.g [Ox]. In many situations it is not possible to give a complete characterization of the subset F. Then arises the problem to find, for instance, the category of F. Is it of first category or of second? This question is usually studied in the theory of ordinary differential equations (ODE in short), as well as, in stochastic differential equations (SDE in short), ergodic theory, spectral theory of operators, fixed point theorems, points of derivability of continuous functions, etc... (see e.g. [O, H, R, LY, V1, V2, DM1, DM2, DM3, Ku, S2, Z, He, Si, BMO1, BMO2]).

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In the present paper we discuss genericity of pathwise uniqueness and strong existence (via approximation) of the solutions of ODE and SDE, as well as the relations between pathwise uniqueness and convergence of approximation with time delay. The part of our results which concerns genericity is closely related to those of [O, Ox, LY, DM1, DM2, DM3, S2, H, BMO1, BMO2], and improves on them. While the approximation part is related to those of [KN, EO, MB]. The equations under consideration are the following:

$$(E^f) \qquad X^f(t) = 0 \quad (t \leq 0) \quad \text{and} \quad X^f(t) = \int_0^t f\big(s, X^f(s)\big) ds \qquad (t > 0)$$

and

$$(E^\sigma) \qquad X^\sigma(t) = 0 \quad (t \leq 0) \quad \text{and} \quad X^\sigma(t) = \int_0^t \sigma\big(s, X^\sigma(s)\big) dB(s) \quad (t > 0)$$

where f and σ are measurable in t for all $x \in \mathbb{R}^d$, continuous in x for almost all $t \in \mathbb{R}^+$.

Genericity property (or prevalence) seems to be first studied in [O] for ODE. This study has been extended in [LY] and in [DM2] to ODE assuming values in an infinite dimensional Banach space. In SDE, the notion of genericity is used in [S2] to study the dependance on a parameter of solutions. The probabilistic method given in [S2] seems not to be related to those of deterministic equations. The genericity of strong existence and uniqueness of the solution of equation (E^σ) has been discussed in [He] by adapting an idea used in [LY], and, in [BMO1] by adapting the method used in [DM2]. In [BMO1], the genericity of convergence of Picard's approximation as well as of Euler's approximation are studied also. In all the above papers, it is assumed that the coefficients f and/or σ are continuous with respect to their two arguments and uniformly bounded. In [He], the continuity of the coefficients is not assumed, in return the diffusion coefficient σ must be non degenerate.

Here the continuity in the arguments as well as the uniform boundness of the coefficients will be dropped. Only *measurability with respect to the time variable and continuity with respect to the space variable will be imposed on the coefficients.* For example, the coefficient $f(t,x) = t^{-1/2}x$ is not allowed in [LY, DM] and the coefficient $\sigma(t,x) = t^{-1/4}x$ is not allowed in [S2, He, BMO1, BMO2]. Our result covers this example. In our situation, the difficulties stay first in the fact that the coefficients are neither uniformly bounded nor continuous and next in the choice of a convenient space of coefficients in which the subset of locally Lipshitz functions is dense.

In the first part of the paper, we prove that convergence of the approximations with time delay as well as pathwise uniqueness are generic properties in both ODE and SDE. In the second part, we deal with existence of (weak) solution as well as the relation between pathwise uniqueness and L^2-approximation of the solutions of SDE. We show that pathwise uniqueness implies the L^2-convergence of the time delayed processes to the solution of (E^σ). This is done by using a method closely related to that introduced in [KN], where similar result on Euler's approximation is stated. Other results about approximation and stability are given in [KY, GK, BMO2].

The paper is organized as follows. In section 1, we prove that, for an appropriate metric, the set of functions which are measurable in t and locally Lipschitz in x is dense in the set of functions which are measurable in t and continuous in x. In section 2, we prove that for ODE convergence of the approximations with time delay is generic. A similar result is stated in section 3 for SDE. In section 4, weak existence for SDE is proved. It is also established that pathwise uniqueness of the solution of (E^σ) implies L^2-convergence of the delayed processes to the unique solution of (E^σ). As a consequence we give a simple proof of the Yamada-Watanabe theorem about the relation between pathwise uniqueness and strong existence of solutions.

I. Approximation by Lipschitz functions.

For $1 \leq q < \infty$, we denote by $L^q_{loc}(\mathbb{R}^+; C_b(\mathbb{R}^d))$ the set of functions $f = f(t,x)$ from $\mathbb{R}^+ \times \mathbb{R}^d$ into $\mathbb{R}$ which are measurable in t for each $x \in \mathbb{R}^d$, continuous in x for almost every $t \in \mathbb{R}^+$ and such that the function

$$N_0[f](t) := \sup_{x \in \mathbb{R}^d} |f(t,x)|$$

belongs to the Lebesgue space $L^q_{loc}(\mathbb{R}^+)$. For $M > 0$ we set,

$$K[f,M](t) := \sup \left\{ \frac{|f(t,y) - f(t,x)|}{|y-x|} : |x| \leq M, \ |y| \leq M, \ x \neq y \right\}$$

where $| \, . \, |$ denotes the Euclidean norm.

Lemma 1. *Let $f \in L^q_{loc}(\mathbb{R}^+; C_b(\mathbb{R}^d))$ with $1 \leq q < \infty$. For each $\varepsilon > 0$ there exists $f_\varepsilon \in L^q_{loc}(\mathbb{R}^+; C_b(\mathbb{R}^d))$ such that $K[f_\varepsilon, M] \in L^\infty_{loc}(\mathbb{R}^+)$ for every $M > 0$ and*

$$\int_0^\infty N_0^q[f - f_\varepsilon](t)dt < \varepsilon.$$

Proof. Without loss of generality, we assume $f \geq 0$. Let $\{\varphi_n : n \in \mathbb{N}^*\}$ be a locally finite partition of unity in $\mathbb{R}^d$. Given any $k > 0$ we set

$$f_k^{(n)}(t,x) := \inf_{y \in \mathbb{R}^d} \{ \, \varphi_n(y) f(t,y) + k|y-x| \, \}.$$

Since f is measurable in t, continuous in y, the function $f_k^{(n)}$ is measurable in t. The fact that $y \longrightarrow \varphi_n(y) f(t,y)$ is bounded continuous implies that $f_k^{(n)}$ satisfies a Lipschitz condition in x with constant k and $\lim_{k \uparrow \infty} f_k^{(n)}(t,x) = \varphi_n(x) f(t,x)$ for every x and almost every t. Since $f_k^{(n)}$ is nondecreasing in k and φ_n is compactly supported, we deduce from the inequality $0 \leq f_k^{(n)} \leq \varphi_n f$ and the Dini theorem that

$$\lim_{k \uparrow \infty} N_0 \left[\varphi_n f - f_k^{(n)} \right](t) = 0$$

for almost every $t \geq 0$. Let $\varepsilon > 0$ be fixed. Since $0 \leq N_0[\varphi_n f - f_k^{(n)}] \leq N_0[f]$ and

$N_0[f] \in L^q_{loc}(\mathbb{R}^+)$, it follows from the dominated convergence theorem that for each $p \in \mathbb{N}^*$ there exists $k(n,p) > 0$ such that

$$\int_{p-1}^{p} 2^{nq} N_0^q \left[\varphi_n f - f^{(n)}_{k(n,p)}\right](t)dt < \frac{\varepsilon}{C_q 2^{n+p}} \qquad \text{where} \qquad C_q = \left(\sum_{n=1}^{\infty} 2^{\frac{nq}{1-q}}\right)^{q-1}$$

and $C_q = 1$ if $q = 1$. We define f_ε on $\mathbb{R}^+ \times \mathbb{R}^d$ by

$$f_\varepsilon(t,x) := \sum_{n=1}^{\infty} f^{(n)}_{k(n,p)}(t,x) \quad \text{if} \quad (t,x) \in [p-1,p) \times \mathbb{R}^d.$$

Let us prove that $K[f_\varepsilon, M] \in L^\infty_{loc}(\mathbb{R}^+)$ for every $M > 0$. Since the partition is locally finite, for each $M > 0$ there exists $n(M) \in \mathbb{N}^*$ (not depending on p) such that

$$f_\varepsilon(t,x) := \sum_{n=1}^{n(M)} f^{(n)}_{k(n,p)}(t,x) \quad \text{if} \quad (t,x) \in [p-1,p) \times B_M,$$

where B_M is the closed ball in $\mathbb{R}^d$ with radius M. Then we have

$$|f_\varepsilon(t,y) - f_\varepsilon(t,x)| \le \left(\sum_{n=1}^{n(M)} k(n,p)\right)|y-x| \quad \text{if} \quad (t,x,y) \in [p-1,p) \times B_M \times B_M.$$

This implies that $K[f_\varepsilon, M](t) \le \max\{\sum_{n=1}^{n(M)} k(n,i) : 1 \le i \le p\}$ for every $p \in \mathbb{N}^*$ and $t \in [0,p)$. Hence $K[f_\varepsilon, M] \in L^\infty_{loc}(\mathbb{R}^+)$. The fact that $f_\varepsilon \in L^q_{loc}(\mathbb{R}^+; C_b(\mathbb{R}^d))$ follows from the inequality $0 \le f_\varepsilon \le f$. Now we use the Hölder inequality to get

$$\begin{aligned}
\int_0^\infty N_0^q[f - f_\varepsilon](t)dt &= \sum_{p=1}^{\infty} \int_{p-1}^{p} N_0^q\left[\sum_{n=1}^{\infty} \left(\varphi_n f - f^{(n)}_{k(n,p)}\right)\right](t)dt \\
&\le \sum_{p=1}^{\infty} \int_{p-1}^{p} \left(\sum_{n=1}^{\infty} N_0\left[\varphi_n f - f^{(n)}_{k(n,p)}\right](t)\right)^q dt \\
&\le C_q \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \int_{p-1}^{p} 2^{nq} N_0^q\left[\varphi_n f - f^{(n)}_{k(n,p)}\right](t)dt \\
&< C_q \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varepsilon}{C_q 2^{n+p}} = \varepsilon.
\end{aligned}$$

Lemma 1 is proved. □

For $1 \le q < \infty$ we denote by $L^q_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big)$ the set of functions $f = f(t,x)$, defined on $\mathbb{R}^+ \times \mathbb{R}^d$ with values in $\mathbb{R}^d$, which are measurable in t for each $x \in \mathbb{R}^d$, continuous in x for almost every $t \in \mathbb{R}^+$ and such that the function

$$N_1[f](t) := \sup_{x\in\mathbb{R}^d} \frac{|f(t,x)|}{1+|x|}$$

belongs to the Lebesgue space $L^q_{loc}(\mathbb{R}^+)$. Endowed with the metric

$$\rho_1(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\left(\int_0^k N_1^q[g-f](t)dt \right)^{1/q}}{1+\left(\int_0^k N_1^q[g-f](t)dt \right)^{1/q}}$$

the space $L^q_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big)$ is a complete metric space. A sequence (f_n) converges to some f in $L^q_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big)$ if and only if $\lim_{n\uparrow\infty} \int_I N_1^q[f_n - f](s)ds = 0$ for every bounded interval I of $\mathbb{R}^+$. Our space of locally Lipschitz functions is defined as follows.

$$Lip_{loc,q} := \{ f \in L^q_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big) : K[M,f] \in L^\infty_{loc}(\mathbb{R}^+) \text{ for every } M > 0 \}.$$

Corollary 2. *If $1 \le q < \infty$ then $Lip_{loc,q}$ is a dense subset of $L^q_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big)$.*

Proof. Let $f \in L^q_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big)$ and $g(t,x) := \big(f(t,x) - f(t,0)\big)/\big(1+|x|\big)$. By Lemma 1 there exists a sequence (g_n) in $L^q_{loc}\big(\mathbb{R}^+; C_b(\mathbb{R}^d;\mathbb{R}^d)\big)$ such that $K[g_n, M] \in L^\infty_{loc}(\mathbb{R}^+)$ for every $M > 0$ and $\lim_{n\uparrow\infty} \int_{\mathbb{R}^+} N_0^q[g_n - g](t)dt = 0$. Let us define

$$f_n(t,x) := f(t,0) + (1+|x|)\big(g_n(t,x) - g_n(t,0)\big).$$

Clearly $f_n \in L^q_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big)$ and for almost every t and every x we have

$$\frac{|f_n(t,x) - f(t,x)|}{1+|x|} \le |g_n(t,x) - g(t,x)| + |g_n(t,0) - g(t,0)|.$$

Hence $N_1[f_n - f](t) \le 2N_0[g_n - g](t)$ which implies that f_n converges to f with respect to the metric ρ_1. For almost every t and every $x, y \in B_M$ we have

$$|f_n(t,y) - f_n(t,x)| \le \big(1+|x|\big)|g_n(t,y) - g_n(t,x)| + \big|\,|y| - |x|\,\big||g_n(t,y) - g_n(t,0)|.$$

Hence $K[f_n, M] \le \big(1+2M\big)K[g_n, M]$ which implies that $f_n \in Lip_{loc,q}$. □

Remark 2. The interest of Lemma 1 and Corollary 2 lies in the fact that the approximation is uniform in $\mathbb{R}^d$ with respect to the space variable and not only uniform on each compact set of $\mathbb{R}^d$.

Proposition 3. *If $1 \leq q < \infty$ then $Lip_{loc,q}$ is meager in $L^q_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$.*

Proof. For each integer p we set

$$\mathcal{L}^q_p := \Big\{ f \in L^q_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d)) : K[f,1](t) \leq p \text{ for } a.e.\ t \in [0,1] \Big\}.$$

Clearly $\mathcal{L}^q_p$ is a closed subset of $L^q_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ and $Lip_{loc,q} \subset \bigcup_p \mathcal{L}^q_p$. Define $\varphi(t,x) = (1/\sqrt{|x|})x$ if $x \neq 0$ and $\varphi(t,0) = 0$. For every $f \in \mathcal{L}^q_p$ the function $f_n := f - (1/n)\varphi$ satisfies $f_n \in L^q_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$, $f_n \notin \mathcal{L}^q_p$ and f_n converges to f with respect to the metric ρ_1. Hence $\mathcal{L}^q_p$ is a nowhere dense subset of $L^q_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$. Proposition 3 is proved. □

II. Some properties of approximation with delay in ODE.

Given $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ we denote by $\{X^f_r\}_{r>0}$ the collection defined by

$$(E^f_r) \quad X^f_r(t) = 0 \quad (t \leq 0) \quad \text{and} \quad X^f_r(t) = \int_0^t f(s, X^f_r(s-r))ds \quad (t > 0).$$

As easily seen, X^f_r is an absolutely continuous function on every bounded interval of $\mathbb{R}$, i.e. X^f_r belongs to the usual Sobolev space $W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$. Therefore, notions such as "*compactness*" and "*convergence*" are to be understood in the sense of the strong topology of $W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$. Recall that this topology is metrizable and a sequence (X_n) converges to some X in $W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$ if and only if

$$\lim_{n\uparrow\infty} \Big(\sup_{t\in I} |X_n(t) - X(t)| \Big) = 0 \quad \text{and} \quad \lim_{n\uparrow\infty} \Big(\int_I |\dot{X}_n(t) - \dot{X}(t)|dt \Big) = 0,$$

for every bounded interval I of $\mathbb{R}$.

Lemma 4. (compactness) *If $f \in L^1_{loc}(R^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ then for every $t \in \mathbb{R}^+$*

$$(1) \qquad 1 + \sup_{0\leq u\leq t} |X^f_r(u)| \leq \exp \int_0^t N_1[f](s)ds,$$

and $\{X^f_r : r > 0\}$ is a relatively compact subset of $W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$.

Proof. Let $t \geq 0$. Since

$$1 + |X^f_r(t)| \leq 1 + \int_0^t N_1[f](s)\big(1 + \sup_{0\leq u\leq s} |X^f_r(u)|\big)ds$$

inequality (1) is obtained by using Gronwall's lemma. We shall prove that the collection $\{X^f_r\}_{r>0}$ is relatively compact with respect to the strong topology of $C([0,T]; \mathbb{R}^d)$ for every $T > 0$. Let $T > 0$ be fixed. Thanks to (1) and Ascoli's theorem, we just have to prove that the collection $\{X^f_r\}_{r>0}$ is equicontinuous. If

$0 \le t_1 < t_2 \le T$ then

$$|X_r^f(t_2) - X_r^f(t_1)| \le \int_{t_1}^{t_2} N_1[f](s)\big(1 + |X_r^f(s-r)|\big)ds \le C \int_{t_1}^{t_2} N_1[f](s)ds,$$

where the constant $C = C(f,T)$ is given by $C := \exp \int_0^T N_1[f](s)ds$. Since C does not depends on t_1, t_2 and $N_1[f] \in L^1[0,T]$, equicontinuity follows. We finally prove that the collection $\{X_r^f\}_{r>0}$ is relatively compact with respect to the strong topology of $W_{loc}^{1,1}(\mathbb{R};\mathbb{R}^d)$. Let (r_n) be a sequence of positive real numbers. Using a diagonal process, we deduce from the first part of the proof that there exists a subsequence (r'_n) and $(r,X) \in [0,\infty] \times C(\mathbb{R};\mathbb{R}^d)$ such that:

$$\lim_{n\uparrow\infty} r'_n = r \quad \text{and} \quad \lim_{n\uparrow\infty} \big(\sup_{t\in I} |X_{r'_n}^f(t) - X(t)|\big) = 0$$

for every bounded interval I in $\mathbb{R}$. If $r = \infty$ then $(X_{r'_n}^f)$ clearly converges in $W_{loc}^{1,1}(\mathbb{R};\mathbb{R}^d)$ to the function X defined by:

$$X(t) = 0 \quad (t \le 0) \qquad \text{and} \qquad X(t) = \int_0^t f(s,0)ds \quad (t > 0).$$

If $0 \le r < \infty$ then $\lim_{n\uparrow\infty} f\big(t, X_{r'_n}^f(t-r'_n)\big) = f\big(t, X(t-r)\big)$ for almost every $t \in \mathbb{R}^+$. Therefore inequality (1) and dominated convergence theorem imply together that $(X_{r'_n}^f)$ converges in $W_{loc}^{1,1}(\mathbb{R};\mathbb{R}^d)$ to X and

$$X(t) = 0 \quad (t \le 0) \qquad \text{and} \qquad X(t) = \int_0^t f\big(s, X(s-r)\big)ds \quad (t > 0).$$

The proof is complete. □

Part (a) of Proposition 5 is immediately deduced from the above proof.

Proposition 5. *Let $f \in L_{loc}^1\big(\mathbb{R}^+; C_l(\mathbb{R}^d;\mathbb{R}^d)\big)$. Then the following two properties hold*

-(a)- for every sequence (r_n) of positive numbers converging to 0, there exists a subsequence (r'_n) and $X \in W_{loc}^{1,1}(\mathbb{R};\mathbb{R}^d)$ such that $(X_{r'_n}^f)$ converges in $W_{loc}^{1,1}(\mathbb{R};\mathbb{R}^d)$ to some X which is a solution of (E^f).

-(b)- If moreover the solution of (E^f) is unique then X_r^f converges to this solution in $W_{loc}^{1,1}(\mathbb{R};\mathbb{R}^d)$ as r tends to 0.

Proof. Let (r_n) be any sequence of positive real numbers converging to 0. By Lemma 4, $\{X_{r_n}^f\}_{n\in\mathbb{N}}$ is a relatively compact subset of $W_{loc}^{1,1}(\mathbb{R};\mathbb{R}^d)$. By (a) every converging subsequence $(X_{r'_n}^f)$ converges to the solution of (E^f), which proves (b). □

Some continuity property with respect to f of the set of solutions of (E^f) is stated in Proposition 6 below. This property will be used in the proof of Theorem 7. We also give a direct proof of some continuity of the mapping $f \longrightarrow X_r^f$. For $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ we use the following notation

$$R_1[f](t) := \exp \int_0^t N_1[f](s) ds.$$

Recall that by Lemma 4 we have $1 + \sup_{0 \le u \le t} |X_r^f(u)| \le R_1[f](t)$ for every $t \ge 0$

Lemma 6. *If $g \in Lip_{loc,1}$ then Equation (E^g) admits a unique solution $X_0^g \in W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$. Moreover, for each $t, \varepsilon > 0$, there exists $\delta > 0$ satisfying:*

$$\sup_{0 \le u \le t} |X(u) - X_0^g(u)| < \varepsilon \quad \text{and} \quad \sup_{r>0} \left(\sup_{0 \le u \le t} |X_r^f(u) - X_r^g(u)| \right) < \varepsilon$$

for every $(f, X) \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d)) \times W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$ such that X is solution of (E^f) and $\int_0^t N_1[f - g](s) ds < \delta$.

Proof. Existence of a solution was stated in Proposition 5 and uniqueness follows from Gronwall's Lemma. For $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$, $r > 0$ and $t \ge 0$, we set

$$I_r^{(1)}(t) := \int_0^t \left| f\big(s, X_r^f(s-r)\big) - g\big(s, X_r^f(s-r)\big) \right| ds$$

and

$$I_r^{(2)}(t) := \int_0^t \left| g\big(s, X_r^f(s-r)\big) - g\big(s, X_r^g(s-r)\big) \right| ds$$

Since $R_1[f](t) \le R_1[g](t) R_1[f-g](t)$, Lemma 4 implies that

$$(2) \quad I_r^{(1)}(t) \le R_1[g](t) R_1[f-g](t) \int_0^t N_1[f-g](s) ds$$

Since $\max\{1 + \sup_{0 \le u \le t} |X_r^f(u)|, 1 + \sup_{0 \le u \le t} |X_r^g(u)|\} \le R_1[g](t) R_1[f-g](t)$ we also have

$$(3) \quad I_r^{(2)}(t) \le \int_0^t \big(K[g, R[g](t) R_1[f-g](t)](s) \sup_{0 \le u \le s} |X_r^f(u) - X_r^g(u)| \big) ds$$

Let $t, \varepsilon > 0$ be fixed. Since $|X_r^f(u) - X_r^g(u)| \le I_r^{(1)}(t) + I_r^{(2)}(t)$ for every $u \in [0, t]$, we deduce from (2), (3) and Gronwall's Lemma that there exists $\delta_1 > 0$ satisfying:

$$\sup_{r>0} \left(\sup_{0 \le u \le t} |X_r^f(u) - X_r^g(u)| \right) < \varepsilon$$

for every $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ such that $\int_0^t N_1[f-g](s) ds < \delta_1$. Replacing X_r^f by a solution X of (E^f) and X_r^g by X_0^g in the above proof, we deduce that there exists $\delta_2 > 0$ satisfying:

$$\sup_{0 \le u \le t} |X(u) - X_0^g(u)| < \varepsilon$$

for every $(f, X) \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d)) \times W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$ such that X is a solution of (E^f) and $\int_0^t N_1[f-g](s)ds < \delta_2$. The proof is complete. □

Remark 6. With minor modifications of the above proof, a stronger result can be stated, i.e. if $g \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ be such that $K[g, M] \in L^1_{loc}(\mathbb{R}^+)$ for every $M > 0$ and (f_n) be a sequence in $L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ converging to g with respect to the metric ρ_1 then $(X_r^{f_n})$ converges (uniformly in r) to X_r^g in $W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$. However, we only need the version stated in Lemma 6.

The main result of the present section is the following.

Theorem 7. *-(a)- Let $\mathcal{G}_0$ be the set of functions $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ such that the solution (X^f) of equation (E_r^f) is unique. Then $\mathcal{G}_0$ is residual in $L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$.*
-(b)- Let $\overline{\mathcal{G}}_0$ be the set of functions $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ such that (X_r^f) converges (to some X satisfying (E^f)) in $W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$ as r tends to 0. Then $\overline{\mathcal{G}}_0$ is residual in $L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$.

Proof. By Lemma 6, for each integers $n, k \geq 1$ and $g \in Lip_{loc,1}$, there exists $\delta(n,k,g) > 0$ satisfying: $|X(t) - X_0^g(t)| \leq 1/k$ for every $(t, f, X) \in (-\infty, n] \times L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d)) \times W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^d)$ such that X is a solution of (E^f) and $\int_0^n N_1[f-g](s)ds < \delta(n,k,g)$. We consider the following subset of $L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$:

$$\mathcal{G} = \bigcap_{n,k} \bigcup_{g \in Lip_{loc,1}} \left\{ f : \int_0^n N_1[f-g](s)ds < \delta(n,k,g) \right\}.$$

Corollary 2 implies that $\mathcal{G}$ is a G_δ dense subset of $L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$. If $f \in \mathcal{G}$ and X, Y are two solutions of (E^f) then for every positive integers n, k, there exists $g_{n,k} \in Lip_{loc,1}$ such that $|X(t) - X_0^{g_{n,k}}(t)| + |Y(t) - X_0^{g_{n,k}}(t)| < 2/k$ where $X_0^{g_{n,k}}$ denotes the solution of $(E^{g_{n,k}})$. This implies that $|X(t) - Y(t)| \leq \frac{2}{k}$ for every integer $k \geq 1$ and $t \in (-\infty, n]$, then $X = Y$. Thus $\mathcal{G} \subset \mathcal{G}_0$ then $\mathcal{G}_0$ is residual. To prove that $\overline{\mathcal{G}}_0$ is residual, it is enough to show that $\mathcal{G}_0 \subset \overline{\mathcal{G}}_0$, which is the statement of Proposition 5-(b). Theorem 7 is proved. □

Remark 7. The fact that $\overline{\mathcal{G}}_0$ is residual can be proved without using the fact that $\overline{\mathcal{G}}_0 \supset \mathcal{G}_0$. To do this, redefine the set $\mathcal{G}$ with $\delta(n,k,g) > 0$ satisfying:

$$\sup_{r>0} \left(\sup_{0 \leq u \leq t} |X_r^f(u) - X_r^g(u)| \right) < \frac{1}{k}$$

for every $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ such that $\int_0^t N_1[f-g](s)ds < \delta(n,k,g)$.

III. Some properties of approximation with delay in SDE.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space endowed with a filtration satisfying the usual conditions and B be an $\mathbb{R}^m$-valued Brownian motion defined on it. We denote by $\mathbb{R}^{d\times m}$ the space of $d \times m$ matrices. For each function $\sigma = \sigma(t,x)$ from $\mathbb{R}^+ \times \mathbb{R}^d$ to $\mathbb{R}^{d\times m}$ we denote by

$$N_2[\sigma](t) := \sup_{x\in\mathbb{R}^d} \frac{|\sigma(t,x)|}{\sqrt{1+|x|^2}} \qquad \text{and} \qquad R_2[\sigma](t) := \exp \int_0^t N_1^2[\sigma](s)ds.$$

We write $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})$ when σ is measurable in t for all $x \in \mathbb{R}^d$, continuous in x for almost all $t \in \mathbb{R}^+$ and such that $N_2[\sigma] \in L^2_{loc}(\mathbb{R}^+)$. Given some $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})$, we denote by $\{X^\sigma_r\}_{r>0}$ the family of d-dimensional continuous processes defined by

$$(E^\sigma_r) \qquad X^\sigma_r(t) = 0 \quad (t<0) \quad \text{and} \quad X^\sigma_r(t) = \int_0^t \sigma(s, X^\sigma_r(s-r))dB(s) \quad (t \geq 0).$$

Thanks to the fact that $N_2[\sigma] \in L^2_{loc}(\mathbb{R}^+)$ and $|\sigma(t,x)|^2 \leq N_2^2[\sigma](t)\big(1+|x|^2\big)$, the family of delayed processes X^σ_r, introduced in (E^σ_r), is well defined for any t. For instance, it may be constructed recursively on $[0,r]$, $[r,2r]$, $[2r,3r]$...

Lemma 8. -(*a*)- *For any $p \in [1,\infty[$ there exists a positive constant $C(p)$ such that*

$$(4) \qquad 1 + E\Big(\sup_{0\leq u\leq t} |X^\sigma_r(u)|^{2p}\Big) \leq \Big(R_2[\sigma](t)\Big)^{C(p)}$$

for every $\sigma \in L^2_{loc}\big(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})\big)$, $r>0$ and $t \geq 0$.

-(*b*)- *Assertion (a) remains true when X^σ_r is replaced by any solution of equation (E^σ).*

Proof. We use Itô's formula and Fubini theorem to get

$$\begin{aligned} E\big(|X^\sigma_r(t)|^{2p}\big) \leq & E\Big(\int_0^t p(2p-1)|X^\sigma_r(s)|^{2p-2}|\sigma(s, X^\sigma_r(s-r))|^2 ds\Big) \\ & \leq p(2p-1)\int_0^t N_2^2[\sigma](s) E\Big(|X^\sigma_r(s)|^{2p-2}\big(1+|X^\sigma_r(s-r)|^2\big)\Big)ds \\ & \leq 2p(2p-1)\int_0^t N_2^2[\sigma](s) E\Big(1+\sup_{0\leq u\leq s}|X^\sigma_r(u)|^{2p}\Big)ds \end{aligned}$$

Since $E(\sup_{0<u\leq t}|X^\sigma_r(u)|^{2p}) \leq (2p/2p-1)^{2p}E(|X^\sigma_r(t)|^{2p}$ we conclude that there exists a positive constant $C(p)$ (which doesn't depend on (σ, r, t)) such that

$$1+E\Big(\sup_{0\leq u\leq t}|X^\sigma_r(u)|^{2p}\Big) \leq 1 + C(p)\int_0^t N_2^2[\sigma](s)\Big(1+E\Big(\sup_{0\leq u\leq s}|X^\sigma_r(u)|^{2p}\Big)\Big)ds$$

Now, Gronwall's lemma and Fatou's lemma give inequality (4). The proof of part (*b*) is similar. □

Let us specify the metric spaces under consideration in the study of generic properties. For each $t > 0$ let $\mathcal{F}_t^B$ be the σ-field generated by $\{B(s) : 0 \leq s \leq t\}$ completed with the P-null sets of Ω. We denote by $\mathcal{E}_2$ the set of $\mathbb{R}^d$-valued processes X defined on $\mathbb{R}^+ \times \Omega$ which are $\mathcal{F}_t^B$-adapted, continuous and such that $E(\sup_{0\leq s\leq t} |X(s)|^2) < \infty$ for every $t > 0$. We set

$$d_2(X_1, X_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\{E(\sup_{0\leq s\leq n} \mid X_1(s) - X_2(s) \mid^2)\}^{1/2}}{1 + \{E(\sup_{0\leq s\leq n} \mid X_1(s) - X_2(s) \mid^2)\}^{1/2}}$$

and

$$\rho_2(\sigma_1, \sigma_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\{\int_0^n N_2^2[\sigma_1 - \sigma_2](s)ds\}^{1/2}}{1 + \{\int_0^n N_2^2[\sigma_1 - \sigma_2](s)ds\}^{1/2}}.$$

The space $\mathcal{E}_2$ (resp. $L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})))$ endowed with the metric d_2 (resp. ρ_2) is a complete metric space. Our space of locally Lipschitz functions is defined as follows.

$$Lip_{loc,2} := \{\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})) : K[\sigma, M] \in L^\infty_{loc}(\mathbb{R}^+) \text{ for every } M > 0\}.$$

where $K[\sigma, M]$ is defined as in section 1. The following result will be used in the proof of Theorem 11.

Lemma 9. *The following properties hold.*
-(*a*)- *Let* $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d : \mathbb{R}^{d\times m}))$ *and* $X \in \mathcal{E}_2$. *If there exists a sequence* (r_n) *converging to* 0 *such that* $(X^\sigma_{r_n})$ *converges to* X *in* $(\mathcal{E}_2, d_2)$ *then* X *is a solution of* (E^σ).
-(*b*)- *If* $\sigma \in Lip_{loc,2}$ *then the mapping* $r \longrightarrow X_r^\sigma$ *from* $(0, \infty)$ *into* $(\mathcal{E}_2, d_2)$, *is uniformly continuous and for every sequence* (r_n) *converging to* 0 *the sequence* $(X^\sigma_{r_n})$ *converges in* $(\mathcal{E}_2, d_2)$ *to the unique solution of* (E^σ).

Proof. Assertion (a) is proved by showing that

$$\lim_{n\uparrow\infty} E\Big(\Big|\int_0^t \sigma(s, X^\sigma_{r_n}(s - r_n))dB(s) - \int_0^t \sigma(s, X(s))dB(s)\Big|^2\Big) = 0.$$

By Corollary 2 there exists a sequence (σ_p) in $Lip_{loc,2}$ such that (σ_p) converges to σ with respect to the metric ρ_2. Let n, M and p be any positive integers. Let

$$\theta_2(t) := 1 + \sup_{0\leq u\leq t} E\big(|X(u)|^2\big) \quad \text{and} \quad \theta_{2,n}(t) := 1 + \sup_{0\leq u\leq t} E\big(|X^\sigma_{r_n}(u)|^2\big)$$

$$\theta_4(t) := 1 + \sup_{0\leq u\leq t} E\big(|X(u)|^4\big) \quad \text{and} \quad \theta_{4,n}(t) := 1 + \sup_{0\leq u\leq t} E\big(|X^\sigma_{r_n}(u)|^4\big)$$

We use Cauchy-Schwarz' inequality and Chebyshev's inequality to get,

$$E\left(\left|\int_0^t \sigma(s, X_{r_n}(s-r_n))dB(s) - \int_0^t \sigma(s, X(s))dB(s)\right|^2\right) \leq$$

$$\leq 4E\int_0^t |\sigma_p(s, X_{r_n}^{\sigma}(s-r_n)) - \sigma_p(s, X(s-r_n))|^2 ds$$

$$+ 4E\int_0^t |\sigma_p(s, X(s-r_n)) - \sigma_p(s, X(s))|^2 ds$$

$$+ 4E\int_0^t |\sigma(s, X_{r_n}^{\sigma}(s-r_n)) - \sigma_p(s, X_{r_n}^{\sigma}(s-r_n))|^2 ds$$

$$+ 4E\int_0^t |\sigma_p(s, X(s)) - \sigma(s, X(s))|^2 ds$$

$$\leq 4\int_0^t K^2[\sigma_p, M](s)ds \sup_{0\leq u\leq t} E\left(|X_{r_n}^{\sigma}(u) - X(u)|^2\right)$$

$$+ \frac{16}{M}\int_0^t N_2^2[\sigma_p](s)ds \sqrt{\theta_{2,n}(t) + \theta_2(t)}\sqrt{\theta_{4,n}(t) + \theta_4(t)}$$

$$+ 4\int_0^t K^2[\sigma_p, M](s)ds \sup_{0\leq u\leq t} E\left(|X(u-r_n) - X(u)|^2\right)$$

$$+ \frac{16}{M}\int_0^t N_2^2[\sigma_p](s)ds \sqrt{2\theta_2(t)}\sqrt{2\theta_4(t)}$$

$$+ 4\int_0^t N_2^2[\sigma - \sigma_p](s)ds \sup_{0\leq u\leq t} E\left(1 + |X_{r_n}^{\sigma}(u)|^2\right)$$

$$+ 4\int_0^t N_2^2[\sigma - \sigma_p](s)ds \sup_{0\leq u\leq t} E\left(1 + |X(u)|^2\right)$$

Hence, Fatou's Lemma and Lemma 8(a) imply together that

$$E\left(\left|\int_0^t \sigma(s, X_{r_n}(s-r_n))dB(s) - \int_0^t \sigma(s, X(s))dB(s)\right|^2\right) \leq$$

$$\leq 4\sup_{0\leq u\leq t} E\left(|X_{r_n}^{\sigma}(u) - X(u)|^2\right)\int_0^t K^2[\sigma_p, M](s)ds$$

$$+ 4\sup_{0\leq u\leq t} E\left(|X(u-r_n) - X(u)|^2\right)\int_0^t K^2[\sigma_p, M](s)ds$$

$$+ \frac{64}{M}\left(R_2[\sigma](t)\right)^{\frac{c(1)+c(2)}{2}}\int_0^t N_2^2[\sigma_p](s)ds$$

$$+ 8\left(R_2[\sigma](t)\right)^{C(1)}\int_0^t N_2^2[\sigma - \sigma_p](s)ds$$

where $C(1)$ (resp. $C(2)$) is the constant of lemma 8 corresponding to $p=1$ (resp. $p=2$). We now conclude by passing to the limit at ∞ successively on n, M and p.

Let us prove assertion (b). For arbitrarily positive constants r, r' and M let

$\tau = \tau(r, r', M)$ be defined by: $\tau := \inf\{u \in \mathbb{R}^+ : 2 + |X_r^\sigma(u)|^2 + |X_{r'}^\sigma(u)|^2 > M^2\}$.
Let us prove that

$$(5) \qquad E\Big(|X_r^\sigma(t) - X_r^\sigma(t\wedge\tau)|^2\Big) \leq \frac{4\sqrt{2}}{M}\Big(R_2[\sigma](t)\Big)^{\frac{C(1)+C(2)}{2}}$$

By Cauchy-Schwarz' inequality and Lemma 8 (a) we have

$$\begin{aligned}
&E\big(|X_r^\sigma(t) - X_r^\sigma(t\wedge\tau)|^2\big) \leq \\
&\leq \big\{E\big(|X_r^\sigma(t) - X_r^\sigma(t\wedge\tau)|^4\big)\big\}^{1/2}\big\{P(\tau < t)\big\}^{1/2} \\
&\leq \frac{4}{M}\big\{E\big(\sup_{0\leq u\leq t}|X_r^\sigma(u)|^4\big)\big\}^{1/2}\big\{E\big(2 + \sup_{0\leq u\leq t}|X_r^\sigma(u)|^2 + \sup_{0\leq u\leq t}|X_{r'}^\sigma(u)|^2\big)\big\}^{1/2} \\
&\leq \frac{4\sqrt{2}}{M}\big(R_2[\sigma](t)\big)^{\frac{C(1)+C(2)}{2}}.
\end{aligned}$$

where $C(1)$ (resp. $C(2)$) is the constant of lemma 8 corresponding to $p = 1$ (resp. $p = 2$).
We now prove that for each $M > 0$,

$$(6) \qquad \varepsilon_M[r, r'](t) := E\Big(\sup_{0\leq u\leq t}|X_r^\sigma(u\wedge\tau) - X_{r'}^\sigma(u\wedge\tau)|^2\Big) \longrightarrow 0 \quad \text{as} \quad |r' - r| \longrightarrow 0$$

Let us denote $\theta[r, r'](t) := \sup\{\int_a^b N_2^2[\sigma](u)du : a, b \in [0, t],\ |b - a| \leq |r - r'|\}$. By Doob's inequality and lemma 8 we have

$$\begin{aligned}
\varepsilon_M[r, r'](t) &\leq 4E\Big(\int_0^{t\wedge\tau} K^2[\sigma, M](s)|X_r^\sigma(s - r) - X_{r'}^\sigma(s - r')|^2 ds\Big) \\
&\leq 8E\Big(\int_0^{t\wedge\tau} K^2[\sigma, M](s)|X_r^\sigma(s - r) - X_r^\sigma(s - r')|^2 ds\Big) \\
&+ 8E\Big(\int_0^{t\wedge\tau} K^2[\sigma, M](s)\sup_{0\leq u\leq s}|X_r^\sigma(u\wedge\tau) - X_{r'}^\sigma(u\wedge\tau)|^2 ds\Big) \\
&\leq 8\big(R_2[\sigma](t)\big)^{C(1)}\theta[r, r'](t)\int_0^t K^2[\sigma, M](s)ds \\
&+ 8\int_0^t K^2[\sigma, M](s)\varepsilon_M[r, r'](s)ds.
\end{aligned}$$

Since $N_2^2[\sigma]$ and $K^2[M, \sigma] \in L^1_{loc}(\mathbb{R}^+)$ the proof of (6) is completed by using Gronwall's lemma. We use Doob's inequality, and (5) to get

$$E\Big(\sup_{0\leq u\leq t}|X_r^\sigma(u) - X_{r'}^\sigma(u)|^2\Big) \leq \frac{96\sqrt{2}}{M}\big(R_2[\sigma](t)\big)^{\frac{C(1)+C(2)}{2}} + 12\varepsilon_M[r, r'](t).$$

The fact that the mapping $r \longrightarrow X_r^\sigma$ is uniformly continuous is then a consequence of (6). Since the space $(\mathcal{E}_2, d_2)$ is complete, there exists $X_0^\sigma \in \mathcal{E}_2$ such that for each sequence (r_n) converging to 0, the sequence $(X_{r_n}^\sigma)$ converges to X_0^σ in $(\mathcal{E}_2, d_2)$. We deduce from Lemma 9 (a) that X_0^σ is a solution of (E^σ). The fact that (E^σ) admits at most one solution is an easy consequence of Gronwall's Lemma. □

We now state a continuity property of the mappings $\sigma \longrightarrow X_r^\sigma$.

Lemma 10. *Let $\sigma' \in Lip_{loc,2}$ and (σ_n) be a sequence in $L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$. Assume that (σ_n) converges to σ' with respect to the metric ρ_2. Then the following two properties hold.*

-(a)- the sequence $(X_r^{\sigma_n})$ converges to $X_r^{\sigma'}$ in $(\mathcal{E}_2, d_2)$ uniformly in r,

-(b)- if X_n is an arbitrary solution of equation (E^{σ_n}) and $X^{\sigma'}$ is the unique solution of equation $(E^{\sigma'})$ then the sequence (X_n) converges to $X^{\sigma'}$ in $(\mathcal{E}_2, d_2)$.

Proof. For positive numbers r, M and a function $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ we define $\tau = \tau(r, M, \sigma)$ by $\tau := \inf\{u \in \mathbb{R}^+ : 2 + |X_r^\sigma(u)|^2 + |X_r^{\sigma'}(u)|^2 > M^2\}$. Using the inequality $R_2[\sigma](t) \leq \big(R_2[\sigma'](t)R_2[\sigma - \sigma'](t)\big)^2$ and arguing as in the proof of (5), we show that

$$E\big(|X_r^\sigma(t) - X_r^\sigma(t\wedge\tau)|^2\big) \leq \frac{4\sqrt{2}}{M}\big(R_2[\sigma'](t)R_2[\sigma - \sigma'](t)\big)^{C(1)+C(2)}.$$

Hence Doob's inequality implies

$$E\bigg(\sup_{0\leq u\leq t}|X_r^\sigma(u) - X_r^{\sigma'}(u)|^2\bigg) \leq$$
$$\leq \frac{96\sqrt{2}}{M}\big(R_2[\sigma'](t)R_2[\sigma - \sigma'](t)\big)^{C(1)+C(2)} + 12\varepsilon_M[\sigma, \sigma'](t)$$

where

$$\varepsilon_M[\sigma, \sigma'](t) := \sup_{r>0} E\bigg(\sup_{0\leq u\leq t}|X_r^\sigma(u\wedge\tau) - X_r^{\sigma'}(u\wedge\tau)|^2\bigg).$$

Therefore Lemma 10(a) will be proved by showing that $\varepsilon_M[\sigma, \sigma'](t) \longrightarrow 0$ as $\rho_2(\sigma, \sigma') \longrightarrow 0$. It follows from Doob's inequality and lemma 8 that

$$E\bigg(\sup_{0\leq u\leq t} \mid X_r^\sigma(u\wedge\tau) - X_r^{\sigma'}(u\wedge\tau)\mid^2\bigg) \leq$$
$$\leq 8\int_0^t E\big(\mid \sigma(s, X_r^\sigma(s-r)) - \sigma'(s, X_r^\sigma(s-r))\mid^2\big)ds$$
$$+ 8E\bigg(\int_0^{t\wedge\tau} \mid \sigma'(s, X_r^\sigma(s-r)) - \sigma'(s, X_r^{\sigma'}(s-r))\mid^2 ds\bigg)$$
$$\leq 8\big(R_2[\sigma'](t)R_2[\sigma-\sigma'](t)\big)^{2C(1)}\int_0^t N_2^2[\sigma-\sigma'](s)ds$$
$$+ 8\int_0^t K^2[\sigma', M](s)E\bigg(\sup_{0\leq u\leq s}|X_r^\sigma(u\wedge\tau) - X_r^{\sigma'}(u\wedge\tau)|^2\bigg)ds$$

The conclusion now follows from Gronwall's lemma. We omit the proof of Lemma 10(b) which is similar to the proof of Lemma 9(b). □

Remark 10. The conclusion of Lemma 10 is also true when the assumption $\sigma' \in$

$Lip_{loc,2}$ is replaced by $\sigma' \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ and $K^2[\sigma, M] \in L^1_{loc}(\mathbb{R}^+)$ for every $M > 0$. However we only need Lemma 10 as stated above.

The main result of the present section is the following.

Theorem 11. *Let $\mathcal{G}_2$ be the set of those functions σ in $L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ such that (X^σ_r) converges (to a solution of equation (E^σ)) in $(\mathcal{E}_2, d_2)$ as r tends to 0. Let $\overline{\mathcal{G}}_2$ be the set of those functions σ in $L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ for which (E^σ) has a unique solution in $(\mathcal{E}_2, d_2)$.*

-(a)- The set $\mathcal{G}_2$ is residual in $\big(L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})), \rho_2\big)$.

-(b)- The set $\overline{\mathcal{G}}_2$ is residual in $\big(L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})), \rho_2\big)$.

Proof. Lemma 10(a) implies that for each positive integers k and each $\sigma' \in Lip_{loc,2}$, there exists $\delta(k,\sigma') > 0$ satisfying: $d_2(X^\sigma_r, X^{\sigma'}_r) \le 1/k$ for every $r > 0$ and $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ such that $\rho_2(\sigma, \sigma') < \delta(k, \sigma')$. Then by Corollary 2 the set $\mathcal{G}$ defined by

$$\mathcal{G} := \bigcap_k \bigcup_{\sigma' \in Lip_{loc,2}} \Big\{\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})); \quad \rho_2(\sigma, \sigma') < \delta(k, \sigma')\Big\}$$

is a G_δ dense subset of $L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$. If $\sigma \in \mathcal{G}$ then for each positive integer k there exists $\sigma'_k \in Lip_{loc,2}$ such that $d_2(X^\sigma_r, X^{\sigma'_k}_r) < 1/k$ for each $r > 0$. We then have $d_2(X^\sigma_r, X^\sigma_{r'}) \le 2/k + d_2(X^{\sigma_k}_r, X^{\sigma_k}_{r'})$ for every r, $r' > 0$. Hence Lemma 9 implies that (X^σ_r) converges to some solution of (E_σ) in $(\mathcal{E}_2, d_2)$ as $r \downarrow 0$. Therefore $\mathcal{G} \subset \mathcal{G}_2$ which implies that $\mathcal{G}_2$ is residual.

Let us prove that $\overline{\mathcal{G}_2}$ is residual. Lemma 10(b) implies that, for each positive integer k and each $\sigma' \in Lip_{loc,2}$, there exists $\delta(k,\sigma') > 0$ satisfying: $d_2(X, X^{\sigma'}_0) < 1/k$ for every $(\sigma, X) \in \mathcal{G}_2 \times \mathcal{E}_2$ such that $\rho_2(\sigma, \sigma') < \delta(k, \sigma')$ and X is a solution of (E^σ) (where $X^{\sigma'}_0$ denotes the unique solution of $(E^{\sigma'})$). We consider the subset $\mathcal{G}'$ of $L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ defined by

$$\mathcal{G}' = \bigcap_k \bigcup_{\sigma' \in Lip_{loc,2}} \{\sigma \in \mathcal{G}_2; \quad \rho_2(\sigma, \sigma') < \delta(k, \sigma')\}$$

Clearly $\mathcal{G}'$ is residual in $\big(L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m})), \rho_2\big)$. Let $\sigma \in \mathcal{G}'$ and X_1, X_2 be two solutions of equation (E^σ). For each positive integer k there exists $\sigma_k \in Lip_{loc,2}$ such that $d_2(X_1, X^{\sigma_k}_0) + d_2(X_2, X^{\sigma_k}_0) < 2/k$ where $X^{\sigma_k}_0$ denotes the unique solution of (E^{σ_k}). Hence $X_1 = X_2$.Therefore $\mathcal{G}' \subset \overline{\mathcal{G}}_2$ which implies that $\overline{\mathcal{G}}_2$ is residual. □

IV. Weak solution and L^2-approximation.

Let us specify the terminology used in the present section. Let σ be a function in $L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$. We say that (E^σ) admits a weak solution if there exists a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, an $\mathcal{F}_t$-Brownian motion B, a continuous $\mathcal{F}_t$-adapted process X such that

$$X(t) = \int_0^t \sigma(s, X(s))dB(s) \qquad P\text{-almost surely.}$$

for every $t \in \mathbb{R}^+$. In such a case, (B, X) is called a weak solution of (E^σ). We say that pathwise uniqueness holds for (E^σ) if whenever (B, X) and $(\overline{B}, \overline{X})$ are any two weak solutions of (E^σ) defined on the same filtered space then

$$B(t) = \overline{B}(t) \text{ } a.s. \text{ for every } t \in \mathbb{R}^+ \quad \Longrightarrow \quad X(t) = \overline{X}(t) \text{ } a.s. \text{ for every } t \in \mathbb{R}^+.$$

Before stating the main result of the present section, let us recall that the approximation with time delay X_r^σ depends on the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and the $\mathcal{F}_t$-Brownian motion B under consideration.

Proposition 12. *If $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ then (E^σ) admits a weak solution.*

To prove Proposition 12 we need the following Lemma.

Lemma 13. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered space and B be an $\mathcal{F}_t$-Brownian motion defined on it. If $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ then $(X_{r_n}^\sigma)$ is tight for every sequence (r_n) of positive numbers.*

Proof of Lemma 13. Thanks to Lemma 8, it is sufficient to show that the sequence of quadratic variations processes $(\langle X_{r_n}^\sigma \rangle)$ is tight, that is for each $T > 0$ and $\varepsilon > 0$

$$\lim_{h\downarrow 0} \sup_n P\left\{ \sup_{\delta\in[0,h]} \sup_{t\in[0,T]} \langle X_{r_n}^\sigma \rangle(t+\delta) - \langle X_{r_n}^\sigma \rangle(t) > \varepsilon \right\} = 0.$$

Let $0 \le h \le 1$. We use Lemma 8 to get

$$\begin{aligned} E\Big(\sup_{\delta\in[0,h]} \sup_{t\in[0,T]} \langle X_{r_n}^\sigma \rangle(t+\delta) - \langle X_{r_n}^\sigma \rangle(t) \Big) = \\ &= E\Big(\sup_{\delta\in[0,h]} \sup_{t\in[0,T]} \int_t^{t+\delta} |\sigma(s, X_{r_n}^\sigma(s - r_n))|^2 ds \Big) \\ &\le \Big(\sup_{\substack{t\in[0,T] \\ \delta\in[0,h]}} \int_t^{t+\delta} N_2^2[\sigma](s) ds \Big) E\Big(1 + \sup_{0\le u\le T+1} |X_{r_n}^\sigma(u)|^2 \Big) \\ &\le \big(R_2[\sigma](T+1) \big)^{C(1)} \sup_{\substack{t\in[0,T] \\ \delta\in[0,h]}} \int_t^{t+\delta} N_2^2[\sigma](s) ds. \end{aligned}$$

Since $N_2^2[\sigma] \in L^1_{loc}(\mathbb{R}^+)$, the proof is completed by using Chebyshev's inequality. □

Proof of Proposition 12. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered space, B an $\mathcal{F}_t$-Brownian motion defined on it and $\{X_r^\sigma : r > 0\}$ the corresponding family of approximations with time delay. Let (r_n) be a sequence of positive numbers such that $r_n \downarrow 0$. By Lemma 13 $\{(X_{r_n}^\sigma(t), X_{r_n}^\sigma(t - r_n), B(t))\}_n$ is tight. Hence, Skorokhod's representation theorem ([F]) shows that there exist a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$, a sequence of stochastic processes $(\overline{X}_n(t), \overline{X}'_n(t), \overline{B}_n(t))$ and a stochastic process $(\overline{X}(t), \overline{X}'(t), \overline{B}(t))$ defined on it such that:

(7) $(\overline{X}_n(t), \overline{X}'_n(t), \overline{B}_n(t)) = (X^\sigma_{r_n}(t), X^\sigma_{r_n}(t - r_n), B(t))$ in law for every n,

(8) there exists a subsequence (n') such that $(\overline{X}_{n'}, \overline{X}'_{n'}, \overline{B}_{n'})$ converges to $(\overline{X}, \overline{X}', \overline{B})$ uniformly on every finite time interval $\overline{P}$-almost surely.

Let $\overline{\mathcal{F}}^n_t$ (resp.$\overline{\mathcal{F}}_t$) be the σ-algebra $\sigma(\overline{X}'_n(s), \overline{B}_n(s); s \leq t)$ (resp. $\sigma(\overline{X}'(s), \overline{B}(s); s \leq t)$) completed with $\overline{P}$-null sets. Hence $(\overline{B}_n(t), \overline{\mathcal{F}}^n_t)$ and $(\overline{B}(t), \overline{\mathcal{F}}_t)$ are Brownian motions and the processes $(\overline{X}_n)$, $(\overline{X}'_n)$ and $(\overline{X})$, $(\overline{X}')$ are adapted to $\overline{\mathcal{F}}^n_t$ and $\overline{\mathcal{F}}_t$ respectively. Let us prove that $(\overline{X}, \overline{B})$ is a weak solution of (E^σ). We may assume that (8) holds without extracting a subsequence of $(\overline{X}_n, \overline{X}'_n, \overline{B}_n)$. Property (7) implies that,

$$\text{(9)} \qquad \overline{X}_n(t) = \int_0^t \sigma(s, \overline{X}'_n(s)) d\overline{B}_n(s) \qquad (\overline{P}\text{-almost surely}).$$

Since $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$, Property (7), Doob's inequality and Lemma 8 imply that $\overline{E}(|\overline{X}_n(t) - \overline{X}'_n(t)|^2) \leq (R_2[\sigma](t))^{C(1)} \int_{0\vee(t-r_n)}^t N_2^2[\sigma](s)ds$. We use Property (8) and Fatou's lemma to get $\overline{E}(|\overline{X}(t) - \overline{X}'(t)|^2) \leq \liminf_n \overline{E}(|\overline{X}_n(t) - \overline{X}'_n(t)|^2) = 0$. Hence

$$\text{(10)} \qquad \overline{X} = \overline{X}' \qquad (\overline{P}\text{-almost surely}).$$

Taking account (9) and (10), it is sufficient to show that

$$\text{(11)} \qquad \lim_{n\uparrow\infty} \int_0^t \sigma(s, \overline{X}'_n(s)) d\overline{B}_n(s) = \int_0^t \sigma(s, \overline{X}'(s)) d\overline{B}(s) \qquad (\text{in probability } \overline{P}).$$

Let (σ_p) be a sequence of $\mathbb{R}^{d\times m}$- valued functions, defined on $\mathbb{R}^+ \times \mathbb{R}^d$, which are globally Lipschitz in their two arguments and such that, for each $M > 0$ and $t \in \mathbb{R}^+$,

$$\lim_{p\uparrow\infty} \int_0^t \sup_{|x|\leq M} |\sigma_p(s,x) - \sigma(s,x)|^2 ds = 0 \text{ and } \lim_{p\uparrow\infty} \int_0^t N_2^2[\sigma_p](s)ds = \int_0^t N_2^2[\sigma](s)ds.$$

Let t and ε be fixed positive numbers. We use triangular inequality to get,

$$\text{(12)} \quad \overline{P}\{|\int_0^t \sigma(s, \overline{X}'_n(s)) d\overline{B}_n(s) - \int_0^t \sigma(s, \overline{X}'(s)) d\overline{B}(s)| > \varepsilon\} \leq \sum_{i=1}^{i=3} I_i(n,p)$$

where

$$I_1(n,p) = \overline{P}\left\{\left|\int_0^t \sigma(s, \overline{X}'_n(s)) d\overline{B}_n(s) - \int_0^t \sigma_p(s, \overline{X}'_n(s)) d\overline{B}_n(s)\right| > \frac{\varepsilon}{3}\right\}$$

$$I_2(n,p) = \overline{P}\left\{\left|\int_0^t \sigma_p(s, \overline{X}'_n(s)) d\overline{B}_n(s) - \int_0^t \sigma_p(s, \overline{X}'(s)) d\overline{B}(s)\right| > \frac{\varepsilon}{3}\right\}$$

$$I_3(n,p) = \overline{P}\left\{\left|\int_0^t \sigma_p(s, \overline{X}'(s)) d\overline{B}(s) - \int_0^t \sigma(s, \overline{X}'(s)) d\overline{B}(s)\right| > \frac{\varepsilon}{3}\right\}.$$

Skorohod's theorem ([S1] page 32) shows that for each p,

(13) $\lim_{n\to\infty} I_2(n,p) = 0$

Let us prove that

(14) $\lim_{p\to\infty} (\sup_n I_1(n,p)) = 0.$

Let $M > 0$ and set $A_n^M(s) = \{1 + \sup_{0\le u\le s} |\overline{X}'_n(u)|^2 > M^2\}$ and $B_n^M(s) = \Omega \setminus A_n^M(s)$. For a given set E, let $\mathcal{X}_E$ denotes the indicator function of E. We successively use Chebyshev's inequality, Doob's inequality, Cauchy-Schwarz' inequality, again Chebyshev's inequality, Property (7), and Lemma 8(a) to get

$$\begin{aligned}
I_1(n,p) &= \overline{P}\left\{\left|\int_0^t \sigma_p(s,\overline{X}'_n(s))d\overline{B}_n(s) - \int_0^t \sigma(s,\overline{X}'_n(s))d\overline{B}_n(s)\right| > \frac{\varepsilon}{3}\right\} \\
&\le \frac{9}{\varepsilon^2}\overline{E}\int_0^t |\sigma_p(s,\overline{X}'_n(s)) - \sigma(s,\overline{X}'_n(s))|^2 ds \\
&\le \frac{9}{\varepsilon^2}\overline{E}\int_0^t |\sigma_p(s,\overline{X}'_n(s)) - \sigma(s,\overline{X}'_n(s))|^2 \mathcal{X}_{A_n^M(s)} ds \\
&\quad + \frac{9}{\varepsilon^2}\overline{E}\int_0^t |\sigma_p(s,\overline{X}'_n(s)) - \sigma(s,\overline{X}'_n(s))|^2 \mathcal{X}_{B_n^M(s)} ds \\
&\le \frac{18}{\varepsilon^2}\Big(\int_0^t N_2^2[\sigma](s) + N_2^2[\sigma_p](s)ds\Big)\overline{E}\Big(\big(1 + \sup_{0\le u\le t}|\overline{X}'_n(u)|^2\big)\mathcal{X}_{A_n^M(s)}\Big) \\
&\quad + \frac{9}{\varepsilon^2}\overline{E}\int_0^t \sup_{|x|\le M} |\sigma_p(s,x) - \sigma(s,x)|^2 \mathcal{X}_{B_n^M(s)} ds \\
&\le \frac{18\sqrt{2}}{\varepsilon^2 M}(R_2[\sigma](t))^{\frac{C(1)+C(2)}{2}}\Big(\int_0^t N_2^2[\sigma](s)ds + \int_0^t N_2^2[\sigma_p](s)ds\Big) \\
&\quad + \frac{9}{\varepsilon^2}\int_0^t \sup_{|x|\le M} |\sigma_p(s,x) - \sigma(s,x)|^2 ds.
\end{aligned}$$

Take the supremum over n, then pass to the limit, successively on p and M, to get (14). The same arguments and Fatou's lemma allow us to prove that

(15) $\lim_{p\to\infty} (\sup_n I_3(n,p)) = 0$

We now use (12), (13), (14) and (15) to get (11). Proposition 12 is proved. □

Proposition 14. *Let $\sigma \in L^2_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^{d\times m}))$ such that pathwise uniqueness holds for equation (E^σ). Then for each filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and each $\mathcal{F}_t$-Brownian motion B defined on it, there exists a continuous $\mathcal{F}_t$-adapted process X such that (B, X) is a weak solution of (E^σ) and*

$$\lim_{r\downarrow 0} E\bigg(\sup_{0\le s\le t} |X^\sigma_r(s) - X(s)|^2 \bigg) = 0$$

for every $t \in \mathbb{R}^+$. Moreover X is $\mathcal{F}^B_t$-adapted.

Proof. Suppose that the conclusion of Proposition 14 is false. Then there exist $t > 0$, $\delta > 0$ and two sequences $r_n \downarrow 0$ and $r'_n \downarrow 0$ such that

$$(16) \qquad E\bigg(\sup_{0\le u\le t} |X^\sigma_{r_n}(u) - X^\sigma_{r'_n}(u)|^2 \bigg) \ge \delta$$

for every n. By Lemma 13, $(X^\sigma_{r_n})$ and $(X^\sigma_{r'_n})$ are tight. Then Skorokhod's representation theorem shows that there exists a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$, a sequence of stochastic processes $\big(\overline{X}_n(t), \overline{X}'_n(t), \overline{Z}_n(t), \overline{Z}'_n(t), \overline{B}_n(t)\big)$ and a process $\big(\overline{X}(t), \overline{X}'(t), \overline{Z}(t), \overline{Z}'(t), \overline{B}(t)\big)$ defined on it such that

(17) $\big(\overline{X}_n(t), \overline{X}'_n(t), \overline{Z}_n(t), \overline{Z}'_n(t), \overline{B}_n(t)\big) = \big(X_{r_n}(t), X_{r'_n}(t), X_{r_n}(t-r_n), X_{r'_n}(t-r'_n), B(t)\big)$ in the sense of probability law,

(18) there exists a subsequence n' such that $\big(\overline{X}_{n'}(t), \overline{X}'_{n'}(t), \overline{Z}_{n'}(t), \overline{Z}'_{n'}(t), \overline{B}_{n'}(t)\big)$ converges to $\big(\overline{X}(t), \overline{X}'(t), \overline{Z}(t), \overline{Z}'(t), \overline{B}(t)\big)$ uniformly on every finite time interval $\overline{P}$-a.s.

We argue as in the proof of proposition 12 to show that $(\overline{B}, \overline{X})$ and $(\overline{B}, \overline{X}')$ are two weak solutions of (E^σ) then we use inequality (16) and property (17) to get

$$0 < \delta \le \liminf_{n\uparrow\infty} E\big(\sup_{0\le u\le t} |X^\sigma_{r_n}(u) - X^\sigma_{r'_n}(u)|^2 \big) = \overline{E}\big(\sup_{0\le u\le t} |\overline{X}(u) - \overline{X}'(u)|^2 \big).$$

Since pathwise uniqueness holds, we get a contradiction. Since (X^σ_r) is $\mathcal{F}^B_t$-adapted, so is X. The fact that (X, B) satisfies (E^σ) is deduced from Lemma 9(a). □

Remark 14. The fact that pathwise uniqueness implies $\mathcal{F}^B_t$-measurability of the solution was first proved in [YW]. However the proof is rather complicated. A simpler proof of this mesurability property was given in [KN]. Here, this result is immediately deduced from the L^2-convergence of the approximation with delay.

Remark 15. All the previous results remain true for stochastic differential equations with an arbitrary drift $f \in L^1_{loc}(\mathbb{R}^+; C_l(\mathbb{R}^d; \mathbb{R}^d))$ and arbitrary initial data in $\mathbb{R}^d$. Only minor modifications are required in the proofs.

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