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Canonical Lift and Exit Law of the Fundamental Diffusion Associated with a Kleinian Group

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Abstract

Let Γ be a geometrically finite Kleinian group, relative to the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, and let δ denote the Hausdorff dimension of its limit set. Denote by Φ the eigenfunction of the hyperbolic Laplacian Δ, associated with its first eigenvalue $2\lambda_0 = \delta(\delta - d)$, and by Z_t^Φ the associated diffusion on $\mathbb{H}$, whose generator is $\frac{1}{2}\Delta^\Phi := \frac{1}{2}\Phi^{-1}\Delta \circ \Phi - \lambda_0$. We give a simple construction of Z_t^Φ through its canonical lift to the frame bundle $\mathcal{O}\mathbb{H}$, that allows to determine directly its asymptotic behavior.

Keywords : diffusion process, hyperbolic space, Patterson measure.

AMS-classification 2000 : 60 J 60, 37 D 40, 58 J 65.

1 Introduction

Consider the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, endowed with some geometrically finite Kleinian group Γ. The Hausdorff dimension $\delta \in [0, d]$ of its limit set (see [P], [Su1] or [Su2]) plays a fundamental role. When δ is larger than $d/2$, $\delta(\delta - d)$ is the highest eigenvalue of the Laplacian on a fundamental domain. The associated eigenstate Φ plays an important role in the study of the quotient $\Gamma \setminus \mathbb{H}$ and of its geodesic flow. The corresponding fundamental diffusion Z_t^Φ, which we call "Φ-diffusion", is then also a natural object and tool in this framework : see [Su1], [E-F-LJ-1], [E-F-LJ-2].

Now Φ is classically represented as the mass of the celebrated Patterson measure, and thus makes sense also when it is not square-integrable. So that the Φ-diffusion Z_t^Φ on $\mathbb{H}$ can naturally be considered for all values of δ. It is ergodic on $\Gamma \setminus \mathbb{H}$ if and only if $\delta > d/2$.

The aim of this article is to give a simple construction of Z_t^Φ, from which can be immediately deduced the asymptotic behaviour of Z_t^Φ on $\mathbb{H}$, that exhibits an interesting dichotomy : whereas the almost sure limit point $Z_\infty^\Phi \in \partial\mathbb{H}$ has a singular law when $\delta \geq d/2$, namely the normalized Patterson measure (which appears thus as an harmonic measure), it happens to have an absolutely continuous (explicit) law when $\delta < d/2$.

We note however that this asymptotic behaviour of Z_t^Φ could also be deduced from the general theory of Martin boundary, see remark 7 below.
Our method starts from the Brownian motion on $\mathbb{H}$ modified by a constant drift, then uses the group action to define a diffusion on the stable leaves of the orthonormal frame bundle $\mathcal{O}\mathbb{H}$, whose projection onto $\mathbb{H}$ will be the Φ-diffusion.
This method could likely work in the general case of a symmetric space of non compact type and rank one.

2 Notations and basic data

Let $\mathbb{H}$ denote the hyperbolic space $\mathbb{H}^{d+1}$, with boundary $\partial\mathbb{H}$, unitary tangent bundle $T^1\mathbb{H}$, orthonormal frame bundle $\mathcal{O}\mathbb{H}$, Riemannian area dV , and (hyperbolic) Laplacian Δ .
Given (z, z', u) in $\mathbb{H} \times \mathbb{H} \times \partial\mathbb{H}$, denote by $\log[B_u(z, z')]$ the Busemann function, that is to say the algebraic hyperbolic distance, on any geodesic ending at u, from the stable horocycle $H(z, u)$ determined by z to the stable horocycle $H(z', u)$.
In the Poincaré half-space model, we have $B_u(z, z') = p(z', u)/p(z, u)$, $p(z, u)$ denoting the Poisson kernel : $p(z, u) = \mathcal{I}m(z) \times |z - u|^{-2}$ if $u \neq \infty$ and $p(z, \infty) = \mathcal{I}m(z)$. We have the cocycle property : $B_u(z, z") = B_u(z, z') \times B_u(z', z")$.

Let Γ be a discrete (non-elementary) group of Möbius isometries of $\mathbb{H}$, that we suppose geometrically finite. Let $\Lambda = \Lambda(\Gamma)$ denote its limit set, with Hausdorff dimension say δ . Recall that δ is also the critical convergence exponent of the Poincaré series relative to Γ ; (see for example ([Su2], Theorem 1)). Obviously $\delta \leq d$.

Let $\{\mu_z \mid z \in \mathbb{H}\}$ denote the family of Patterson (finite) measures on Λ associated with Γ . It can be defined, up to a multiplicative constant (that we definitively fix), as the only family of measures on Λ satisfying the following geometric "conformal density" property :

$$d\mu_{z'}(u) = B_u^\delta(z, z')\, d\mu_z(u) \quad \text{for any } z\,, z' \text{ in } \mathbb{H}$$

together with the invariance property by the group Γ, in the sense that :

$$\gamma^*\mu_z = \mu_{\gamma z} \quad \text{for any } \gamma \text{ in } \Gamma \text{ and } z \text{ in } \mathbb{H}\ ,$$

with the convention $\gamma^*\mu := \mu \circ \gamma^{-1}$.
See for example ([P], Lecture 2), [Su2], or ([Ni], Sections 3.4 and 4.7).

Set $\quad \Phi(z) := \int d\mu_z = \mu_z(\partial\mathbb{H}) = \mu_z(\Lambda)\ ,\quad$ and $\quad \lambda_0 := \delta(\delta - d)/2\ .$
This is a function on $\mathbb{H}$ that verifies $\Delta\Phi = 2\,\lambda_0\,\Phi$. See ([P], theorem 1 page 301).
Note that for every γ in Γ we have $\Phi(\gamma z) = \gamma^*\mu_z(\partial\mathbb{H}) = \mu_z(\gamma^{-1}(\partial\mathbb{H})) = \Phi(z)$.
Moreover, when $\delta > d/2$ then Φ is square-integrable with respect to dV on the fundamental domains of $\Gamma \setminus \mathbb{H}$, and it is the fundamental eigenstate on $\Gamma \setminus \mathbb{H}$.
See ([P], Theorem 1 page 301), or ([P-S], page 177).

Note that a consequence is that the volume of $\Gamma \setminus I\!H$ is finite if and only if $\delta = d$.

Let π denote the canonical projection from $T^1 I\!H$ onto $I\!H$, π_1 denote the canonical projection from $\mathcal{O}I\!H$ onto $T^1 I\!H$, and $\pi_2 = \pi \circ \pi_1$ denote the canonical projection from $\mathcal{O}I\!H$ onto $I\!H$.

We shall use on the unitary tangent bundle $T^1 I\!H$ the two following systems of coordinates :
_ firstly, $(z,u) \in I\!H \times \partial I\!H$, the geodesic running from z to u determining the unitary tangent vector at the base point z; this identifies $T^1 I\!H$ with $I\!H \times \partial I\!H$;
_ secondly, given a reference point $z_0 \in I\!H$, the point (z,u) of $T^1 I\!H$ (just defined above) can be represented by the triple $(u,v,s) \in \partial I\!H \times \partial I\!H \times I\!R$, where
_ v is the starting point of the geodesic ending at u and running through z ;
_ s is the algebraic hyperbolic distance from z to the orthogonal projection z_1 of z_0 onto the geodesic $\overrightarrow{vu}$.

The $PSO(d+1,1)$ model for both $\mathcal{O}I\!H$ and the Möbius isometries of $I\!H$ allows to identify $T^1 I\!H$ with $PSO(d+1,1)/SO_d$ and $I\!H$ with $PSO(d+1,1)/SO_{d+1}$, and to use on $\mathcal{O}I\!H$ the coordinates system $(z,u,r) \in I\!H \times \partial I\!H \times SO_d$.

Denote by $dist(\zeta, uv)$ the hyperbolic distance from $\zeta \in I\!H$ to the geodesic $\overrightarrow{vu}$. The following well-known identity is valid for any ζ in $I\!H$, any distinct u, v in $\partial I\!H$, and any z on the geodesic $\overrightarrow{vu}$ running from v to u.

$$(*) \qquad \mathrm{ch}^2(dist(\zeta, uv)) \;=\; B_u(\zeta,z)\, B_v(\zeta,z) \;.$$

(Indeed, since this is an intrinsic formula, we may consider the half-space model with $u = \infty$ and $v = 0$. Denoting then by (X,Y) the Euclidean coordinates of ζ in this model, and by $(0,y)$ those of z, it is elementary that $B_u(\zeta,z) = y/Y$, $B_v(\zeta,z) = (|X|^2+Y^2)/(yY)$, and, using the classical formula for the distance (see [P]), that $\mathrm{ch}^2(dist(\zeta,uv)) = \mathrm{ch}^2(dist(\zeta,(0,|\zeta|)) = (|X|^2+Y^2)^2/(Y|\zeta|)^2 = 1 + |X|^2/Y^2 = B_u(\zeta,z)B_v(\zeta,z)$.)

The Liouville measure $\tilde{\lambda}$ on $T^1 I\!H$ can be expressed for any reference point z_0, by :

$$d\tilde{\lambda}(u,v,s) \;=\; \mathrm{ch}^{2d}(dist(z_0,uv))\, d\mu^h_{z_0}(u)\, d\mu^h_{z_0}(u)\, ds ,$$

where μ^h_z denotes the harmonic measure at z. Recall that we have in the half-space model : $\quad d\mu^h_z(u) \;=\; p^d(z,u)\, du$.
Note that the above geometric property holds for harmonic measures, by changing δ into d : $\quad d\mu^h_{z'}(u) = B^d_u(z,z')\, d\mu^h_z(u) \quad$ for any z , z' in $I\!H$.
This and the identity $(*)$ show the irrelevance of the reference point z_0 in the expression of the Liouville measure $\tilde{\lambda}$ above. As can be verified by a direct elementary computation, the expression of $\tilde{\lambda}$ in the (z,u) coordinates is : $d\tilde{\lambda}(z,u) \;=\; d\mu^h_z(u)\, dV(z)$.

$\tilde{\lambda}$ is naturally lifted to the Liouville measure λ' on $\mathcal{O}I\!H$, by taking λ' uniform on each fibre SO_d. λ' is finite on a fundamental domain of $\Gamma \setminus \mathcal{O}I\!H$ only when $\delta = d$.

Observe from the two expressions above for the Liouville measure the following formula

$$(1) \qquad \int F\,dV = \int F\circ\pi(u,v,s)\,B_v^d(z_0,\pi(u,v,s))\,d\mu_{z_0}^h(v)\,ds\;,$$

valid for any $u\in\partial\mathbb{H}$, any $z_0\in\mathbb{H}$, and any test function F on $\mathbb{H}$.

Let θ_t and θ_x^+ denote respectively the geodesic and the positive horocycle flows on the orthonormal frame bundle $\mathcal{O}\mathbb{H}$. Moreover for any $z=(x,y)\in\mathbb{R}^d\times\mathbb{R}_+^*$, set

$$T_z := \theta_x^+\,\theta_{Log\,y}\;.$$

Observe the following important classical relation :

$$(2) \qquad T_{(x,y)}\,T_{z'} = T_{(x,0)\,+\,y\,z'}\;.$$

This means in particular that the set $\{T_z\,|\,z\in\mathbb{R}^d\times\mathbb{R}_+^*\}$ constitutes a group, isomorphic to a subgroup of the affine group of $\mathbb{R}^d$.

In the $PSO(d+1,1)$ model, the Liouville measure λ' is the Haar measure, and the flows are expressed by right multiplication by some matrices of $PSO(d+1,1)$. Thus the Liouville measure λ' is invariant by the horocycle and geodesic flows.

We can decompose the horocycle flow according to the canonical basis of $\mathbb{R}^d$: so for $x=(x_1,..,x_d)\in\mathbb{R}^d$, we set : $\theta_x^+=\theta_{x_1}^1..\theta_{x_d}^d$.

Let us introduce then the Lie derivatives : for any smooth function F on $\mathcal{O}\mathbb{H}$, any ξ in $\mathcal{O}\mathbb{H}$, $1\le j\le d$ we set :

$$(3) \qquad \mathcal{L}_0F(\xi) := \frac{d_o}{dt}F(\xi\,\theta_t)\;\;,\;\;\mathcal{L}_jF(\xi) := \frac{d_o}{dt}F(\xi\,\theta_t^j)\;.$$

$\frac{d_o}{dt}$ means and will mean the derivative at $t=0$ with respect to t.
We immediately see that : (4) $[\mathcal{L}_0\,,\mathcal{L}_j]=\mathcal{L}_j$, $[\mathcal{L}_j\,,\mathcal{L}_{j'}]=0$, and

$$(5) \qquad \mathcal{L}_0F(\xi T_{(x,y)}) = y\,\frac{\partial}{\partial y}F(\xi T_{(x,y)})\;\;,\;\;\mathcal{L}_jF(\xi T_{(x,y)}) = y\,\frac{\partial}{\partial x_j}F(\xi T_{(x,y)})\;.$$

Note that since the flows act on the right hand side, while Γ acts on the left hand side, these two operations commute.

Note that while the geodesic flow still makes sense on $T^1\mathbb{H}$, the horocycle flow makes sense only on $\mathcal{O}\mathbb{H}$.

By identifying Möbius isometries of $\mathbb{H}$ and orthonormal frames on $\mathbb{H}$, we deduce the following relations : $\pi_2(\xi T_z)=\xi(z)$, and $\pi_1(\xi T_z)=\xi(\vec{z})$, for any $\xi\in\mathcal{O}\mathbb{H}$ and $z\in\mathbb{H}$, $\vec{z}\in T^1\mathbb{H}$ denoting the line element based at z and pointing at ∞ .

Let us call "Φ-diffusion" and denote by Z_t^Φ the diffusion on $\mathbb{H}$ associated to the fundamental state Φ , that is to say having infinitesimal generator

$$\frac{1}{2}\Delta^\Phi := \frac{1}{2\Phi}\Delta\circ\Phi-\lambda_0\;.$$

This diffusion was already considered by Sullivan in [Su1] and for $d=1$ in [E-F-LJ-1,2].

3 An intrinsic measure on $T^1\mathbb{H}$

We introduce an intrinsic measure ν on $T^1\mathbb{H}$, which was already used for $d=1$ in [E-F-LJ-1,2]. Its interest is to be smooth along the stable leaves and quasi-invariant under the geodesic and positive horocycle flows, and to be an invariant measure for two dual diffusions on $\mathcal{O}\mathbb{H}$, which are both projected by π_2 onto the Φ-diffusion.

Definition 1 *Let $\tilde{\nu}$ be the measure on $T^1\mathbb{H}$ defined by :*

$$d\tilde{\nu}(z,u) = \Phi(z)\, d\mu_z(u)\, dV(z) \ .$$

Denote by ν' the unique measure on $\mathcal{O}\mathbb{H}$ which has marginal $\tilde{\nu}$ on $T^1\mathbb{H}$ and whose conditional laws on the fibres are the normalized Haar measure on $SO_d \equiv \mathcal{O}\mathbb{H}/T^1\mathbb{H}$. Set also $dV^{\Phi}(z) := \Phi^2(z)\, dV(z)$.

Remark 1 Observe that the Γ-invariance of Φ, $\tilde{V}$ and the geometric property of (μ_z) imply the Γ-invariance of $\tilde{\nu}$ (and then of ν'). Observe also that by definition of Φ we have $\pi_2^*\nu' = \pi^*\tilde{\nu} = V^{\Phi}$, and then that $\tilde{\nu}$ and ν' are finite above a fundamental domain relating to Γ if and only if $\delta > d/2$, with mass $\|\Phi\|_2^2$.

Remark 2 In the finite volume case, we have $\delta = d$, Φ constant, $d\mu(u)$ is proportional to the uniform measure du, and then our measure $\tilde{\nu}$ is proportional to the Liouville measure $\tilde{\lambda}$.

Proposition 1 *The measure ν' is quasi-invariant under the geodesic and positive horocycle flows :*

$$\frac{d(T_z * \nu')}{d\nu'}(\xi) = y^{d-\delta} \times \frac{\Phi\circ\pi_2(\xi T_z^{-1})}{\Phi\circ\pi_2(\xi)} \quad \textit{for any } \xi\in\mathcal{O}\mathbb{H} \textit{ and } z=(x,y)\in\mathbb{R}^d\times\mathbb{R}_+^* .$$

Note that this quasi-invariance property is what remains from the invariance of the Liouville measure λ' under the flows, in the finite volume case. The proof was already given for $d=1$ in [E-F-LJ-2]. We write it now for any d and for selfcontainedness.

Proof Let us use the invariance of the Liouville measure and of the coordinate u under the flows, and the expression of the Liouville measure in the coordinates system $\xi = \xi(z,u,r) \in \mathcal{O}\mathbb{H}$. We get for any $\zeta\in\mathbb{R}^d\times\mathbb{R}_+^*$ and any test functions H on $\partial\mathbb{H}$ and G on $\mathcal{O}\mathbb{H}$:

$$\int G(\xi(z,u,r)T_\zeta)\, H(u)\, d\mu_z^h(u)\, dV(z)\, dr = \int G(\xi(z,u,r))\, H(u)\, d\mu_z^h(u)\, dV(z)\, dr \ .$$

Thus we obtain for any $u\in\partial\mathbb{H}$, $z_0\in\mathbb{H}$, and any test function G on $\mathcal{O}\mathbb{H}$:

$$\int G(\xi(z,u,r)T_\zeta)\, B_u^d(z_0,z)\, dV(z)\, dr = \int G(\xi(z,u,r))\, B_u^d(z_0,z)\, dV(z)\, dr \ .$$

Whence using the definition 1 of $\tilde{\nu}$, a reference point $z_0\in\mathbb{H}$, the geometric property of (μ_z), and the (z,u,r)-coordinates on $\mathcal{O}\mathbb{H}$, we get :

$$\int G(\xi T_\zeta)\, d\nu'(\xi) = \int G(\xi(z,u,r)T_\zeta)\, \Phi(z)\, B_u^\delta(z_0,z)\, d\mu_{z_0}(u)\, dV(z)\, dr \ =$$

$$\int G(\xi(z,u,r))\Phi\circ\pi_2(\xi(z,u,r)T_\zeta^{-1})B_u^{\delta-d}(z_0,\pi_2(\xi(z,u,r)T_\zeta^{-1}))B_u^d(z_0,z)\,d\mu_{z_0}(u)dV(z)dr$$
$$= \int G(\xi)\times\frac{\Phi\circ\pi_2(\xi T_\zeta^{-1})}{\Phi\circ\pi_2(\xi)}\times\frac{B_u^{\delta-d}(z_0,\pi_2(\xi T_\zeta^{-1}))}{B_u^{\delta-d}(z_0,\pi_2(\xi))}\,d\nu'(\xi)$$
$$= \int G(\xi)\times\frac{\Phi\circ\pi_2(\xi T_\zeta^{-1})}{\Phi\circ\pi_2(\xi)}\times B_{\xi\theta_\infty}^{\delta-d}(\pi_2(\xi),\pi_2(\xi T_\zeta^{-1}))\,d\nu'(\xi)\,.$$

The result follows, since writing $\zeta=(x,y)$, we clearly have by the definition of B_u :

$$B_{\xi\theta_\infty}^{\delta-d}(\pi_2(\xi),\pi_2(\xi T_\zeta^{-1})) = B_{\xi\theta_\infty}^{d-\delta}(\pi_2(\xi\theta_{-\log y}),\pi_2(\xi)) = y^{d-\delta}\,.\ \diamond$$

4 Diffusions on $I\!H$ and on $OI\!H$

Let us from now on identify $I\!H$ with its Poincaré half-space model $I\!R^d\times I\!R_+^*$, and denote by $z=(x,y)$ the current point. Recall that

$$\Delta = y^2\times\left(\frac{\partial^2}{\partial y^2}+\frac{1-d}{y}\times\frac{\partial}{\partial y}+\sum_{j=1}^d\frac{\partial^2}{\partial x_j^2}\right).$$

4.1 The diffusions Z_t^δ , ξ_t^δ and Z_t^Φ

Let (w_t,W_t) denote a Brownian motion on $I\!R\times I\!R^d$, defined on $(\Omega,\mathcal{F},I\!P)$. Set

$$y_t := \exp\left[w_t+(\delta-d/2)\,t\right]\ ,\quad x_t := \int_0^t y_s\,dW_s\quad ,\quad Z_t^\delta := (x_t\,,y_t)\in I\!H\,.$$

For all δ, Z_t^δ is the diffusion on $I\!H$ starting from $e_o:=(0,1)$, with invariant measure $y^{2\delta-2d}dxdy$, and generator

$$\frac{1}{2}\Delta^\delta := \frac{1}{2}\Delta+\delta\,y\,\frac{\partial}{\partial y} = \frac{y^2}{2}\left(\frac{\partial^2}{\partial y^2}+\frac{2\delta+1-d}{y}\times\frac{\partial}{\partial y}+\sum_{j=1}^d\frac{\partial^2}{\partial x_j^2}\right).$$

Similarly, denote by $Z_t^b=(x_t^b,y_t^b)$ the analogous process with δ replaced by $b\in[0,d]$. In particular, Z_t^0 is the Brownian motion on $I\!H$.

Recall the classical identification between $OI\!H$ and the set of Möbius isometries of $I\!H$, and that in this identification we have for any $\xi\in OI\!H$ and any $z\in I\!H$: $\xi(z)=\pi_2(\xi T_z)$.

In particular, we see that $\pi_2(\xi T_{Z_t^0})=\xi(Z_t^0)$ is a Brownian motion on $I\!H$, started from $\pi_2(\xi)$, for any $\xi\in OI\!H$. As a consequence, denoting by P_t the Brownian semi-group on $I\!H$, we have $I\!E\big(f\circ\pi_2(\xi T_{Z_t^0})\big)=P_tf(\pi_2(\xi))$.

Observe then that $T_{Z_t^0}$ is a right Brownian motion on a subgroup of the affine group of $I\!R^d$. Indeed, for any $b\in[0,d]$,

$$T_{Z_t^b}^{-1}T_{Z_{t+s}^b} = T_{\left(\frac{x_{t+s}^b-x_t^b}{y_t^b},\,\frac{y_{t+s}^b}{y_t^b}\right)} = T_{Z_s^b}\circ\Theta_t$$

is independent of the sub-σ-field $\mathcal{F}_t$ generated by the coordinates until time t .

Definition 2 *For any* $\xi \in O\mathbb{H}$ *, set* $\xi_t^\delta := \xi\, T_{Z_t^\delta}$.
Set $\Delta^\Phi := \Phi^{-1}\,\Delta \circ \Phi - 2\lambda_0 = \Delta + 2\,(\nabla \log \Phi)\cdot \nabla$.
Denote by Z_t^Φ *and call "*Φ*-diffusion" the diffusion on* $\mathbb{H}$ *with generator* $\frac{1}{2}\Delta^\Phi$.

By the preceding observation, ξ_t^δ is a diffusion on $O\mathbb{H}$, starting from ξ.
From (5) we get $\Delta^\delta[F(\xi\, T_z)] = (D^\delta F)(\xi\, T_z)$, where

$$D^\delta := \sum_{j=0}^{d} \mathcal{L}_j{}^2 + (2\delta - d)\,\mathcal{L}_0 = D^0 + 2\,\delta\,\mathcal{L}_0 .$$

Then the generator of the diffusion ξ_t^δ is $\frac{1}{2}D^\delta$.

Moreover, note that the Φ-diffusion is symmetrical and has invariant measure V^Φ.
Note also that it happens to be the diffusion already considered in [Su1].
In the finite volume case $\delta = d$, this is just the Brownian motion.

Remark 3 We have for any test-function F on $O\mathbb{H}$:
$D^0(F\circ\pi_2)(\xi\, T_\cdot) = \Delta[F\circ\pi_2(\xi\, T_\cdot)] = \Delta(F\circ\xi) = (\Delta F)\circ\xi = (\Delta F)\circ\pi_2(\xi\, T_\cdot)$, whence
$D^0(F\circ\pi_2) = (\Delta F)\circ\pi_2$.

4.2 ν' as an invariant measure

We deduce now from the quasi-invariance property of ν' an adjonction property for ν' , and thus its invariance with respect to two diffusions on $O\mathbb{H}$.

Proposition 2 *We have for all* δ *and all test functions* F, G *on* $O\mathbb{H}$ *:*

$$\int (D^\delta F)\, G\, d\nu' = \int F\,(D^\Phi G)\, d\nu' , \qquad \textit{where}$$

$D^\Phi := \sum_{j=0}^d \mathcal{L}_j^2 - d\,\mathcal{L}_0 + 2\sum_{j=0}^d (\mathcal{L}_j \log \Phi\circ\pi_2)\,\mathcal{L}_j = (\Phi\circ\pi_2)^{-1}\, D^0\circ(\Phi\circ\pi_2) - 2\,\lambda_0$.

<u>Proof</u> We deduce directly from proposition 1 the infinitesimal expression of the quasi-invariance of ν' : we have for $j \in \{0,..,d\}$:

$$\int \mathcal{L}_j F\, d\nu' = -\int F \times (\mathcal{L}_j \log \Phi\circ\pi_2)\, d\nu' + 1_{\{j=0\}}\,(d-\delta)\int F\, d\nu' .$$

This implies immediately (writing Φ for $\Phi\circ\pi_2$) :

$$\mathcal{L}_j^* = -\mathcal{L}_j - \mathcal{L}_j(\log\Phi) + 1_{\{j=0\}}\,(d-\delta) ,$$

the adjoint being relating to ν'; whence (using that $D^0\,\Phi = (\Delta\Phi)\circ\pi_2 = 2\lambda_0\,\Phi$) :

$$(D^\delta)^* = D^\Phi + (d-\delta)\delta - d\,\mathcal{L}_0 \log\Phi + \sum_{j=0}^d \Big((\mathcal{L}_j \log\Phi)^2 + \mathcal{L}_j^2 \log\Phi \Big)$$

$$= D^\Phi + \Phi^{-1}\,(D^0\Phi - 2\lambda_0\Phi) = D^\Phi . \ \diamond$$

Corollary 1 *For $\xi \in \mathcal{O}I\!\!H$ and each δ we have :*

(i) under $I\!\!P$, the diffusion ξ_t^δ admits the invariant measure ν' ;

(ii) under $\nu' \otimes I\!\!P$, ξ_t^δ extends to a stationary diffusion defined for all real t, and ξ_{-t}^δ is the stationary diffusion associated with the infinitesimal generator $\frac{1}{2} D^\Phi$, say ξ_t^Φ.

Remark 4 (*i*) We see from remark 3 and from the h-process form of D^Φ in proposition 2 above that we have for any test-function F on $\mathcal{O}I\!\!H$:

$$D^\Phi(F \circ \pi_2) = (\Delta^\Phi F) \circ \pi_2 \,.$$

(*ii*) The h-process form of D^Φ shows that ξ_t^Φ can be defined by the following formula, where $0 = t_0 < .. < t_n$, $F_0, .., F_n$ are test-functions on $\mathcal{O}I\!\!H$, and $\xi_t^0 = \xi T_{Z_t^0}$:

$$\int\int \prod_{j=0}^n F_j(\xi_{t_j}^\Phi)\, d\nu'(\xi)\, dI\!\!P = \int\int \frac{e^{-\lambda_o t_n}}{\Phi \circ \pi_2(\xi)} \times \Phi \circ \pi_2\,(\xi_{t_n}^0) \times \prod_{j=0}^n F_j(\xi_{t_j}^0)\, d\nu'(\xi)\, dI\!\!P \,.$$

(*iii*) Recall that the identification between $I\!\!H$ and the half-space $I\!\!R^d \times I\!\!R_+^*$, respectively between $T^1 I\!\!H$ and $PSO(d+1,1)/SO_d$, amounts to fixing one point of $I\!\!H$ to be e_o , respectively of $T^1 I\!\!H$ to be $\vec{e_o}$, and similarly between $\mathcal{O}I\!\!H$ and $PSO(d+1,1)$. Note that although the process Z_t^δ depends on this identification, the diffusions Z_t^Φ , ξ_t^δ and ξ_t^Φ do not and are intrinsic.

By means of the coordinates (z,u) , we have for each $z \in I\!\!H$ an identification between $\partial I\!\!H$ and $T_z^1 I\!\!H$. This allows to consider the Patterson measure μ_z as a measure on $T_z^1 I\!\!H$, and then to localize the measure ν' as follows :

Definition 3 *For each $z \in I\!\!H$, set $\tilde{\nu}_z := \delta_z \otimes \mu_z / \Phi(z)$. This is a probability measure on $T^1 I\!\!H$, concentrated on $T_z^1 I\!\!H$.*

Denote also by ν'_z the probability measure on $\mathcal{O}I\!\!H$ concentrated on $\pi_2^{-1}(z)$, uniform on each fibre $\pi_1^{-1} \circ \pi_1(\xi)$, and projected on $\tilde{\nu}_z$ by π_1 .
So that we have the disintegration : $\nu' = \nu'_z \, dV^\Phi(z)$ on $\mathcal{O}I\!\!H$.

4.3 $\pi_2(\xi_t^\delta)$ is the Φ-diffusion

We obtain here the Φ-diffusion by projection of the diffusion ξ_t^δ .
We begin with the stationary case.

Proposition 3 *Under $\nu' \otimes I\!\!P$, the projection $\pi_2(\xi_t^\delta)$ of the stationary diffusion ξ_t^δ on $\mathcal{O}I\!\!H$ is the stationary Φ-diffusion.*

Proof Consider $0 = t_0 < t_1 < .. < t_n$ and test-functions $f_0, .., f_n$ on $I\!\!H$. Using corollary $(1, i, ii)$ and remark $(4, ii)$, we have :

$$\int\int \prod_{j=0}^n f_j \circ \pi_2(\xi_{t_j}^\delta)\, d\nu'(\xi)\, dI\!\!P = \int\int \prod_{j=0}^n f_j \circ \pi_2(\xi_{t_j - t_n}^\delta)\, d\nu'(\xi)\, dI\!\!P$$

$$= \int\int e^{-\lambda_o t_n} \times \frac{\Phi \circ \pi_2(\xi_{t_n}^0)}{\Phi \circ \pi_2(\xi_0^0)} \times \prod_{j=0}^n f_j \circ \pi_2(\xi_{t_n - t_j}^0)\, d\nu'(\xi)\, dI\!\!P$$

$$= \int\int e^{-\lambda_o t_n} \times \frac{\Phi\big(\xi(Z^0_{t_n})\big)}{\Phi \circ \pi_2(\xi)} \times \prod_{j=0}^{n} f_j\big(\xi(Z^0_{t_n - t_j})\big)\, d\pi_2^* \nu'(\xi)\, d\mathbb{P}$$

(since $\pi_2(\xi T_z) = \xi(z)$ and since $\xi(Z^0_t)$ is a Brownian motion starting from $\xi(e_o) = \pi_2(\xi)$, ξ being an isometry)

$$= \mathbb{E}_{V^\Phi}\left(\prod_{j=0}^{n} f_j(Z^\Phi_{t_n - t_j})\right)$$

(since $\pi_2^* \nu' = V^\Phi$ and by the h-process form of Δ^Φ in definition 2)

$$= \mathbb{E}_{V^\Phi}\Big(\prod_{j=0}^{n} f_j(Z^\Phi_{t_j})\Big)$$

by symmetry and invariance of Z^Φ with respect to V^Φ. $\diamond$

We can localize this result as follows.

Corollary 2 *Under the probability law $\nu'_z \otimes \mathbb{P}$, the projection $\pi_2(\xi^\delta_t)$ on $\mathbb{H}$ of the diffusion ξ^δ_t on $\mathcal{O}\mathbb{H}$ is the Φ-diffusion Z^Φ_t starting from z.*

Proof Let us apply the preceding proposition with $f_0(z) := 1_{\{dist(z,z')<\varepsilon\}}$. We get

$$\int_{\{dist(z,z')<\varepsilon\}} \int \mathbb{E}\left(\prod_{j=1}^{n} f_j \circ \pi_2(\xi^\delta_{t_j})\right) d\nu'_z(\xi) dV^\Phi(z) = \int_{\{dist(z,z')<\varepsilon\}} \mathbb{E}_z\left(\prod_{j=1}^{n} f_j(Z^\Phi_{t_j})\right) dV^\Phi(z).$$

Now the semi-group P^Φ_t of the Φ-diffusion is clearly Fellerian, and then the map $z \longmapsto \mathbb{E}_z\left(\prod_{j=1}^n f_j(Z^\Phi_{t_j})\right)$ is continuous. Thus the result will follow from the continuity of the map $z \longmapsto \int \mathbb{E}\Big(\prod_{j=1}^{n} f_j \circ \pi_2(\xi^\delta_{t_j})\Big)\, d\nu'_z(\xi)$, for compactly supported continuous functions f_j.

Now $(\xi, z) \longmapsto f_j \circ \pi_2(\xi T_z)$ is (uniformly) continuous on $T^1\mathbb{H} \times \mathbb{H}$, and then $\xi = (z, u, r) \longmapsto F(\xi) := \mathbb{E}\Big(\prod_{j=1}^{n} f_j \circ \pi_2(\xi^\delta_{t_j})\Big)$ is continuous and bounded on $\mathcal{O}\mathbb{H}$. Thus

$$\Big|\int F(\xi) d\nu'_z(\xi) - \int F(\xi) d\nu'_{z'}(\xi)\Big| = \Big|\int \left(\frac{F(z,u,r) B^\delta_u(z',z)}{\Phi(z)} - \frac{F(z',u,r)}{\Phi(z')}\right) d\mu_{z'}(u)\, dr\Big|$$

goes to zero as $z \to z'$ by dominated convergence, since $B_\cdot(z',z)$ is uniformly bounded for bounded $dist(z,z')$. $\diamond$

This allows to deduce immediately the asymptotic behaviour of the Φ-diffusion in the case $\delta \geq d/2$.

Corollary 3 *The Φ-diffusion Z^Φ_t starting from $z \in \mathbb{H}$ converges almost surely as $t \to \infty$ to a random point belonging to $\partial\mathbb{H}$, whose law is $\mu_z/\Phi(z)$.*

<u>Proof</u> It is clear from the definition that y_t, and then Z_t^δ, goes almost surely to ∞ with t when $\delta > d/2$. When $\delta = d/2$, setting $r_t := \left(|x_t|^2 + y_t^2\right)^{1/2}$, we see from the expression of the generator $\frac{1}{2}\Delta^{d/2}$ that $r_t = B^{d+2}(Y_t)$, where $B^{d+2}(t)$ is some $(d+2)$-dimensional Bessel process and $Y_t := \int_0^t y_s^2\, ds = \int_0^t e^{2w_s}\, ds$, which both go almost surely to ∞ with t, thereby showing that r_t, and then $Z_t^{d/2}$, also does.

Now corollary 2 above and the convergence of Z_t^δ to ∞ imply the convergence of $\pi_2(\xi_t^\delta)$ to $\xi(\infty) = u$, whose law under $\nu_z' \otimes \mathbb{P}$ is $\mu_z/\Phi(z)$, by definition of ν_z'. $\diamond$

Remark 5 It is however false that $\pi_2(\xi_t^\delta)$ has under $\mathbb{P}$ the law of the Φ-diffusion. Indeed, as observed above for corollary 3, $\pi_2(\xi_t^\delta)$ is conditionned to exit at $\xi(\infty)$.

5 Exit measure of the Φ-diffusion if $\delta < d/2$

For this whole section, we consider the only case : $\delta \in [0, d/2[$.

Lemma 1 $\int_{\partial\mathbb{H}} \mathrm{ch}^{2\delta}(dist(z,uw))\, d\mu_z^h(w)$ *does not depend on* $(z,u) \in \mathbb{H} \times \partial\mathbb{H}$, *but only on* d *and* δ (*and equals* $\pi^{d/2} \times \dfrac{\Gamma(d/2-\delta)}{\Gamma(d-\delta)}$).

<u>Proof</u> Indeed, the left hand side of this formula being clearly intrinsic, it is sufficient to work with the model of the ball centered at z. Then the harmonic measure μ_z^h is uniform, which in turn implies the independence with respect to u.

This independence is sufficient for our future purpose, however let us now perform the computation of the constant, interesting for the intertwining formula in remark 6 below. Taking polar coordinates in $\partial\mathbb{H} \equiv \mathbb{S}^d$, with leading angle the one between u and w, say α, we have $\mathrm{ch}\,(dist(z,uw)) = 2/|u-w| = (\sin\alpha/2)^{-1}$ and thus we get:

$$\int_{\partial\mathbb{H}} \mathrm{ch}^{2\delta}(dist(z,uw))\, d\mu_z^h(w) = \int_{\mathbb{S}^d} (\sin\alpha/2)^{-2\delta}\, d\mu_O^h$$

$$= |\mathbb{S}^{d-1}| \times \int_0^\pi (\sin\alpha/2)^{-2\delta} (\sin\alpha)^{d-1}\, d\alpha \times |\mathbb{S}^d|^{-1} \times |\mu^h_\cdot|$$

$$= \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \times 2^d \int_0^\infty (1+r^2)^{\delta-d} r^{d-2\delta-1} dr \times |\mathbb{S}^d|^{-1} \times \int_{\mathbb{R}^d} p^d(z,v)\, dv$$

$$= \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \times 2^{d-1} \int_0^1 s^{d/2-1}(1-s)^{d/2-\delta-1} ds \times |\mathbb{S}^d|^{-1} \times \int_{\mathbb{R}^d} (1+|w|^2)^{-d}\, dw$$

$$= 2^d\, \pi^{d/2} \times \frac{\Gamma(d/2-\delta)}{\Gamma(d-\delta)} \times |\mathbb{S}^d|^{-1} \times |\mathbb{S}^{d-1}| \times \frac{\Gamma(d/2)^2}{2\,\Gamma(d)} = \pi^{d/2} \times \frac{\Gamma(d/2-\delta)}{\Gamma(d-\delta)}\,. \ \diamond$$

Remark 6 Another form of the formula of the preceding lemma is the following intertwining formula between $p^{d-\delta}$ and p^δ, which expresses the unextremal kernel p^δ as a mean of the extremal kernels $p^{d-\delta}$: For all $(z,u) \in \mathbb{H} \times \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} p^{d-\delta}(z,w)\, |u-w|^{-2\delta}\, dw \ = \ \pi^{d/2} \times \frac{\Gamma(d/2-\delta)}{\Gamma(d-\delta)} \times p^\delta(z,u)\,.$$

Indeed, to see this it is sufficient to apply the formula $(*)$, to express the harmonic measure and the Busemann function by means of the Poisson kernel, and to observe that for ζ at the top of the geodesic $\overrightarrow{uw}$ we clearly have : $p(\zeta,u)p(\zeta,w) = |u-w|^{-2}$. A complicated proof of this intertwining formula was written in ([M], epilogue).
M. Babillot and J.P. Otal already knew a simple proof for it. See also ([G], p. 386).

Definition 4 *Set for* $w \in \partial I\!H$ *et* $z \in I\!H$:

$$q(z,w) := \int_{\partial I\!H} \mathrm{ch}^{2\delta}(dist(z,uw))\, d\mu_z(u)\,,\ d\check{\mu}_z(w) := \pi^{-d/2}\,\frac{\Gamma(d-\delta)}{\Gamma(d/2-\delta)}\, q(z,w)\, d\mu_z^h(w)\,,$$

and on $T^1 I\!H$:

$$\check{\nu}\,(dz,du) := \Phi(z)\,\check{\mu}_z(du)\,dV(z)\quad,\qquad and\qquad \check{\nu}_z := \delta_z \otimes \check{\mu}_z/\Phi(z)\,.$$

Denote by $\check{\nu}'$ *(respectively* $\check{\nu}'_z$ *) the measure on* $\mathcal{O}I\!H$ *projected onto* $\check{\nu}$ *(respectively* $\check{\nu}_z$ *) by* π_1 *and uniform on each fibre* $\pi_1^{-1}\circ\pi_1(\xi)$.

Lemma 2 *(i) For all* $\gamma \in \Gamma$, $z \in I\!H$ *and* $w \in \partial I\!H$, *we have* $q(\gamma z, \gamma w) = q(z,w)$.
(ii) $\check{\mu}_z(\partial I\!H) = \Phi(z)$, *and then* $\check{\nu}_z$ *is a probability measure.*
(iii) The family of measures $(\check{\mu}_z)$ *satisfies the invariance property and the geometrical property of exponent* $(d-\delta)$:
$\gamma^*\check{\mu}_z = \check{\mu}_{\gamma z}$ *for all* $\gamma \in \Gamma$, *and* $d\check{\mu}_{z'}(u) = B_u^{d-\delta}(z,z')\, d\check{\mu}_z(u)$ *for all* $z,z' \in I\!H$.
(iv) $\check{\nu}\,(dz,du) = \check{\nu}_z(du)\, V^{\Phi}(dz)$ *has an infinite mass on a fundamental domain, and* $\pi^*\check{\nu} = V^{\Phi}$.

Proof (i) follows directly from the invariance property of (μ_z);

(ii) follows directly from Lemma 1 ;

(iii) The invariance formula follows from the invariance property of the harmonic measure and from (i) above, and the second formula follows from the formula $(*)$ and from the cocycle property of the Busemann function ;

(iv) is straightforward. $\diamond$

As the proof of the quasi-invariance formula for ν' (see proposition 1) uses only the geometrical property of (μ_z), we deduce the quasi-invariance formula for $\check{\nu}$ below (using the above lemma (2, iii)) merely by changing δ into $d-\delta$.

Proposition 4 *The measure* $\check{\nu}'$ *is quasi-invariant under the geodesic and positive horocycle flows :*

$$\frac{d(T_z{}^*\,\check{\nu}')}{d\check{\nu}'}(\xi) = y^{\delta} \times \frac{\Phi\circ\pi_2(\xi T_z^{-1})}{\Phi\circ\pi_2(\xi)} \qquad \text{for all } \xi \in \mathcal{O}I\!H \text{ and } z=(x,y) \in I\!H\ .$$

As for Proposition 2, we deduce :

Corollary 4 *For all test-functions* F, G *we have :*

$$(i)\quad \int \mathcal{L}_j F\, d\check{\nu}' = -\int F \times (\mathcal{L}_j \log \Phi\circ\pi_2)\, d\check{\nu}' + 1_{\{j=0\}}\,\delta \int F\, d\check{\nu}'\,;$$

$$(ii)\quad \int (D^{d-\delta} F)\, G\, d\check{\nu}' = \int F\, (D^{\Phi} G)\, d\check{\nu}'\,.$$

As for Corollary 1, we deduce immediately :

Corollary 5 *For all $\delta < d/2$ we have :*

(i) under $\mathbb{P}$, $\xi_t^{d-\delta}$ is the diffusion on $\mathcal{O}\mathbb{H}$ starting from ξ and with generator $\frac{1}{2} D^{d-\delta}$; it admits the invariant measure $\check{\nu}'$;

(ii) under $\check{\nu}' \otimes \mathbb{P}$, $\xi_t^{d-\delta}$ extends to a stationary diffusion defined for all real t, and $\xi_{-t}^{d-\delta}$ equals the stationary diffusion ξ_t^{Φ} on $\mathcal{O}\mathbb{H}$ (with generator $\frac{1}{2} D^{\Phi}$).

We have also the analogue of Proposition 3, using Proposition 4 instead of Proposition 1. Indeed it is sufficient to adapt the proof of proposition 3, changing everywhere δ into $d-\delta$ and ν' into $\check{\nu}'$, and noticing that $\pi_2^* \check{\nu}' = V^{\Phi}$.

Proposition 5 *For all $\delta < d/2$, under $\check{\nu}' \otimes \mathbb{P}$, the projection $\pi_2(\xi_t^{d-\delta})$ of the stationary diffusion $\xi_t^{d-\delta}$ on $\mathcal{O}\mathbb{H}$ is the stationary Φ-diffusion.*

Using this proposition 5 instead of Proposition 3, we get (by the same argument, merely adapted by changing respectively δ into $d-\delta$, ν_z' into $\check{\nu}_z'$, and $\mu_{z'}$ into $\check{\mu}_{z'}$) the analogue of Corollary 2.

Corollary 6 *Under the probability law $\check{\nu}_z' \otimes \mathbb{P}$, the projection $\pi_2(\xi_t^{d-\delta})$ on $\mathbb{H}$ of the diffusion $\xi_t^{d-\delta}$ on $\mathcal{O}\mathbb{H}$ is the Φ-diffusion Z_t^{Φ} starting from z.*

This allows to deduce the asymptotic behaviour of the Φ-diffusion in the case $\delta < d/2$. The proof is the same as for Corollary 3 (when $\delta > d/2$).

Corollary 7 *When $\delta < d/2$, the Φ-diffusion Z_t^{Φ} starting from $z \in \mathbb{H}$ converges almost surely as $t \to \infty$ to a random point of $\partial\mathbb{H}$, whose law is $\check{\mu}_z/\Phi(z)$.*

Remark 7 The general theory of Martin boundary for rank one symmetric spaces can be applied to yield Corollaries 3 and 7.
Let us briefly outline the main steps of such a proof.
a) Characterization of the Martin boundary of $(\mathbb{H}, \Delta - \lambda I)$ as $\partial\mathbb{H}$, and of its Martin kernels as $B_u(z,z')^{\delta}$ when $\delta \geq d/2$.
This work has been done for example in Proposition 3.2 of [L-MG-T] and in [G].
b) Application of an extended version of the Fatou-Naïm-Doob theorem, as the one given in Theorem 3.1 of [A].
c) Use of the intertwining formula (see our remark 6), as we used above our lemma 1, for the case $\delta < d/2$, in which the $B_u(z,z')^{\delta}$ are not extremal (see also [G], p. 386).

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REFERENCES

[A] ANCONA A. *Théorie du potentiel sur les graphes et les variétés.*
Ecole d'été de probabilités de Saint-Flour XVIII, Lecture Notes 1427, Springer 1990.

[E] ENRIQUEZ N. Thèse de l'Université Paris Sud , 3^{eme} partie, 91405 Orsay, Septembre 1995.

[E-F-LJ-1] ENRIQUEZ N., FRANCHI J., LE JAN Y. *Enroulements des géodésiques sous la mesure de Patterson-Sullivan.* C.R.A.S. Paris, tome 326, Série 1, 723-726, 1998.

[E-F-LJ-2] ENRIQUEZ N., FRANCHI J., LE JAN Y. *Stable windings on hyperbolic surfaces.* Prépublication, September 1999.

[E-LJ] ENRIQUEZ N., LE JAN Y. *Statistic of the winding of geodesics on a Riemann surface with finite area and constant negative curvature.* Rev. Mat. Iberoamericana, Vol. 13, 2, 377-401, 1997.

[F] FRANCHI J. *Asymptotic singular homology of a complete hyperbolic 3-manifold of Finite Volume.* Proc. London Math. Soc. (3) n^o 79, 451-480, 1999.

[G] GUIVARC'H Y. *Sur la représentation intégrale des fonctions harmoniques et des fonctions propres positives dans un espace riemannien symétrique.* Bull. Sci. Math., 2^e série, n^o 108, 373-392, 1984.

[L] LEDRAPPIER F. *Harmonic 1-forms on the stable foliation.* Bol. Soc. Bras. Math. 25, n^o 2, 121-138, 1994.

[LJ1] LE JAN Y. *The central limit theorem for the geodesic flow on non compact manifolds of constant negative curvature.* Duke Math. J. 74 n^o 1, 159-175, 1994.

[LJ2] LE JAN Y. *Free energy for Brownian and geodesic homology.* Probab. Theory Rel. Fields 102, 57-61, 1995.

[L-MG-T] LYONS T.J., MAC GIBBON K.B., TAYLOR J.C. *Projection theorems for hitting probabilities and a theorem of Littlewood.* Journal of Functional Analysis n^o 59, 470-489, 1984.

[M] MANDOUVALOS N. *Scattering operator, Eisenstein series, inner product formula and Maass-Selberg relations for Kleinian groups.* Memoirs of the A.M.S. vol 78, n^o 400, 1989.

[Ni] NICHOLLS P.J. *The ergodic theory of discrete groups.* London Math. Society, Lecture Note Series n^o 143, Cambridge University Press, 1989.

[P] PATTERSON S.J. *Lectures on measures on limit sets of Kleinian groups.* Analytical and geometrical aspects of hyperbolic space, D. Epstein editor, 281-323, London Math. Society, Lecture Note Series n^o 111, Cambridge University Press, 1987.

[P-S] PHILLIPS R.S., SARNAK P. *The Laplacean for domains in hyperbolic space and limit sets of Kleinian groups.* Acta Math. 155, 173-241, 1985.

[S-V] STRATMAN B., VELANI S.L. *The Patterson measure for geometrically finite groups with parabolic elements, new and old.* Proc. London Math. Soc. (3) n^o 71, 197-220, 1995.

[Su1] SULLIVAN D. *The density at infinity of a discrete group of hyperbolic motions.* Publ. Math. I.H.E.S. n^o 50, 171-209, 1979.

[Su2] SULLIVAN D. *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups.* Acta Math. 153, 259-277, 1984.

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