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A simple proof of the L^p continuity of the higher order Riesz Transforms with respect to the Gaussian measure γ_d

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Abstract. In this paper we will give a simple proof of the $L^p(\gamma_d)$ continuity of the higher order Riesz transforms with respect to the Gaussian measure γ_d, with constant independent of the dimension, by means of a multiplier theorem of P. A. Meyer.

Résumé. Dans cet article nous donnons une démonstration simple de la continuité $L^p(\gamma_d)$ des transformations de Riesz d'ordre supérieur par rapport à la mesure gaussienne avec constante indépendante de la dimension. La méthode de preuve est basée sur un théorème de multiplicateur de P. A. Meyer.

1 Introduction

Let us consider the Gaussian measure $\gamma_d(dx) = \frac{1}{\pi^{d/2}} e^{-|x|^2} dx$ in $\mathbb{R}^d$, and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2}\Delta - \langle x, \nabla_x \rangle.$$

The higher order Riesz transforms associated with this operator are defined as

$$\mathcal{R}_\alpha = D_x^\alpha (-L)^{-|\alpha|/2}, \tag{1}$$

The study of the $L^p(\gamma_d)$ boundedness of $\mathcal{R}_\alpha$ with constant independent of the dimension goes back to the work of P. A. Meyer [4] whose proof is based on probabilistic methods. R. Gundy gave another probabilistic proof in [1]. On the other hand, G. Pisier in [5], proved these inequalities analytically by using the transference method due to A. P. Calderón. By using pointwise estimates of the kernel, W. Urbina in [6] proved the result with constant depending on the dimension. Lately, C. Gutiérrez, C. Segovia and J. L. Torrea in [3], proved

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the $L^p(\gamma_d)$ boundedness of $\mathcal{R}_\alpha$ as a consequence of the $L^p(\gamma_d)$ boundedness with constant independent of dimension of the Euclidean norm of the vector $\vec{R}_k = \{\mathcal{R}_\alpha\}_{|\alpha|=k}$. To prove this last result they use an extension of the Littlewood-Paley-Stein theory of g functions of higher order to the vector-valued case, and inequalities previously proved by C. Gutiérrez [2] in the context of the Riesz transforms of order one.

In this paper we will give a simple proof of the $L^p(\gamma_d)$ boundedness of $\mathcal{R}_\alpha$, with constant independent of the dimension, by means of a multiplier theorem of P. A. Meyer. More explicitly the main result of this paper is the following:

Theorem 1 *The higher order Riesz transforms $\mathcal{R}_\alpha$ with respect to the Gaussian measure are bounded on $L^p(\gamma_d)$, $1 < p < +\infty$, with constant independent of the dimension, that is, for each α there exists a positive constant $C_{\alpha,p}$, depending on α and p, such that*

$$||\mathcal{R}_\alpha f||_{p,\gamma_d} \leq C_{\alpha,p}||f||_{p,\gamma_d}. \tag{2}$$

for all $f \in L^p(\gamma_d)$.

Before proving this theorem let us introduce some notations which will be used in the multiplier theorem of P. A. Meyer as well as in the proof of this result.

Let $H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$ be the one-dimensional Hermite polynomial of order n with $n \in \mathbb{N} \cup \{0\}$. These verify the following differential equation

$$-\frac{1}{2}H_n''(t) + tH_n'(t) = nH_n(t). \tag{3}$$

Besides

$$H_n'(t) = 2nH_{n-1}(t) \tag{4}$$

and

$$||H_n||^2_{2,\gamma_d} = \sqrt{\pi}\, n!\, 2^n. \tag{5}$$

Letting $\beta = (\beta_1, \ldots, \beta_d)$ be a multi-index with non-negative integer entries, the d-dimensional Hermite polynomial of order β is defined as

$$H_\beta(x) = \prod_{i=1}^{d} H_{\beta_i}(x_i),$$

with $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then

$$||H_\beta||^2_{2,\gamma_d} = \pi^{\frac{d}{2}} \beta!\, 2^{|\beta|},$$

where $\beta! = \beta_1! \ldots \beta_d!$ and $|\beta| = \beta_1 + \cdots + \beta_d$. Now we define the normalized d-dimensional Hermite polynomial of order β as

$$h_\beta(x) = \frac{H_\beta(x)}{||H_\beta||_{2,\gamma_d}} = \prod_{i=1}^{d} h_{\beta_i}(x_i),$$

which is the product of one-dimensional normalized Hermite polynomials.

These polynomials are the eigenvectors of $-L$, that is

$$(-L)h_\beta = |\beta| h_\beta.$$

$(-L)^{-\frac{k}{2}}$ is the Riesz Potential of order k which is defined on every d-dimensional Hermite polynomial by $(-L)^{-\frac{k}{2}} h_\beta = \frac{1}{|\beta|^{\frac{k}{2}}} h_\beta$ with $|\beta| > 0$ and extended by linearity on every polynomial $f(x)$ such that $\int_{\mathcal{R}^n} f(x) d\gamma(x) = 0$. From (1), the higher order Riesz transform of order α is

$$\mathcal{R}_\alpha h_\beta = \frac{1}{|\beta|^{\frac{|\alpha|}{2}}} D_x^\alpha h_\beta = \frac{1}{|\beta|^{\frac{|\alpha|}{2}}} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} h_\beta, \tag{6}$$

and by linearity is extended to every polynomial on $I\!R^d$.

Theorem 1 is a consequence of the following Theorem due to P. A. Meyer (see [4])

Theorem 2 *(P. A. Meyer) Let ϕ be a real function analytic around the origin and let us consider a multiplier operator*

$$T_\varphi\Big(\sum_\beta c_\beta h_\beta\Big)(x) = \sum_\beta \varphi(|\beta|) c_\beta h_\beta(x), \tag{7}$$

where $\varphi(n) = \phi(\frac{1}{n})$ for $n \geq n_0$ and n_0 large enough. Then T_φ admits a bounded extension to $L^p(\gamma_d)$, $1 < p < \infty$, that is, for any $f \in L^p(\gamma_d)$

$$||T_\varphi f||_{p,\gamma_d} \leq C_\varphi ||f||_{p,\gamma_d}. \tag{8}$$

Its proof is basically based on the hypercontractivity property of the Ornstein-Uhlenbeck semigroup and it holds with constant independent of dimension. Moreover it is true in infinite dimensions, for more details see [4].

2 Proof of the main result

P. A. Meyer proved this result in [4] by using probabilistic methods and his multiplier theorem. Indeed, let the vector $\nabla^k = (D_x^\alpha)_{|\alpha|=k}$, then $\vec{R}_k = \nabla^k(-L)^{-\frac{k}{2}}$; he proved the following inequality by means of Khintchine's inequality

$$||\ |\vec{R}_k f|\ ||_{p,\gamma_d} \leq ||\ |(-L)^{\frac{1}{2}} \nabla^{k-1} (-L)^{-\frac{k}{2}}|\ ||_{p,\gamma_d}$$

and then he says that if $(-L)^{\frac{1}{2}}$ and ∇^{k-1} could be interchanged then the result would follow by induction on k. This is not true but there is a multiplier that relates both operators, i.e. $(-L)^{\frac{1}{2}}\nabla^{k-1} = T_k\nabla^{k-1}(-L)^{\frac{1}{2}}$ and therefore

$$\begin{aligned} \| \, |\vec{R}_k f| \, \|_{p,\gamma_d} &\leq \| \, |T_k \nabla^{k-1}(-L)^{-\frac{k-1}{2}} f| \, \|_{p,\gamma_p} \\ &= \| \, |T_k \vec{R}_{k-1} f| \, \|_{p,\gamma_p} \\ &\leq C_{p,k} \| \, |\vec{R}_{k-1} f| \, \|_{p,\gamma_d}, \end{aligned}$$

thus the result follows by induction on k and by taking into account that the Riesz transforms of order one are bounded operators on $L^p(\gamma_d)$ with constant independent of the dimension.

Here, we write the higher order Riesz transform $\mathcal{R}_\alpha$ as the composition of powers of Riesz transforms of order one together with a multiplier operator which by Theorem 2 is a bounded operator on $L^p(\gamma_d)$ with constant independent of the dimension. Then the theorem follows from de $L^p(\gamma_d)$ continuity of the Riesz transforms of order one and that of the multiplier operator.

Let us consider a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ fixed. The action of $\mathcal{R}_\alpha$ on the Hermite polynomials h_β is as follows

$$\mathcal{R}_\alpha h_\beta(x) = \left(\frac{2^{|\alpha|}}{|\beta|^{|\alpha|}}\right)^{1/2} \left[\prod_{i=1}^{d} \beta_i(\beta_i - 1)\cdots(\beta_i - \alpha_i + 1)\right]^{1/2} h_{\beta-\alpha}(x), \quad (9)$$

with $\beta_i \geq \alpha_i$ for all $i = 1, \cdots, d$.

Now, for the same multi-index α, let us consider the operator

$$\mathcal{R}_1^{\alpha_1}\mathcal{R}_2^{\alpha_2}\ldots\mathcal{R}_d^{\alpha_d}, \quad (10)$$

that is, the iteration of powers of Riesz transforms of order one. Then

$$\mathcal{R}_1^{\alpha_1}\mathcal{R}_2^{\alpha_2}\ldots\mathcal{R}_d^{\alpha_d} h_\beta(x) = 2^{|\alpha|/2} \prod_{i=1}^{d} \left[\frac{\beta_i(\beta_i - 1)\cdots(\beta_i - \alpha_i + 1)}{|\beta|(|\beta| - 1)\ldots(|\beta| - \alpha_i + 1)}\right]^{1/2} h_{\beta-\alpha}(x). \quad (11)$$

Comparing (9) and (11) let us consider the multiplier operator T_α, defined on the Hermite polynomials as

$$\begin{aligned} T_\alpha h_\beta(x) &= \left[\frac{\prod_{i=1}^{d} |\beta|(|\beta| - 1)\ldots(|\beta| - \alpha_i + 1)}{|\beta|^{|\alpha|}}\right]^{1/2} h_\beta(x) \\ &= \left[\prod_{i=1}^{d} (1 - \frac{1}{|\beta|})\ldots(1 - \frac{(\alpha_i - 1)}{|\beta|})\right]^{1/2} h_\beta(x). \end{aligned}$$

Then, T_α satisfies the conditions of the multiplier theorem of P. A. Meyer with

$$\phi(x) = \left[\prod_{i=1}^{d}(1-x)\dots(1-(\alpha_i-1)x)\right]^{1/2},$$

and clearly, by definition,

$$\mathcal{R}_\alpha = (\mathcal{R}_1^{\alpha_1}\mathcal{R}_2^{\alpha_2}\dots\mathcal{R}_d^{\alpha_d})\circ T_\alpha \tag{12}$$

Therefore $L^p(\gamma_d)$-continuity of $\mathcal{R}_\alpha$ can be obtained immediately from the $L^p(\gamma_d)$-boundedness with constant independent of the dimension of the Riesz transforms $\mathcal{R}_i$ and Meyer's result, and the constant depends only on α and p.

References

1. Gundy, R. *Sur les transformations de Riesz pour le semigroupe d'Ornstein-Uhlenbeck*, C.R. Acad. Sci. 303 (Série I) (1986), 967-970.
2. Gutiérrez, C. *On the Riesz transforms for the Gaussian measure.* J. Fourier Anal. 120 (1) (1994) 107-134.
3. Gutiérrez, C., Segovia C.& J. L. Torrea. *On higher Riesz transforms for the Gaussian measure.* J. Fourier Anal. Appl. Vol 2 #6 (1996) 583-596.
4. Meyer, P.A. *Transformations de Riesz pour les lois gaussiennes.* Lectures Notes in Math. 1059 (1984) Springer-Verlag. Berlin. 179-193 .
5. Pisier, G. *Riesz transform: a simpler analytic proof of P. A. Meyer inequality.* Lectures Notes in Math 1321. Springer-Verlag (1988) 485-501.
6. Urbina, W. *Singular Integrals with respect to the Gaussian measure.* Scuola Normale Superiore di Pisa. Classe di Science. Serie IV Vol XVIII, 4 (1990) 531-567.