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Malgorsata Kuchta
Michal Morayne
Slawomir Solecki

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A MARTINGALE PROOF OF THE THEOREM BY JESSEN, MARCINKIEWICZ AND ZYGMUND ON STRONG DIFFERENTIATION OF INTEGRALS

Małgorzata Kuchta, Michał Morayne, Sławomir Solecki

Abstract

We give a martingale proof of the theorem by Jessen, Marcinkiewicz and Zygmund on almost everywhere strong differentiability of functions on $\mathbf{R}^n$ belonging to $\mathrm{L}(\mathrm{Log}^+ \mathrm{L})^{n-1}$. The proof is based on Cairoli's theorem on convergence of multi-indexed martingales.

There are a few (independently obtained but similar) martingale proofs of the Lebesgue integral differentiation theorem in $\mathbf{R}^n$ ([Ch], [M], [MS]). The main tool in these proofs is Lévy's martingale convergence theorem. They substantially simplify geometric considerations involved in the standard proof of Lebesgue's theorem via Vitali's covering theorem. This approach, however, does not seem to have been used to prove the Jessen, Marcinkiewicz and Zygmund theorem ([JMZ]) on strong differentiability of integrals. It turns out that this too can be done if one uses Cairoli's theorem on convergence of multi-indexed martingales (instead of Lévy's theorem). The proof given here goes very much along the lines of [MS]; it adapts the techniques used there to the case of strong differentiation of integrals.

We shall use the following standard notation. $\mathbf{Z}$ will denote the set of all integers, $\sigma(\mathcal{A})$ the *σ–field* generated by a family of sets $\mathcal{A}$. The *n–dimensional Lebesgue measure in* $\mathbf{R}^n$ will be denoted by λ (we omit the exponent n here as there will be no danger of confusion). Let $(\Omega, \mathcal{F}, P)$ be a probability space. By $\mathrm{L}_1(\Omega)$ we shall denote the family of all real, $\mathcal{F}$-measurable functions such that $E|f| < \infty$. In the case when Ω is an open subset of $\mathbf{R}^n$ and $\lambda(\Omega) = 1$, by $\mathcal{F}$ we shall always mean the family of Lebesgue (or Borel) measurable sets and $P = \lambda \mid \mathcal{F}$. If A is a subset of $\mathbf{R}^n$, the set of Borel subsets of $\mathcal{A}$ will be denoted by $\mathcal{B}(A)$. If $\mathcal{A}$ is a family of sets and X a set, $\mathcal{A} \mid X$ denotes the set $\{A \cap X : A \in \mathcal{A}\}$. For a subset X of $\mathbf{R}^n$ and a vector $x \in \mathbf{R}^n$, we put $x + X = \{x + y : y \in X\}$. A parallelepiped in $\mathbf{R}^n$ is the product of n open non-empty intervals. By $\delta(A)$ we denote the diameter of A. We say that a point x of an open set $U \subseteq \mathbf{R}^n$ is a *strong Lebesgue point* for f if

$$\frac{1}{\lambda(Q_m)} \int_{Q_m} |f(s) - f(x)|\, ds \to 0$$

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for each sequence of parallelepipeds Q_m such that $x \in Q_m \subseteq U$ for each $m \leq 1$ and $\delta(Q_m) \to 0$. (Without loss of generality, we can assume that x is the center of Q_m, i.e., $Q_m = x + \big((-\delta_1, \delta_1) \times \ldots \times (-\delta_n, \delta_n)\big)$ for some $\delta_1, \ldots, \delta_n$.)

Here is the theorem of Jessen, Marcinkiewicz and Zygmund.

Theorem 1 (Jessen, Marcinkiewicz, Zygmund). *Let $f : \mathbf{R}^n \to \mathbf{R}$ be such that $|f|(\log^+ |f|)^{n-1} \in \mathrm{L}_1$. Almost every point of $\mathbf{R}^n$ is a strong Lebesgue point of f.*

The following theorem proved in [C] will be our main tool to prove Theorem 1.

Theorem 2 (Cairoli). *For $i = 1, \ldots, n$, let $(\Omega_i, \mathcal{F}_i, P_i)$ be a probability space and $\{\mathcal{F}_{i,j} : j \in \mathbf{N}\}$ a sequence of σ-algebras such that $\mathcal{F}_{i,j} \subseteq \mathcal{F}_{i,j+1}$ and $\sigma(\bigcup_{j\in\mathbf{N}} \mathcal{F}_{i,j}) = \mathcal{F}_i$. Call $(\Omega, \mathcal{F}, P)$ the product space $\prod_{i=1}^n (\Omega_i, \mathcal{F}_i, P_i)$, and for $J = (j_1, \ldots, j_n) \in \mathbf{N}^n$, call $\mathcal{F}_J$ the sub-σ-field $\mathcal{F}_{1,j_1} \times \ldots \times \mathcal{F}_{n,j_n}$ of $\mathcal{F}$.*

If $f : \Omega \to \mathbf{R}$ is $\mathcal{F}$-measurable and $E|f|(\log^+ |f|)^{n-1} < \infty$, then conditional expectations $f_J = E(f \mid \mathcal{F}_J)$ converge almost surely to f when $J \to \infty$ (that is, when $j_1 \to \infty, \ldots, j_n \to \infty$).

We derive first the following corollary which was suggested as one of the simplifications by the referee. P.-A. Meyer informed us that it is an extension of Hunt's lemma [DM, Chapter V, Theorem 45].

Corollary 1. *With the same notation and assumptions as in Theorem 2, $E\big(|f - f_J| \mid \mathcal{F}_J\big) \to 0$ almost surely when $J \to \infty$.*

Proof. For each $a \in \mathbf{R}$, one has almost surely

$$\limsup_{J\to\infty} E\big(|f - f_J| \mid \mathcal{F}_J\big) \leq \limsup_{J\to\infty} E\big(|f - a| \mid \mathcal{F}_J\big) + \limsup_{J\to\infty} |f_J - a| = 2\,|f - a|\,,$$

wherefrom, almost surely, $\limsup_{J\to\infty} E\big(|f - f_J| \mid \mathcal{F}_J\big) \leq 2 \inf_{a\in\mathbf{Q}} |f - a| = 0.$■

Corollary 2. *In the situation of Theorem 2, assume furthermore that each $\mathcal{F}_{i,j}$ is generated by a finite or countable partition of Ω_i. For almost all $\omega \in \Omega$, $E\big(|f - f(\omega)| \mid \mathcal{F}_J\big)(\omega)$ tends to 0 when $J \to \infty$.*

Proof. By the countability assumption, conditional expectations with respect to $\mathcal{F}_J$ are defined everywhere, and not only almost everywhere; so the expression $E\big(|f - f(\omega)| \mid \mathcal{F}_J\big)(\omega)$ is meaningful. One has

$$E\big(|f - f(\omega)| \mid \mathcal{F}_J\big) \leq E\big(|f - f_J(\omega)| \mid \mathcal{F}_J\big) + |f_J(\omega) - f(\omega)|\,.$$

The second term tends to zero for almost all ω by Theorem 2; the first term, when evaluated at ω, is equal to $E\big(|f - f_J| \mid \mathcal{F}_J\big)(\omega)$, which tends to 0 for almost all ω by Corollary 1.■

For $t = (t_1, ..., t_n) \in \mathbf{R}^n$ and $J = (j_1, .., j_n) \in \mathbf{N}^n$, we shall denote by $\mathcal{A}_J^t$ the partition of $\mathbf{R}^n$ consisting of all sets of the form

$$\prod_{i=1}^{n} \left(t_i + \tfrac{k_i}{2^{j_i}}\,,\ t_i + \tfrac{k_i+1}{2^{j_i}}\right]$$

where $(k_1, ..., k_n)$ ranges over $\mathbf{Z}^n$. For $x = (x_1, ..., x_n) \in \mathbf{R}^n$, the element of $\mathcal{A}_J^t$ that contains x will be denoted by $I_J^t(x)$. If $n = 1$, we let $I_j^t(x)$, $t, x \in \mathbf{R}$, $j \in \mathbf{N}$, stand for $I_{(j)}^{(t)}((x))$. For $\Delta = (\delta_1, ..., \delta_n) \in (0, \infty)^n$, we shall call $Q_\Delta(x)$ the parallelepiped $\prod_{i=1}^n (x_i - \delta_i\,,\ x_i + \delta_i)$; its center is x and its diameter is minorized by $\sup_i \delta_i$.

All the geometry we need is contained in the following simple lemma, where T denotes the set $\{0, \frac{1}{3}\}^n$.

Lemma. *Suppose $J = (j_1, ..., j_n) \in \mathbf{N}^n$ and $\Delta = (\delta_1, ..., \delta_n) \in (0, \infty)^n$ satisfy $3\delta_i \le 2^{-j_i}$ for each $i \in \{1, ..., n\}$. Then, for every $x \in \mathbf{R}^n$, one has*

$$Q_\Delta(x) \subseteq \bigcup_{t \in T} I_J^t(x)\,.$$

Proof. As $\bigcup_{t\in T} I_J^t(x)$ is the Cartesian product of the one-dimensional sets $I_{j_i}^0(x_i) \cup I_{j_i}^{1/3}(x_i)$, it suffices to establish $(x_i - \delta_i\,,\ x_i + \delta_i) \subseteq I_{j_i}^0(x_i) \cup I_{j_i}^{1/3}(x_i)$; in other words, we may suppose $n = 1$ and drop the index i. Now, for any k and ℓ in $\mathbf{Z}$, the distance between $k\,2^{-j}$ and $\ell\,2^{-j} + \frac{1}{3}$ is at least $\frac{1}{3}\,2^{-j}$, hence at least δ. As x belongs to both intervals $I_j^0(x)$ and $I_j^{1/3}(x)$, and as the distance from each end-point of $I_j^0(x)$ to each end-point of $I_j^{1/3}(x)$ is at least δ, the union $I_j^0(x) \cup I_j^{1/3}(x)$ must contain $(x - \delta, x + \delta)$.∎

Proof of Theorem 1. Losing no generality, we assume that $|f|\,(\log^+|f|)^{n-1}$ is in $\mathrm{L}_1((0,1)^n)$ and we shall prove that almost every point of the cube $(0,1)^n$ is a strong Lebesgue point of f. By modifying the Lebesgue-measurable function f on a negligible set, we also assume, with no loss of generality, that f is Borel-measurable.

We have to show that, for almost every $x \in (0,1)^n$,

$$\frac{1}{\lambda(Q_\Delta(x))} \int_{Q_\Delta(x)} |f(s) - f(x)|\, ds$$

tends to 0 when the diameter of $Q_\Delta(x)$ tends to 0.

Call $\mathcal{F}_J^t$ the finite σ-field on $(0,1)^n$ generated by the restriction $\mathcal{A}_J^t \mid (0,1)^n$ of the partition $\mathcal{A}_J^t$ to the cube $(0,1)^n$; notice that $\mathcal{F}_J^t$ is a product σ-field with factors $\mathcal{F}_j^t$ satisfying $\mathcal{F}_j^t \subseteq \mathcal{F}_{j+1}^t$ and $\sigma\big(\bigcup_j \mathcal{F}_j^t\big) = \mathcal{B}\big((0,1)\big)$ for fixed t. So for fixed t we are in the situation of Theorem 2. To each $\Delta = (\delta_1, ..., \delta_n)$, associate $J = (j_1, ..., j_n)$ such that $2^{-j_i-1} < 3\delta_i \le 2^{-j_i}$. According to the above lemma, the inequality $3\delta_i \le 2^{-j_i}$ implies $Q_\Delta(x) \subseteq \bigcup_{t\in T} I_J^t(x)$; on the other hand, the inequality $2^{-j_i-1} < 3\delta_i$ easily gives $\lambda\big(I_J^t(x)\big) \le 3^n\,\lambda\big(Q_\Delta(x)\big)$. So one can write

$$\begin{aligned}\frac{1}{\lambda\big(Q_\Delta(x)\big)}\int\limits_{Q_\Delta(x)}|f(s)-f(x)|\,ds &\le \frac{1}{\lambda\big(Q_\Delta(x)\big)}\sum_{t\in T}\int\limits_{I_J^t(x)}|f(s)-f(x)|\,ds\\ &\le 3^n\sum_{t\in T}\frac{1}{\lambda\big(I_J^t(x)\big)}\int\limits_{I_J^t(x)}|f(s)-f(x)|\,ds\\ &= 3^n\sum_{t\in T}E\big(|f-f(x)|\,\big|\,\mathcal{F}_J^t\big)(x)\,.\end{aligned}$$

(The latter equality requires $I_J^t(x) \subseteq (0,1)^n$; this can be obtained for instance by restricting x to belong to $(\varepsilon, 1-\varepsilon)^n$, and by taking $\delta_i < \varepsilon/6$.)

Now, when the diameter of $Q_\Delta(x)$ tends to 0, $\sup_i \delta_i$ tends to 0, and $J \to \infty$. As the factor 3^n and the finite set T do not depend on x, Δ and J, the result follows by applying Corollary 2 for each fixed $t \in T$.■

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1.2. Institute of Mathematics, Wrocław Technical University, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, POLAND
e-mail addresses: kuchta@graf.im.pwr.wroc.pl, morayne@graf.im.pwr.wroc.pl
2. Institute of Mathematics of the Polish Academy of Sciences - Wrocław Branch, Kopernika 18, 51-617 Wrocław, POLAND
3. Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
e-mail address: ssolecki@indiana.edu