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Some invariance properties (of the laws) of Ocone's martingales

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In this note, some properties of continuous martingales shall be investigated, starting from the following important remarks :

from Lévy's characterization of Brownian motion as the continuous martingale (M_t) with increasing process $\langle M\rangle_t = t$, one readily deduces the following invariance properties :

i) if (ε_t) is a predictable process which only takes the values + 1 and -1, then :

$$M^{\varepsilon} \overset{(law)}{=} M, \text{ where : } M_t^{\varepsilon} \overset{def}{=} \int_0^t \varepsilon_s dM_s \ ;$$

ii) for every bounded predictable process (φ_t),

$$D_t^{\varphi} = \exp\left(\int_0^t \varphi_s dM_s - \frac{1}{2}\int_0^t \varphi_s^2 d\langle M\rangle_s\right)$$

is a martingale, and if we denote by $Q \equiv Q^{\varphi}$ the probability such that :

$$Q_{|\mathcal{F}_t} = D_t^{\varphi}\cdot P_{|\mathcal{F}_t}$$

then : $\tilde{M}^{\varphi} \equiv M - \int_0^{\cdot} \varphi_s \, d\langle M\rangle_s$ satisfies : $\{\tilde{M}^{\varphi}, Q\} \overset{(law)}{=} \{M,P\}$

iii) if $S_t^M = \sup_{s\le t} M_s$, then : $S^M - M \overset{(law)}{=} |M|$

To avoid any confusion, let us emphasize again that these identities in law are true for M a $(\mathcal{F}_t)$ Brownian motion. (Indeed, for a general continuous local martingale M, D^{φ} may only be a local martingale...).
In fact, in the sequel, where M is not in general a Brownian motion, it will be convenient for our discussion to consider some adequate variants of ii), precisely :

ii)$_{det}$ Same as ii), but φ is now a deterministic, Borel, bounded, process ;

ii)$_{\langle M\rangle}$ Same as ii), but φ is now a bounded predictable process, which depends only on $\langle M\rangle$.

The rest of this paper consists in discussing which continuous martingales M, other than $(\mathcal{F}_t)$ Brownian motions, satisfy i), or some of the above variants of ii) or iii).
For instance, it is not difficult to prove that, more generally, if (M_t) is a Gaussian martingale, which is, as is well-known, equivalent to :

$$(\langle M\rangle_t, t \geq 0) \text{ is a deterministic process,}$$

then all three properties are still valid.

Pushing those arguments a little further, it is not difficult again to show that these properties are still valid for (M_t) an Ocone martingale that is : a martingale whose Dubins-Schwarz representation : $M_t = \beta_{\langle M\rangle_t}$ features independent β (: Brownian motion) and $\langle M\rangle$.
The reason for our terminology is that Ocone [2] showed that this independence property is equivalent to the above property i). Moreover, a discussion of the interest of Ocone martingales in relation with Lévy's transformation (in other terms, property iii)) is made in [1].
Concerning property ii), we shall now show the

Theorem 1 : *The following properties are equivalent :*

a) (M_t) *is an Ocone martingale.*

b) Property $ii)_{<M>}$ *holds ; c) Property* $ii)_{det}$ *holds.*

Proof : · c) $\Longrightarrow$ a) We assume that $ii)_{det}$ holds, and we consider a deterministic integrand φ, always assumed to be Borel, bounded.

We then use that, for positive functionals F, one has, simply from the definition of Q :

$$E_Q[F(<M>_s, s \le t)] = E_P[F(<M>_s, s \le t) D_t^{\varphi}].$$

Now, as a consequence of $ii)_{det}$, one has :

$$E_Q[F(<M>_s, s \le t)] = E_P[F(<M>_s, s \le t)],$$

so that :

$$E_P[F(<M>_s, s \le t)] = E_P[F(<M>_s, s \le t) D_t^{\varphi}].$$

Obviously, this is equivalent to : $E_P[D_t^{\varphi} \,|\, <M>_s, s \le t] = 1$,

hence also to :

$$E_P\left[\exp\left(\int_0^t \varphi_s dM_s\right) \Big| <M>_s, s \le t\right] = \exp\left(\frac{1}{2}\int_0^t \varphi_s^2 \, d<M>_s\right). \tag{1}$$

The right-hand side of *(1)* is equal to :

$$E_P\left(\exp\left(\int_0^t \varphi_s d(\gamma_{<M>_s})\right) \Big| <M>_s, s \le t\right),$$

where $(\gamma_u, u \ge 0)$ is a Brownian motion independent of $(<M>_s, s \ge 0)$.

Hence, the identity *(1)* yields :

(2) $$(M_t, \langle M\rangle_t \ ; \ t \geq 0) \overset{(law)}{=} (\gamma_{\langle M\rangle_t}, \langle M\rangle_t \ ; \ t \geq 0).$$

Recall that : $M_t = \beta_{\langle M\rangle_t}$; hence, time-changing both sides of *(2)* with the inverse of $(\langle M\rangle_t, t \geq 0)$, we obtain :

$$\{(\beta_u, u \geq 0) \ ; \ (\langle M\rangle_t, t \geq 0)\} \overset{(law)}{=} \{(\gamma_u, u \geq 0) \ ; \ (\langle M\rangle_t, t \geq 0)\}$$

which shows precisely that β and $\langle M\rangle$ are independent.

· a) $\Longrightarrow$ b) : We start from $(M_t, t \geq 0)$ an Ocone martingale, and we consider an integrand $\varphi(s)$ of the form : $\Phi(\langle M\rangle_u, u \leq s)$, which is predictable and bounded.

Then, we have, denoting simply $\tilde{M}$ for $\tilde{M}^{\varphi}$:

$$E_Q(F(\tilde{M}_u, u \leq t)]$$
$$= E_P\left[F(\tilde{M}_u, u \leq t) \exp\left(\int_0^t \varphi(s)dM_s - \frac{1}{2}\int_0^t \varphi^2(s)d\langle M\rangle_s\right)\right].$$

We then recall :

$$\tilde{M}_u = \beta_{\langle M\rangle_u} - \int_0^{\langle M\rangle_u} \varphi(\tau_v)dv \ , \ u \leq t$$

and we also perform the time-change in the exponential.

Next, within the latter expectation, we condition with respect to the σ-field generated by $(\langle M\rangle_u, u \geq 0)$; then, as a consequence of the well-known property ii) for Brownian motion, we obtain that :
this conditional expectation is equal to

$$E_P[F(\beta_{\langle M\rangle_u}, u \leq t) \ | \ \langle M\rangle]$$

and we denote :

$$G_t = \gamma + \int_0^t \varphi(\langle M\rangle_u) dM_u .$$

On the other hand, we associate to F the $\{\mathcal{N}_t\}$ martingale :

$$F_t \equiv E[F|\mathcal{N}_t] = E[F|\mathcal{M}_t] ,$$

using the first part of the theorem.

We now apply Itô's formula :

$$F_tG_t = F_0\gamma + \int_0^t F_{s-}dG_s + \int_0^t G_s dF_s + [F,G]_t,$$

but, since G is continuous, and (F_t) and (G_t) are orthogonal, we have :

$$[F,G]_t = \langle F^c,G\rangle_t \equiv 0.$$

Thus, finally, $\Phi \equiv FG = F_0\gamma + \int_0^\infty F_{s-}\ \varphi(\langle M\rangle_s)dM_s + \int_0^\infty G_s dF_s$,

which proves the second point. □

We now give some examples of Ocone, and non-Ocone martingales.

Theorem 3 : *Let* $(B_t,t \geq 0)$ *be a* $(\mathcal{F}_t)$ *Brownian motion, and (for simplicity) let* $(\mu_t,t \geq 0)$ *be a* $(\mathcal{F}_t)$ *adapted, continuous process such that* $\mu_s \neq 0$, ds dP *a.s., and* $\int_0^\infty \mu_s^2 ds = \infty$ *a.s.*

Then, $\left\{M_t = \int_0^t \mu_s\, dB_s,\ t \geq 0\right\}$ *is an Ocone martingale iff the Brownian motion* so that finally we have obtained :

$$E_Q[F(\tilde{M}_u, u \le t)] = E_P[F(M_u, u \le t)].$$

· b) $\Longrightarrow$ c) : This is obvious. □

Comment 1 : Note that property ii) only involves martingale densities (D_t^{φ}) which are stochastic integrals with respect to dM_t. In [3] it is remarked in Exercise (1.41), Chap. VIII, that the only martingales (M_t) such that the reinforcement of ii) holds with any possible martingale density are the martingales with deterministic bracket, i.e : the Gaussian martingales, denoted by $\mathcal{G}$ below.

Comment 2 : α) It is quite doubtful that the general property ii) is satisfied for an Ocone martingale ; we postpone investigating this equation in depth.

β) To avoid lengthening the statement of Theorem 1, we did not add there the following equivalent property d), which is nonetheless worth mentioning :

d) for every deterministic bounded process φ, $\{\langle M\rangle, Q^{\varphi}\} \stackrel{(law)}{=} \{\langle M\rangle, P\}$.
The proof of the equivalence : a) $\Longleftrightarrow$ d) uses the same arguments as : a) $\Longleftrightarrow$ c).

Comment 3 : Although an Ocone martingale shares properties i), $ii)_{\langle M\rangle}$ and iii) with Brownian motion, it does not share a priori the important martingale representation property, that is, precisely : every martingale (N_t) , with respect to the natural filtration of (M_t) , is not necessarily a stochastic integral with respect to M.

Indeed, there is the following

Proposition : *An Ocone martingale* $(M_t, t \ge 0)$ *enjoys the martingale representation property (with respect to its natural filtration) iff* $(\langle M\rangle_t, t \ge 0)$ *is a deterministic process.*

Proof : Since, from the definition of an Ocone martingale, $\langle M\rangle$ is independent of β, the DDS Brownian motion associated to M, we can write :

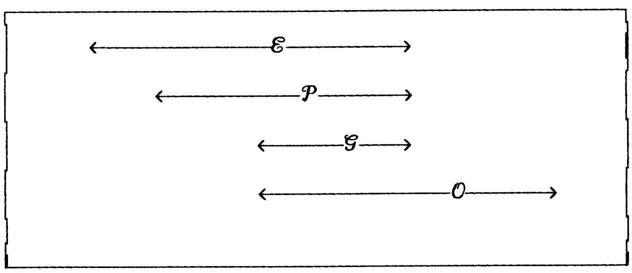
$$(*) \qquad P_M = \int P(\langle M\rangle \in da)\, W^a ,$$

where P_M , resp : W^a , denotes the law of M, resp : the law of the continuous martingale with (deterministic) increasing process $a(\cdot)$.

We now recall (see, e.g. [3], Chap. V) that M enjoys the (martingale) representation property iff P_M is extremal among the set of laws of (continuous) martingales.

Now, from (*) , it follows that P_M is extremal iff $P(\langle M\rangle \in da)$ reduces to a Dirac measure ; in other terms, there exists a deterministic increasing function $a(\cdot)$ such that : $P(\langle M\rangle = a(\cdot)) = 1$. □

At this point, it seems interesting to draw the following diagram, which indicates 4 remarkable classes of continuous (local) martingales :



A classification of continuous local martingales.

The four letters stand for : $\mathcal{E}$: extremal, $\mathcal{P}$: pure, $\mathcal{G}$: gaussian, $\mathcal{O}$: Ocone. And the diagram indicates that : $\mathcal{G} \subset \mathcal{P} \subset \mathcal{E}$, where $\subset$ denotes strict inclusion.

On the other hand, $\mathcal{O} \cap \mathcal{E} = \mathcal{O} \cap \mathcal{P} = \mathcal{G}$.

We now wish to complete the above Proposition by describing all martingales with respect to the natural filtration $\{\mathcal{M}_t\}$ of (M_t). It will be useful to introduce $\{\mathcal{N}_t\}$ the natural filtration of $\{\langle M\rangle_t, t \geq 0\}$.

We now prove the

Theorem 2 : *Let* (M_t) *be an Ocone martingale :*

1) Every $\{\mathcal{N}_t\}$ *martingale* (N_t) *is an* $\{\mathcal{M}_t\}$ *martingale, and it is orthogonal to* (M_t) *, that is :* $(N_t M_t, t \geq 0)$ *is a* $\{\mathcal{M}_t\}$ *local martingale ;*

2) The space of square integrable $\{\mathcal{M}_t\}$ *martingales is the direct sum of the stable space generated by* $\{\mathcal{N}_t\}$ *martingales and of the stable space generated by* $\{M_t\}$.

Proof : 1) Consider (N_t) a uniformly integrable $\{\mathcal{N}_t\}$ martingale.

We shall show : $E[N_\infty | \mathcal{M}_t] = N_t \; (= E[N_\infty | \mathcal{N}_t])$

which proves the first point of the first assertion of the theorem.

With obvious notation, one has :

$$
\begin{aligned}
& E[N_\infty \, f(M_s, s \leq t)] \\
&= E[N_\infty \, f(\beta_{\langle M\rangle_s}, s \leq t)] \\
&= \int W(d\omega) \; E[N_\infty \, f(\omega(\langle M\rangle_s), s \leq t)] \\
&= \int W(d\omega) \; E[N_t \, f(\omega(\langle M\rangle_s), s \leq t)] \\
&= E[N_t \, f(M_s , s \leq t)].
\end{aligned}
$$

Similarly, we now show that $(N_t M_t, t \geq 0)$ is a $\{\mathcal{M}_t\}$ martingale. Let $s < t$. Then, we have :

$$E[N_t M_t \, f(M_u, u \le s)]$$

$$= \int W(d\beta) \, E[N_t \, \beta_{\langle M\rangle_t} \, f(\beta_{\langle M\rangle_u}, u \le s)].$$

We then use the martingale property for β, and the independence of $\langle M\rangle$ and β ; this yields :

$$E[N_t \, M_t \, f(M_u, u \le s)] = \int W(d\beta) \, E[N_t \, \beta_{\langle M\rangle_s} \, f(\beta_{\langle M\rangle_u}, u \le s)].$$

Next, we use the martingale property for (N_t), with respect to $\{\mathcal{N}_t\}$; we obtain :

$$E[N_t \, M_t \, f(M_u, u \le s)] = \int W(d\beta) \, E[N_s \, \beta_{\langle M\rangle_s} \, f(\beta_{\langle M\rangle_u}, u \le s)]$$

$$= E[N_s \, M_s \, f(M_u, u \le s)].$$

2) To show the second point, it suffices to consider variables $\Phi \in L^2(\mathcal{M}_\infty, P)$, of the form : $\Phi = FG$, where $F \in L^2(\mathcal{N}_\infty, P)$ and $G \in L^2(\mathcal{B}_\infty, P)$, where $\mathcal{B}_\infty = \sigma\{\beta_s, s \ge 0\}$.

As is well known, G may be written in the form :

$$G = \gamma + \int_0^\infty \varphi(s) d\beta_s \, ,$$

for some $\gamma \in \mathbb{R}$, and some $\{\mathcal{B}_t\}$ predictable process φ such that :

$$E\left[\int_0^\infty \varphi^2(s) ds\right] < \infty.$$

Making the time change $s = \langle M\rangle$, we obtain :

$$G = \gamma + \int_0^\infty \varphi(\langle M\rangle_u) dM_u \, ,$$

$\left\{\theta_t \overset{def}{=} \int_0^t \operatorname{sgn}(\mu_s)dB_s, t \geq 0\right\}$ *is independent from the* σ*-field*

$$\mathcal{N}_\infty = \sigma\{|\mu_s|, s \geq 0\}.$$

Proof : As previously discussed, $\{M_t, t \geq 0\}$ is an Ocone martingale iff

$$(3) \qquad E\left[\exp\left(i\int_0^\infty \varphi(s)dM_s\right) \Big| \mathcal{N}_\infty\right] = \exp\left(-\frac{1}{2}\int_0^\infty \varphi^2(s)\, d\langle M\rangle_s\right)$$

for any $\{\mathcal{N}_s\}$ predictable process φ, such that : $\int_0^\infty \varphi^2(s)d\langle M\rangle_s < \infty$.

The identity *(3)* is obviously satisfied if θ is independent from $\mathcal{N}_\infty$.

Conversely, assuming that *(3)* holds, we now take : $\varphi(s) = f(s)\dfrac{1}{|\mu_s|}$ with f a generic, simple, deterministic function, with compact support. Thus, we deduce from *(3)* :

$$E\left[\exp\left(i\int_0^\infty f(s)d\theta_s\right) \Big| \mathcal{N}_\infty\right] = \exp\left(-\frac{1}{2}\int_0^\infty f^2(s)ds\right),$$

which is obviously equivalent to the independence of θ and $\mathcal{N}_\infty$. □

Here is another (fairly general) variant of Theorem 3.

Theorem 3' : *Assume that* $\{\mathcal{N}_t\}$ *is the natural filtration of a* $\{\mathcal{F}_t\}$ *martingale* (N_t) *which is pure, i.e :* $N_t = \gamma_{\langle N\rangle_t}, t \geq 0$, *with* $(\langle N\rangle_t, t \geq 0)$ *measurable with respect to the* σ*-field* $\sigma\{\gamma_u, u \geq 0\}$ *of the Brownian motion* $(\gamma_u, u \geq 0)$.

Then, $(M_t, t \geq 0)$ *is an Ocone martingale as soon as* N *and* M *are orthogonal.*

Proof : It follows immediately from our hypothesis and Knight's theorem on continuous orthogonal martingales (see, e.g, [3]) that β and γ, the respective DDS Brownian motions of M and N are independent. Now, since $\mathcal{N}_\infty$ is, again under our hypothesis equal to the σ-field generated by γ, $\mathcal{N}_\infty$ and β are independent, which finishes the proof. □

To conclude this work, we present a number of simple examples of Ocone, resp : non-Ocone, martingales.

Of course, to avoid trivialities, when looking for Ocone martingales, we exclude the Gaussian examples (one might call the non-Gaussian Ocone martingales "strictly Ocone" martingales).

a) Perhaps, the most simple example of an Ocone martingale is

$$M_t^{(1)} = \int_0^t C_s dB_s \qquad , \; t \geq 0,$$

where B and C are two independent Brownian motions.

The stochastic area of the planar Brownian motion (B_t, C_t), defined as :

$$\mathcal{A}_t = \frac{1}{2} \int_0^t (C_s dB_s - B_s dC_s)$$

is another example of an Ocone martingale.

This follows readily from Theorem 3, since :

$$\langle \mathcal{A} \rangle_t = \frac{1}{4} \int_0^t ds \; R_s^2 \quad , \qquad R_s^2 \equiv B_s^2 + C_s^2 \; ,$$

and we can write :

$$\mathcal{A}_t = \frac{1}{2} \int_0^t R_s d\gamma_s \; ,$$

where $\gamma_t \overset{\text{def}}{=} \int_0^t \frac{C_s dB_s - B_s dC_s}{R_s}$, $t \geq 0$, is a real-valued Brownian motion independent from $(R_t, t \geq 0)$; see, e.g., Yor [9].

b) Here are now some examples of non-Ocone martingales, among which :

$$M_t^{(2)} = \int_0^t B_s dB_s \ , \ t \geq 0$$

and

$$\pi_t \overset{\text{def}}{=} B_t C_t = \int_0^t (C_s dB_s + B_s dC_s).$$

Indeed, $(M_t^{(2)}, t \geq 0)$ is a pure martingale, which is easily seen by time-changing it with the inverse of

$$\langle M^{(2)} \rangle_t = \int_0^t ds \, B_s^2$$

(see, e.g., Stroock-Yor [6]). Since $\langle M^{(2)} \rangle$ is not deterministic, it follows from the above classification that $M^{(2)}$ is not an Ocone martingale.
Also, it is easily seen that the property d) in Comment 2 above is not satisfied for $\varphi(s) \equiv \lambda$ ($\in \mathbb{R}$). In fact, under $Q^{\varphi} \equiv Q^{\lambda}$, the process (B_t) is an Ornstein-Uhlenbeck process with parameter λ, hence :

$$(\langle M^{(2)} \rangle, Q^{\lambda}) \neq (\langle M^{(2)} \rangle, P).$$

The argument we shall use for $\{\pi_t\}$ is somewhat different : first of all, one finds :

$$\langle \pi \rangle_t = \int_0^t ds \, R_s^2 \ ,$$

but the process $\{R_u, u \geq 0\}$ is certainly not independent from the Dubins-Schwarz Brownian motion attached to $\{\pi_t\}$ (hence, $\{\pi_t\}$ is not an Ocone martingale).

We see this non-independence property as follows : since one has, obviously : $|2B_tC_t| \le R_t^2$, then, conditionally on $(R_u, u \ge 0)$, the variable (B_tC_t) cannot be Gaussian, since it is bounded.

Thus, $\{\pi_t\}$ is not an Ocone martingale ; we also remark that it is of the form $\int_0^t (A\,\mathbb{B}_s, d\mathbb{B}_s)$, where $\mathbb{B}_s = \begin{pmatrix} B_s \\ C_s \end{pmatrix}$, and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence, since A is symmetric and has two non-zero, distinct eigenvalues (+1 and -1), it follows from [7] that the natural filtration of $\{\pi_t\}$ is that of a 2-dimensional Brownian motion.

We end up with an example of an Ocone martingale $M_t = \int_0^t \varphi_s \, dB_s$ within the filtration of a 1-dimensional Brownian motion $(B_t, t \ge 0)$, and we also assume that $\varphi_s \ne 0$, ds dP a.s. (otherwise, there are some quite easy examples).

We write : $M_t = \int_0^t \tilde{\varphi}_s \, d\tilde{B}_s$, with $\tilde{B}_s = \int_0^s \operatorname{sgn}(B_u) dB_u$ (hence : $\varphi_s = \tilde{\varphi}_s \operatorname{sgn}(B_s)$)

With the help of Theorem 3, we choose $(\tilde{\varphi}_s, s \ge 0)$ to be a strictly positive process, independent from the Brownian motion $\{\tilde{B}_t\}$, whose natural filtration is identical to that of $\{|B_t|, t \ge 0\}$. This implies that $\{M_t, t \ge 0\}$ is an Ocone martingale ; to be more explicit, we may take, as an example of process $\{\tilde{\varphi}_s\}$:

$$\tilde{\varphi}_s = a_s \, 1_{(s<t_o)} + (1 + 2\operatorname{sgn}(B_{t_o}))1_{(t_o \le s)} ,$$

for some $t_o > 0$, and $\{a_s\}$ a strictly positive deterministic function.

We also note that, as a consequence of Theorem 2, no strictly Ocone martingale (M_t) , such that $(d\langle M\rangle_t)$ is equivalent to Lebesgue measure dt on $\mathbb{R}_+$ can generate the filtration of a 1-dimensional Brownian motion.

Final comment : As a consequence of Theorem 2, the filtration of an Ocone martingale, when it is not Gaussian, corresponds, in Mathematical finance, to an incomplete market. However, the above representation theorem should be useful to discuss further constructions of probability measures now familiar in such cases in Mathematical finance, e.g : the variance-optimal martingale measure for continuous processes [5].

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