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On the Onsager-Machlup functional for elliptic diffusion processes

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Abstract

In this paper, we show, following K. Hara and Y. Takahashi [7], how the stochastic Stokes theorem and the Kunita-Watanabe theorem on orthogonal martingales may be used to produce a general and easy computation of the Onsager-Machlup functional of an elliptic diffusion process, even for large classes of norms in Wiener space.

1 Introduction

In their paper [7], K. Hara and Y. Takahashi present a rather simple and efficient computation of the Onsager-Machlup functional of an elliptic diffusion process for the supremum norm. The first task of this work is to present this simple approach and to show how it extends to various families of norms on Wiener space. In particular, this approach to the Onsager-Machlup functional is rather elementary when the diffusion matrix is the identity matrix and may be applied to the cases of the L^2-norm and L^p-norms with $2 < p < 4$ yielding some new results.

First we introduce the problem of the Onsager-Machlup functional for an elliptic diffusion process. Denote by $\mathcal{C}_0\left([0,1];\mathbb{R}^d\right)$ the space of $\mathbb{R}^d$-valued, continuous functions on $[0,1]$ vanishing at the origin, by $\|\cdot\|_\infty$ the supremum norm on $[0,1]$ and by $\mathcal{B}$ the Borel σ-field of $\left(\mathcal{C}_0\left([0,1];\mathbb{R}^d\right),\|\cdot\|_\infty\right)$. Let P be the Wiener measure defined on $\mathcal{B}$ and $\omega = (\omega_1,\ldots,\omega_d)$ be the canonical Brownian motion on the Wiener space $\left(\mathcal{C}_0\left([0,1];\mathbb{R}^d\right),\mathcal{B},P\right)$. Let $X(t)$ be the elliptic diffusion process solution of the stochastic differential equation

$$dX(t) = \sigma(X(t))\,d\omega(t) + b(X(t))\,dt,\ X(0) = x_0,\ X(t) \in \mathbb{R}^m,$$

where σ is an $m \times d$ matrix of smooth vector fields on $\mathbb{R}^m$, b is a smooth vector field on $\mathbb{R}^m$ and x_0 belongs to $\mathbb{R}^m$. A Riemannian structure is naturally induced by the diffusion coefficients on $\mathbb{R}^m$, such that the generator of the diffusion is $\frac{1}{2}\Delta_M + f$ where Δ_M is the Laplace-Beltrami operator and f is a vector field. To be more precise, equip $\mathbb{R}^m$ with the metric $g = (\sigma\sigma^*)^{-1}$ and the Levi-Cevita connection (we denote by σ^* the transpose of the matrix σ). It is quite natural to consider X as the diffusion on the Riemannian manifold $M = (\mathbb{R}^m, g)$, with generator $\frac{1}{2}\Delta_M + f$, where for every $1 \le i \le m$,

$$f_i(x) = b_i(x) + \frac{1}{2}\sum_{l,j=1}^{m} (\sigma\sigma^*)_{lj}(x) \left\{ {}_{l}{}^{i}{}_{j} \right\},$$

denoting by $\left\{ {}_{l}{}^{i}{}_{j} \right\}$ the Christoffel symbol:

$$\left\{ {}_{l}{}^{i}{}_{j} \right\} = \frac{1}{2}\sum_{q=1}^{m} \left\{ \frac{\partial}{\partial x_l}(\sigma\sigma^*)^{-1}_{qj} + \frac{\partial}{\partial x_j}(\sigma\sigma^*)^{-1}_{lq} - \frac{\partial}{\partial x_q}(\sigma\sigma^*)^{-1}_{lj} \right\} (\sigma\sigma^*)_{iq}.$$

Recall that the Laplace-Beltrami operator is given by

$$\Delta_M F = \sum_{i,j=1}^{m} (\sigma\sigma^*)_{ij} \frac{\partial^2 F}{\partial x_i \partial x_j} - \sum_{i,j,k=1}^{m} (\sigma\sigma^*)_{ij} \left\{ {}_{i}{}^{k}{}_{j} \right\} \frac{\partial F}{\partial x_k}.$$

Let $\rho(x,y)$ be the Riemannian distance on M and let $\|\cdot\|$ be a measurable norm on a subspace of $\mathcal{C}_0([0,1],\mathbb{R})$. Let Φ and Ψ be two smooth M-valued functions on $[0,1]$, starting at x_0. Denote by $\rho(X,\Phi)$ the $\mathbb{R}$-valued function on $[0,1]$ defined by $\rho(X,\Phi) : t \to \rho(X(t),\Phi(t))$. If the limit

$$\lim_{\epsilon\to 0} \frac{P(\|\rho(X,\Phi)\| < \epsilon)}{P(\|\rho(X,\Psi)\| < \epsilon)}$$

exists and can be written as

$$\exp\left[\int_0^1 L\left(\Phi(s),\dot{\Phi}(s)\right) ds - \int_0^1 L\left(\Psi(s),\dot{\Psi}(s)\right) ds\right],$$

such a function L on the tangent bundle TM is called the Onsager-Machlup function of X for the norm $\|\cdot\|$.

Y. Takahashi and S. Watanabe proved in [15] that, if one chooses on $\mathcal{C}_0([0,1];\mathbb{R})$ the supremum norm, the Onsager-Machlup function on the tangent bundle TM is given by

$$L(p,v) = -\frac{1}{2}\|f(p) - v\|_p^2 - \frac{1}{2}\text{div} f(p) + \frac{1}{12}R(p),$$

where, for every p in M, $\|\cdot\|_p$ denotes the Riemannian norm on the tangent space T_pM at p, $\text{div} f(p)$ is the divergence of f at p and $R(p)$ is the scalar curvature at p. They used probabilistic techniques such as Girsanov's formula, stochastic Stokes' theorem (see Lemma 2 below) and the Kunita-Watanabe theorem on orthogonal martingales (see Lemma 3 below). In [5], T. Fujita and S. Kotani obtained the same result by a purely analytical approach. In [4], we proved that this expression of the Onsager-Machlup function is still valid for a large class of norms on $\mathcal{C}_0([0,1];\mathbb{R})$, including in particular Hölder norms $\|\cdot\|_\alpha$ with $0 < \alpha < \frac{1}{2}$. One key idea in [15] consisted in a Besselization technique to come down to small balls of a Bessel process, using Girsanov's transformation. But as the Besselizing drift was singular, the proof was quite complex. Recently, K. Hara has found a new Besselizing drift which is smooth (see [6] and [7]). This makes the proof of [15] simpler and this shorter proof has been presented in [7]. As a consequence of the regularity of the drift, the stochastic Stokes theorem may be applied in greater generality and the proofs do not require anymore bounds on small balls probabilities. In particular, we observe here that this new approach is still valid for large classes of norms on Wiener space, simplifying the early reasonings of [4].

The method of K. Hara and Y. Takahashi is presented in section 3 of this paper. Section 2 consists of a collection of basic lemmas that will be used throughout this work. In the last section, we discuss the case of constant diffusion coefficients and compare recent results for various norms on Wiener space in this context.

2 Fundamental lemmas

Lemma 1 *(see [8]) Let $I_1, \ldots, I_n$ be n random variables on a probability space $(\Omega, \mathcal{B}, P)$. Let $\{A_\epsilon\}_{0<\epsilon<1}$ be a family of events in $\mathcal{B}$. Let $a_1, \ldots, a_n$ be n real numbers. If, for every real number c and every $1 \le i \le n$,*

$$\limsup_{\epsilon \to 0} E\left(\exp\{cI_i\} \middle| A_\epsilon\right) \le \exp(ca_i),$$

then

$$\lim_{\epsilon \to 0} E\left(\exp\left\{\sum_{i=1}^{n} I_i\right\} \middle| A_\epsilon\right) = \exp\left(\sum_{i=1}^{n} a_i\right).$$

The proofs presented in this paper are based on the following two lemmas.

Lemma 2 *(Stochastic Stokes' theorem, see [15] or [7]) Let $Y = (Y_1, \ldots, Y_m)$ be a continuous semimartingale starting at zero and let β be a space-time 1-form defined by*

$$\beta = \sum_{i=1}^{m} \beta_i(s, x) dx_i.$$

Let us define

$$\bar{\beta}_i(s, x) = \int_0^1 \beta_i(s, ux) du,$$

$$S^{ij}(s) = \int_0^s Y_i(u) dY_j(u) - Y_j(u) dY_i(u).$$

Then, we have the following identity (where $\circ$ stands for the Stratonovitch integral) :

$$\begin{aligned}\sum_{i=1}^{m} \int_0^1 \beta_i(s, Y(s)) \circ dY_i(s) &= \frac{1}{2} \sum_{i,j=1}^{m} \int_0^1 \left(\frac{\partial \bar{\beta}_j}{\partial x_i} - \frac{\partial \bar{\beta}_i}{\partial x_j}\right)(s, Y(s)) \circ dS^{ij}(s) \\ &\quad - \sum_{i=1}^{m} \int_0^1 \frac{\partial \bar{\beta}_i}{\partial s}(s, Y(s)) Y_i(s) ds + \sum_{i=1}^{m} \bar{\beta}_i(1, Y(1)) Y_i(1).\end{aligned}$$

To prove Lemma 2, one can apply the classical Stokes theorem for the 1-form β and the random surface $\{(t, uP(t)), 0 \le t \le 1, 0 \le u \le 1\}$, where P is every polygonal line approximating the sample path Y. Then, the formula follows from the fact that the Stratonovitch integral is the limit of the line integral along polygonal lines.

Lemma 3 *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t\}_{0 \le t \le 1}$ be a filtration which satisfies the usual conditions. Let $(Z(t))_{0 \le t \le 1}$ and $(M(t))_{0 \le t \le 1}$ be two continuous square integrable $\mathcal{F}_t$-martingales such that Z has the predictable representation property and $\langle Z, M \rangle = 0$. Let $\mathcal{G} = \sigma\{Z(s), 0 \le s \le 1\}$ be the σ-field generated by Z.*

Let A in $\mathcal{G}$ such that $P(A) > 0$. Let $(F(t))_{0\leq t\leq 1}$ be a continuous $\mathcal{F}_t$-adapted process. Let assume that there exists $C > 0$ such that, for every ω in A and every s in $[0,1]$, $\langle M\rangle_1(\omega) \leq C$ and $|F(s)(\omega)| \leq C$. Then,

$$E\left[\exp\left\{\int_0^1 F(s)dM(s) - \frac{1}{2}\int_0^1 F^2(s)d\langle M\rangle_s\right\} \;\Big|\; A\right] = 1,$$

and consequently

$$E\left[\exp\left\{\int_0^1 F(s)dM(s)\right\} \;\Big|\; A\right] \leq \left\{E\left[\exp\left\{2\int_0^1 F^2(s)d\langle M\rangle_s\right\} \;\Big|\; A\right]\right\}^{\frac{1}{2}}.$$

Proof: Define on $(\Omega, \mathcal{F})$ the probability measure $dQ = \frac{1}{P(A)}\mathbf{1}_A dP$ and denote by E_Q the expectation relative to Q. Since A belongs to $\mathcal{G}$, there exists a process ϕ adapted to the filtration generated by Z such that

$$\mathbf{1}_A = P(A) + \int_0^1 \phi(t)dZ(t).$$

Using the fact that M and $M\int_0^{\cdot}\phi(t)dZ(t)$ are martingales, we obtain that, for all $0 \leq s < t \leq 1$ and every bounded $\mathcal{F}_s$-measurable variable ξ,

$$\begin{aligned} E_Q(M(t)\xi) &= \frac{1}{P(A)}E\left[M(t)\xi\left(P(A) + \int_0^t \phi(u)dZ(u)\right)\right] \\ &= E(M(s)\xi) + \frac{1}{P(A)}E\left(M(s)\xi\int_0^s \phi(u)dZ(u)\right) \\ &= E_Q(M(s)\xi). \end{aligned}$$

We conclude that M is a square integrable martingale under Q. Moreover, the stochastic integrals of F with respect to M under P and under Q are obviously the same Q-almost everywhere. Similarly, the quadratic variations of M under P and under Q are equal Q-almost everywhere. Now, we have

$$E_Q\left[\exp\left(\frac{1}{2}\int_0^1 F^2(s)d\langle M\rangle_s\right)\right] \leq e^{\frac{1}{2}C^3}.$$

Novikov's criterion (see [12]) allows us to conclude that $(N(t))_{0\leq t\leq 1}$, defined by

$$N_t = \exp\left\{\int_0^t F(s)dM(s) - \frac{1}{2}\int_0^t F^2(s)d\langle M\rangle_s\right\},$$

is a martingale under Q and in particular

$$E\left[\exp\left\{\int_0^1 F(s)dM(s) - \frac{1}{2}\int_0^1 F^2(s)d\langle M\rangle_s\right\} \;\Big|\; A\right] = 1,$$

which yields the first assertion of Lemma 3.

Now, write that

$$\int_0^1 F(s)\,dM(s) = \left\{\int_0^1 F(s)\,dM(s) - \int_0^1 F^2(s)\,d\langle M\rangle_s\right\} + \int_0^1 F^2(s)\,d\langle M\rangle_s.$$

Then, using the Cauchy-Schwarz inequality and applying the first point of this lemma to $2F$, we obtain, for every $\epsilon > 0$,

$$E\left[\exp\left\{\int_0^1 F(s)dM(s)\right\} \;\Big|\; A\right] \leq \left\{E\left[\exp\left\{2\int_0^1 F^2(s)d\langle M\rangle_s\right\} \;\Big|\; A\right]\right\}^{\frac{1}{2}}.$$

The proof of Lemma 3 is thus complete.

3 The general case

Let $\|\cdot\|$ be a measurable norm on a subspace of $\mathcal{C}_0([0,1],\mathbb{R})$.

3.1 Theorem

Theorem 1 *Let $X(t)$ be the elliptic diffusion process which is the solution of the stochastic differential equation*

$$dX(t) = \sigma(X(t))\,d\omega(t) + b(X(t))\,dt,\ X(0) = x_0,\ X(t) \in \mathbb{R}^m,$$

where σ is an $m \times d$ matrix of smooth vector fields on $\mathbb{R}^m$, b is a smooth vector field on $\mathbb{R}^m$ and x_0 belongs to $\mathbb{R}^m$. Assume that the norm $\|\cdot\|$ under consideration on $\mathcal{C}_0([0,1],\mathbb{R})$ dominates the supremum norm. Then, the Onsager-Machlup function L is given by

$$L(p,v) = -\frac{1}{2}\|f(p) - v\|_p^2 - \frac{1}{2}divf(p) + \frac{1}{12}R(p).$$

3.2 Reduction of the problem

Let us consider a norm $\|\cdot\|$ on $\mathcal{C}_0([0,1],\mathbb{R})$ which dominates the supremum norm. We will reduce the problem of the Onsager-Machlup function for this norm $\|\cdot\|$ to an evaluation of a conditional exponential moment with regard to small balls of a Bessel process. The approach we will follow here is the one by Y. Takahashi and S. Watanabe in [15] (where they study the case of the supremum norm). However, the smooth Besselizing drift γ we will use has been found by K. Hara in [6] and is different from the drift used by Y. Takahashi and S. Watanabe in [15], which is singular at the origin. Let Φ be a smooth M-valued function on $[0,1]$, starting at x_0.

3.2.1 Introduction of a system of normal coordinates along the curve Φ

On the product manifold $[0,1] \times M$, let $\tilde{\Phi}$ be the curve: $t \in [0,1] \to (t,\Phi(t))$. Let introduce a coordinate system in a neighborhood U of $\tilde{\Phi}$ as follows. Let us choose an orthonormal basis $e^0 = \{e_1^0,\ldots,e_m^0\}$ in the tangent space $T_{\Phi(0)}M$ at $\Phi(0)$. For every $t > 0$, let $e^t = \{e_1^t,\ldots,e_m^t\}$ be the orthonormal basis in $T_{\Phi(t)}M$ obtained as the parallel translate of e^0 along the curve Φ. There exists a neighborhood U of $\tilde{\Phi}$ in $[0,1] \times M$ such that the mapping $(t,x) \in U \to (t,x_1,\ldots,x_m) \in [0,1] \times \mathbb{R}^m$, where $x = \exp_{\Phi(t)}(\sum_{i=1}^m x_i e_i^t)$, is well defined ($s \to \exp_q(sv)$ is the geodesic c with initial conditions $c(0) = q$ and $c'(0) = v$). $\Theta : (t,x) \to (t,x_1,\ldots,x_m)$ is a diffeomorphism of U onto some neighborhood V of the curve $t \to (t,0)$ in $[0,1]\times\mathbb{R}^m$, and for each fixed t, $x \to (x_1,\ldots,x_m)$ is the normal coordinate system N_t in a neighborhood of $\Phi(t)$ with respect to the frame e^t. Denote respectively by $g_{ij}(t,x)$, $g^{ij}(t,x)$, $\Gamma_{ij}^k(t,x)$ and $f_i(t,x)$, the components in the normal coordinates system N_t of the metric, its inverse, the Christoffel symbols and the vector field f. Denote by $\left(\dot{\Phi}_i(t)\right)_{1\le i\le m}$ the components of the tangent vector $\dot{\Phi}_t$ in the frame e^t. Finally, let us consider the space-time process $(t,X(t))$. Its generator is $\frac{\partial}{\partial t} + \frac{1}{2}\Delta_M + f$. This differential operator on $U \subset [0,1]\times M$ is transformed by the above diffeomorphism to the differential operator $\frac{\partial}{\partial t}+\frac{1}{2}\sum_{i,j=1}^m g^{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^m \tilde{b}_i(t,x)\frac{\partial}{\partial x_i}$ on $V \subset [0,1]\times\mathbb{R}^m$, where $\tilde{b}_i(t,x) = f_i(t,x) - \dot{\Phi}_i(t) + \epsilon_i(t,x) - \frac{1}{2}\sum_{l,q=1}^m g^{lq}(t,x)\Gamma_{lq}^i(t,x)$, where $\epsilon_i(t,x)$ is a smooth function satisfying $\sup_{0\leq t\leq 1}|\epsilon_i(t,x)| = O\left(|x|^2\right)$ (cf Lemma 1.2 [5]). Let $T = \inf\{t \geq 0 : (t, X(t)) \notin V\}$ and $\left(t\wedge T, \tilde{X}(t\wedge T)\right) = \Theta(t\wedge T, X(t\wedge T))$. Denoting by $|\cdot|$ the Euclidean norm on $\mathbb{R}^m$, we have for ϵ small enough,

$$P(\|\rho(X,\Phi)\| < \epsilon) = P\left(\left\| \left|\tilde{X}\right| \right\| < \epsilon\right).$$

Moreover, $\tilde{X}$ is given as the solution of the stochastic differential equation

$$d\tilde{X}(t) = \tilde{\sigma}\left(t, \tilde{X}(t)\right) d\tilde{B}(t) + \tilde{b}\left(t, \tilde{X}(t)\right) dt,\ \tilde{X}(0) = 0,\ \tilde{X}(t) \in \mathbb{R}^m,$$

where $\tilde{\sigma}(t,x) = \left(\tilde{\sigma}^{ij}(t,x)\right)_{\substack{1\leq i\leq m\\1\leq j\leq m}}$ is the square root of the inverse of the metric $g(t,x)$ in the normal coordinates N_t, and $\tilde{B}$ is a Brownian motion on $\mathbb{R}^m$.

3.2.2 Use of Girsanov's transformation

For each fixed t in $[0,1]$, since the coordinates system is the normal coordinates system N_t, we have by Gauss lemma, for every $1 \leq i \leq m$,

$$\sum_{j=1}^m g^{ij}(t,x)x_j = x_i, \tag{1}$$

and thus

$$\sum_{j=1}^m \tilde{\sigma}^{ij}(t,x)x_j = x_i.$$

Let us define for every $1 \leq i \leq m$,

$$\gamma_i(t,x) = \frac{1}{2}\sum_{j=1}^m \frac{\partial g^{ij}}{\partial x_j}(t,x).$$

Differentiating both sides of (1), we obtain that γ satisfies

$$\sum_{i=1}^m \left(1 - g^{ii}(t,x)\right) = 2\sum_{j=1}^m \gamma_j(t,x)x_j.$$

Let $Y(t) = (Y_1(t),\ldots,Y_m(t))$ be the solution of the stochastic differential equation on $[0,1]$:

$$dY(t) = \tilde{\sigma}(t,Y(t))\, d\tilde{B}(t) + \gamma(t,Y(t))\, dt,\ Y(0) = 0,\ Y(t) \in \mathbb{R}^m.$$

We get

$$\begin{aligned} d|Y(t)|^2 &= 2\sum_{i=1}^m Y_i(t)dY_i(t) + \sum_{i=1}^m d\langle Y_i\rangle_t \\ &= 2\sum_{i,k=1}^m Y_i(t)\tilde{\sigma}^{ik}(t,Y(t))\, d\tilde{B}_k(t) + 2\sum_{i=1}^m Y_i(t)\gamma_i(t,Y(t))\, dt + \sum_{i=1}^m g^{ii}(t,Y(t))\, dt \\ &= 2\sum_{k=1}^m Y_k(t)d\tilde{B}_k(t) + mdt, \end{aligned}$$

since $\sum_{i=1}^m Y_i(t)\tilde\sigma^{ik}(t,Y(t)) = Y_k(t)$ and $\sum_{i=1}^m (1-g^{ii}(t,Y(t))) = 2\sum_{j=1}^m \gamma_j(t,Y(t))Y_j(t)$. Define $B_t = \sum_{k=1}^m \int_0^t \frac{Y_k(s)}{|Y(s)|} d\tilde B_k(s)$. B is a Brownian motion on $\mathbb{R}$ and, as $d|Y(t)|^2 = 2|Y(t)|\,dB(t) + mdt$, we thus get that $|Y(t)|$ is an m-dimensional Bessel process. Using Girsanov's transformation, we obtain that for every $\epsilon > 0$,

$$P\left(\left\| \left|\tilde X\right| \right\| < \epsilon\right) = E\left(\exp\left\{\sum_{i,j=1}^m \int_0^1 \tilde\sigma_{ij}(t,Y(t))\,\delta^j(t,Y(t))\,d\tilde B_i(t) - \frac{1}{2}\sum_{i,j=1}^m \int_0^1 g_{ij}(t,Y(t))\,\delta^i(t,Y(t))\,\delta^j(t,Y(t))\,dt\right\} \;:\; \|\,|Y|\,\| < \epsilon\right)$$

where $\delta^i(t,x) = \tilde b_i(t,x) - \gamma_i(t,x)$ and $\tilde\sigma_{ij}(t,x)$ denote the components in the normal coordinate system N_t of the inverse matrix of $\tilde\sigma(t,x)$ (i.e the components of the square-root of $g(t,x)$). Thus, we get

$$\begin{aligned} P\left(\left\| \left|\tilde X\right| \right\| < \epsilon\right) &= E\left(\exp\left\{\sum_{i,j=1}^m \int_0^1 \tilde\sigma_{ij}(t,Y(t))\,\delta^j(t,Y(t))\,d\tilde B_i(t) - \frac{1}{2}\sum_{i,j=1}^m \int_0^1 g_{ij}(t,Y(t))\,\delta^i(t,Y(t))\,\delta^j(t,Y(t))\,dt\right\} \;\Big|\; \|\,|Y|\,\| < \epsilon\right) \\ &\quad \times P(\|\,|Y|\,\| < \epsilon). \end{aligned}$$

If we succeed in evaluating the limit of $\frac{P(\|\rho(X,\Phi)\|<\epsilon)}{P(\|\,|Y|\,\|<\epsilon)}$, when ϵ tends to zero, for every smooth path Φ starting at x_0, we will obviously be able to evaluate the limit of $\frac{P(\|\rho(X,\Phi)\|<\epsilon)}{P(\|\rho(X,\Psi)\|<\epsilon)}$, when ϵ tends to zero, for all smooth paths Φ and Ψ starting at x_0. Since, for ϵ small enough,

$$\frac{P(\|\rho(X,\Phi)\| < \epsilon)}{P(\|\,|Y|\,\| < \epsilon)} = E\left(\exp \mathcal{I} \;\Big|\; \|\,|Y|\,\| < \epsilon\right)$$

where

$$\mathcal{I} = \sum_{i,j=1}^m \int_0^1 \tilde\sigma_{ij}(t,Y(t))\,\delta^j(t,Y(t))\,d\tilde B_i(t) - \frac{1}{2}\sum_{i,j=1}^m \int_0^1 g_{ij}(t,Y(t))\,\delta^i(t,Y(t))\,\delta^j(t,Y(t))\,dt,$$

the computation of the Onsager-Machlup functional therefore consists in the asymptotic evaluation of a conditional exponential moment. Thanks to Lemma 1, it suffices to handle the conditional exponential moments of each term appearing in the linear expression of $\mathcal{I}$.

By making the change of drift with respect to the smooth function γ, we will be able to make a general and unique study for every natural norm on the Wiener space.

3.3 Control of the different conditional exponential moments

According to the second theorem of Elie Cartan (see [11]), for each fixed t, since the system of coordinates is the normal coordinates systen , we have a Taylor development of $g_{ij}(t,x)$ at zero (and therefore of $\tilde\sigma_{ij}(t,x)$, $\tilde\sigma^{ij}(\ ,x)$ and $\Gamma^i_{jk}(t,x)$) where the coefficients are universal polynomials in the successive covariant derivatives of the curvature tensor at $\Phi(t)$. All the $O(|x|^q)$'s, $q \geq 0$, which will appear in the developments of the functions of the form $h(t,x)$ will be uniform in t in $[0,1]$. We will use the following developments (see [5], [6], [7], [15]):

$$g^{ij}(t,x) = \delta_{ij} + O(|x|^2),$$

$$g_{ij}(t,x) = \delta_{ij} + O(|x|^2),$$

$$\gamma_i(t,x) = -\frac{1}{6}\sum_{j=1}^{m} R_{ij}(t,0)x_j + O(|x|^2),$$

$$\tilde{b}_i(t,x) = f_i(t,0) - \dot{\Phi}_i(t) + \sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial x_j}(t,0) - \frac{1}{3}R_{ij}(t,0)\right)x_j + O(|x|^2),$$

$$\delta^i = f_i(t,0) - \dot{\Phi}_i(t) + \sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial x_j}(t,0) - \frac{1}{6}R_{ij}(t,0)\right)x_j + O(|x|^2),$$

where $R_{ij}(t,x)$ are the components of the Ricci tensor in N_t. We get

$$g_{ij}(t,Y(t))\,\delta^i(t,Y(t))\,\delta^j(t,Y(t)) = \delta_{ij}\left(f_i(t,0) - \dot{\Phi}_i(t)\right)^2 + O(|Y(t)|).$$

Then we see easily that, for every $1 \leq i \leq m$, every $1 \leq j \leq m$ and every real c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{-\frac{c}{2}\int_0^1 g_{ij}(t,Y(t))\,\delta^i(t,Y(t))\,\delta^j(t,Y(t))\,dt\right\} \Big|\, \|\,|Y|\,\| < \epsilon\right) \leq \exp\left\{-\frac{c}{2}\delta_{ij}\int_0^1\left(f_i(t,0) - \dot{\Phi}_i(t)\right)^2 dt\right\}.$$

By Lemma 1, we can deduce that, for every real c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{-\frac{c}{2}\sum_{i,j=1}^{m}\int_0^1 g_{ij}(t,Y(t))\,\delta^i(t,Y(t))\,\delta^j(t,Y(t))\,dt\right\} \Big|\, \|\,|Y|\,\| < \epsilon\right) \leq \exp\left\{-\frac{c}{2}\sum_{i=1}^{m}\int_0^1\left(f_i(t,0) - \dot{\Phi}_i(t)\right)^2 dt\right\}.$$

Now, write that

$$\sum_{i,j=1}^{m}\int_0^1 \tilde{\sigma}_{ij}(t,Y(t))\,\delta^j(t,Y(t))\,d\tilde{B}_i(t) = \sum_{l,j=1}^{m}\int_0^1 g_{jl}(t,Y(t))\,\delta^j(t,Y(t))\,dY_l(t) - \sum_{l,j=1}^{m}\int_0^1 g_{jl}(t,Y(t))\,\delta^j(t,Y(t))\,\gamma_l(t,Y(t))\,dt.$$

Since $g_{jl}(t,Y(t))\,\delta^j(t,Y(t))\,\gamma_l(t,Y(t)) = O(|Y(t)|)$, for every $1 \leq j \leq m$ and every $1 \leq l \leq m$, we easily get that, for every real c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{c\int_0^1 g_{jl}(t,Y(t))\,\delta^j(t,Y(t))\,\gamma_l(t,Y(t))\,dt\right\} \Big|\, \|\,|Y|\,\| < \epsilon\right) \leq 1.$$

Similarly, write that

$$\int_0^1 g_{jl}(t,Y(t))\,\delta^j(t,Y(t))\,dY_l(t) = \int_0^1 g_{jl}(t,Y(t))\,\delta^j(t,Y(t)) \circ dY_l(t) - \frac{1}{2}\langle g_{jl}(\cdot,Y(\cdot))\,\delta^j(\cdot,Y(\cdot))\,,Y_l(\cdot)\rangle_1.$$

We have

$$\begin{aligned} d\langle g_{jl}(\cdot,Y(\cdot))\,\delta^j(\cdot,Y(\cdot))\,,Y_l(\cdot)\rangle_t &= \sum_{1\le k,q\le m} \tilde\sigma^{qk}(t,Y(t))\,\tilde\sigma^{lk}(t,Y(t))\,\frac{\partial(g_{jl}\delta^j)}{\partial x_q}(t,Y(t))\,dt \\ &= \sum_{1\le q\le m} g^{lq}(t,Y(t))\,\frac{\partial(g_{jl}\delta^j)}{\partial x_q}(t,Y(t))\,dt \\ &= \delta_{jl}\left\{\frac{\partial f_j}{\partial x_j}(t,0)-\frac{1}{6}R_{jj}(t,0)\right\}dt+O(|Y(t)|)dt. \end{aligned}$$

Then, we see easily that for every real number c,

$$\begin{aligned} \limsup_{\epsilon\to 0} E\left(\exp\left\{-\frac{c}{2}\langle g_{jl}(\cdot,Y(\cdot))\,\delta^j(\cdot,Y(\cdot))\,,Y_l(\cdot)\rangle_1\right\}\Big|\,\|\,|Y|\,\|<\epsilon\right) \\ \le \exp\left\{-\frac{c}{2}\delta_{jl}\int_0^1\left\{\frac{\partial f_j}{\partial x_j}(t,0)-\frac{1}{6}R_{jj}(t,0)\right\}dt\right\}. \end{aligned}$$

Consider now $\int_0^1 g_{jl}(t,Y(t))\,\delta^j(t,Y(t))\circ dY_l(t)$. Applying Lemma 2, we get

$$\begin{aligned} \int_0^1 g_{jl}(t,Y(t))\,\delta^j(t,Y(t))\circ dY_l(t) &= \sum_{q=1}^m\int_0^1 \frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(s,Y(s))\circ dS^{ql}(s) \\ &\quad -\int_0^1\frac{\partial\overline{(g_{jl}\delta^j)}}{\partial s}(s,Y(s))\,Y_l(s)ds \\ &\quad +\overline{(g_{jl}\delta^j)}(1,Y(1))\,Y_l(1), \end{aligned}$$

where

$$\overline{(g_{jl}\delta^j)}(s,x)=\int_0^1(g_{jl}\delta^j)(s,ux)du,$$

$$S^{ql}(s)=\int_0^s Y_q(u)dY_l(u)-Y_l(u)dY_q(u).$$

Since moreover

$$\begin{aligned} d\langle\frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(\cdot,Y(\cdot)),S^{ql}\rangle_t &= \sum_{k=1}^m\frac{\partial^2\overline{(g_{jl}\delta^j)}}{\partial x_k\partial x_q}(t,Y(t))\left(g^{kl}(t,Y(t))Y_q(t)-g^{kq}(t,Y(t))Y_l(t)\right)dt \\ &= O(|Y(t)|)dt \end{aligned}$$

we get that

$$\int_0^1(g_{jl}\delta^j)(t,Y(t))\circ dY_l(t)=\sum_{q=1}^m\int_0^1\frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\,dS^{ql}(t)+\int_0^1 O(|Y(t)|)dt+O(|Y(1)|),$$

and the control of the conditional exponential moments of $\int_0^1(g_{jl}\delta^j)(t,Y(t))\circ dY_l(t)$ is equivalent to the control of those of $\sum_{q=1}^m\int_0^1\frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\,dS^{ql}(t)$.

Let M^{ql} be the martingale part of S^{ql}:

$$M^{ql}(t)=\sum_{k=1}^m\int_0^t Y_q(u)\tilde\sigma^{lk}(u,Y(u))\,d\tilde B_k(u)-\sum_{k=1}^m\int_0^t Y_l(u)\tilde\sigma^{qk}(u,Y(u))\,d\tilde B_k(u).$$

We have

$$\int_0^1 \frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\,dS^{ql}(t) = \int_0^1 \frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\,dM^{ql}(t) + \int_0^1 \frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\left\{Y_q(t)\gamma_l(t,Y(t)) - Y_l(t)\gamma_q(t,Y(t))\right\}dt.$$

We see easily that, for every real number c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{c\int_0^1 \frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\left\{Y_q(t)\gamma_l(t,Y(t)) - Y_l(t)\gamma_q(t,Y(t))\right\}dt\right\} \Big| \,\|\,|Y|\,\| < \epsilon\right) \leq 1.$$

Recall that $d|Y(t)|^2 = 2|Y(t)|\,dB(t) + m\,dt$, where $B_t = \sum_{k=1}^m \int_0^t \frac{Y_k(s)}{|Y(s)|}d\tilde{B}_k(s)$. The σ-field generated by B, $\sigma\{B(s), 0 \leq s \leq 1\}$, is the same as the one generated by $|Y|$, $\sigma\{|Y(s)|, 0 \leq s \leq 1\}$; moreover

$$\langle B, M^{ql}\rangle_t = \sum_{k=1}^m \int_0^t \frac{Y_k(s)}{|Y(s)|}\left(Y_q(s)\tilde{\sigma}^{lk}(s,Y(s)) - Y_l(s)\tilde{\sigma}^{qk}(s,Y(s))\right)ds,$$

and since $\sum_{k=1}^m Y_k(s)\tilde{\sigma}^{qk}(s,Y(s)) = Y_q(s)$ and $\sum_{k=1}^m Y_k(s)\tilde{\sigma}^{lk}(s,Y(s)) = Y_l(s)$, we obtain

$$\langle B, M^{ql}\rangle_t = \int_0^t \frac{Y_q(s)Y_l(s) - Y_q(s)Y_l(s)}{|Y(s)|}ds = 0.$$

Thus, by Lemma 3, we get for every real number c and every $0 < \epsilon < 1$,

$$E\left(\exp\left\{c\int_0^1 \frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\,dM^{ql}(t)\right\}\Big| \,\|\,|Y|\,\| < \epsilon\right) \leq \left\{E\left(\exp\left\{2c^2\int_0^1 \left(\frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\right)^2 d\langle M^{ql}\rangle_t\right\}\Big| \,\|\,|Y|\,\| < \epsilon\right)\right\}^{\frac{1}{2}}.$$

Moreover, as

$$d\langle M^{ql}\rangle_t = \sum_{k=1}^m \left(Y_l(t)\tilde{\sigma}^{qk}(t,Y(t)) - Y_q(t)\tilde{\sigma}^{lk}(t,Y(t))\right)^2 dt = O\left(|Y(t)|^2\right)dt,$$

we have

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{c\int_0^1 \frac{\partial\overline{(g_{jl}\delta^j)}}{\partial x_q}(t,Y(t))\,dM^{ql}(t)\right\}\Big| \,\|\,|Y|\,\| < \epsilon\right) \leq 1.$$

By Lemma 1, we deduce that, for every real number c,

$$\limsup_{\epsilon\to 0} E\left[\exp\left\{c\sum_{i,j=1}^m \int_0^1 \tilde{\sigma}_{ij}(t,Y(t))\,\delta^j(t,Y(t))\,d\tilde{B}_i(t)\right\} \Big| \,\|\,|Y|\,\| < \epsilon\right] \leq \exp\left\{-\frac{c}{2}\sum_{j=1}^m \int_0^1 \left\{\frac{\partial f_j}{\partial x_j}(t,0) - \frac{1}{6}R_{jj}(t,0)\right\}dt\right\}.$$

Using once more Lemma 1, the proof of Theorem 1 is thus complete.

4 When σ is the identity matrix

If $m = d$ and if for each x in $\mathbb{R}^d$, $\sigma(x)$ is the identity matrix, the Riemannian structure induced by the diffusion is the Euclidean one. Denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$. So, Theorem 1 gives the limit, when ϵ tends to zero, of $\frac{P(\| \, |X-\Phi| \, \|<\epsilon)}{P(\| \, |X-\Psi| \, \|<\epsilon)}$, for every smooth functions Φ and Ψ and every norm $\|\cdot\|$ on $\mathcal{C}_0([0,1],\mathbb{R})$ which dominates the supremum norm. Actually, studies on the Onsager-Machlup functional in the case $\sigma := I_d$ used to consider the problem in a more general framework and to investigate the limit, when ϵ tends to zero, of $\frac{P(\|X-\Phi\|<\epsilon)}{P(\|X-\Psi\|<\epsilon)}$, for every functions Φ and Ψ such that $\Phi - x_0$ and $\Psi - x_0$ belong to the Cameron-Martin space $\mathcal{H}^d$, and every norm $\|\cdot\|$ on $\mathcal{C}_0\left([0,1],\mathbb{R}^d\right)$. That is to say in particular, the norm under consideration is not assumed a priori of the form: $\|\cdot\| = \| \, |\cdot| \, \|'$ where $\|\cdot\|'$ is a norm on $\mathcal{C}_0([0,1],\mathbb{R})$. In this context, we noticed that the preceding techniques (the stochastic Stokes theorem and the Kunita-Watanabe theorem) give a very quick proof of the following theorem whose framework is a little more general that the one of Theorem 1 for $\sigma := I_d$.

4.1 Theorem

Theorem 2 *Let $X(t)$ be the diffusion process which is the solution of the stochastic differential equation*

$$dX(t) = b(X(t))\,dt + d\omega(t), \;\; X(0) = x_0, \;\; X(t) \in \mathbb{R}^d,$$

where x_0 belongs to $\mathbb{R}^d$ and b is a $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ of class C^2, bounded, such that all its derivatives are bounded and its second derivatives are Lipschitz continuous. Let $\|\cdot\|$ be a measurable norm defined on a subspace F of the Wiener space $\mathcal{C}_0\left([0,1];\mathbb{R}^d\right)$ to which ω and $X-\Phi$ belong (for every Φ in $x_0+\mathcal{H}^d$), such that $(F, \|\cdot\|)$ is separable. If the norm $\|\cdot\|$ dominates the L^2-norm and is such that the random variable $\|\omega\|$ is measurable with respect to the σ-field $\sigma\{|\omega(s)|, 0 \leq s \leq 1\}$ (where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$), then the Onsager-Machlup functional of X for the norm $\|\cdot\|$ exists and is given by

$$L\left(\Phi, \dot{\Phi}\right) = -\frac{1}{2}\sum_{i=1}^{d}\left|\dot{\Phi}_i - b_i(\Phi)\right|^2 - \frac{1}{2}\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i}(\Phi).$$

4.2 Proof

We will need one more lemma.

Lemma 4 *(see [13], [9], [2]) Let f be a deterministic function in $L^2[0,1]$. Define $I_i(f) = \int_0^1 f(t)\,d\omega_i(t)$. If the norm $\|\cdot\|$ dominates the L^1-norm then*

$$\lim_{\epsilon\to 0} E\left(\exp\{|I_i(f)|\} \mid \|\omega\| < \epsilon\right) = 1.$$

Using Girsanov's transformation (see [8]), we obtain, for every $\epsilon > 0$,

$$P(\|X-\Phi\|<\epsilon) = \exp\Big(-\frac{1}{2}\int_0^1 \big|\dot\Phi(s) - b(\Phi(s))\big|^2 ds\Big)$$
$$\times E\Big(\exp\Big\{\frac{1}{2}\int_0^1 |b(\Phi(s))|^2 ds - \frac{1}{2}\int_0^1 |b(\omega(s)+\Phi(s))|^2 ds$$
$$-\sum_{i=1}^d \int_0^1 b_i(\Phi(s))\, d\Phi_i(s) + \sum_{i=1}^d \int_0^1 b_i(\omega(s)+\Phi(s))\, d\Phi_i(s)$$
$$-\sum_{i=1}^d \int_0^1 \dot\Phi_i(s)\, d\omega_i(s)$$
$$+\sum_{i=1}^d \int_0^1 b_i(\omega(s)+\Phi(s))\, d\omega_i(s)\Big\} : \|\omega\|<\epsilon\Big).$$

Therefore

$$\frac{P(\|X-\Phi\|<\epsilon)}{P(\|\omega\|<\epsilon)} = \exp\Big(-\frac{1}{2}\int_0^1 \big|\dot\Phi(s) - b(\Phi(s))\big|^2 ds\Big)\, E_\epsilon,$$

where $E_\epsilon = E\Big(\exp\Big\{\frac{1}{2}\int_0^1 |b(\Phi(s))|^2 ds - \frac{1}{2}\int_0^1 |b(\omega(s)+\Phi(s))|^2 ds$

$$-\sum_{i=1}^d \int_0^1 b_i(\Phi(s))\, d\Phi_i(s) + \sum_{i=1}^d \int_0^1 b_i(\omega(s)+\Phi(s))\, d\Phi_i(s)$$
$$-\sum_{i=1}^d \int_0^1 \dot\Phi_i(s)\, d\omega_i(s) + \sum_{i=1}^d \int_0^1 b_i(\omega(s)+\Phi(s))\, d\omega_i(s)\Big\}\Big|\|\omega\|<\epsilon\Big).$$

The computation of the Onsager-Machlup functional therefore consists in the asymptotic evaluation of E_ϵ. According to Lemma 1, it suffices to handle the conditional exponential moments of each term of the sum inside the exponential map.

By using the fact that b is Lipschitz continuous and bounded and that $\|\cdot\|$ dominates the L^2-norm, we easily get for every real number c,

$$\limsup_{\epsilon\to 0} E\Big(\exp\Big\{c\Big(\frac{1}{2}\int_0^1 |b(\Phi(s))|^2 ds - \frac{1}{2}\int_0^1 |b(\omega(s)+\Phi(s))|^2 ds\Big)\Big\}\Big|\|\omega\|<\epsilon\Big) \le 1$$

and for every $1 \le i \le d$,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{c\left(\int_0^1 (b_i(\omega(s)+\Phi(s)) - b_i(\Phi(s)))\, d\Phi_i(s)\right)\right\}\Big|\|\omega\|<\epsilon\right) \le 1.$$

As an immediate consequence of Lemma 4, we have for every $1 \le i \le d$ and for every real number c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{c\left(\int_0^1 \dot\Phi_i(s)\, d\omega_i(s)\right)\right\}\Big|\|\omega\|<\epsilon\right) \le 1.$$

We are left with the control of the exponential moments of $\int_0^1 b_i(\omega(s)+\Phi(s))\, d\omega_i(s)$ which is the main interest of this proof. We can write

$$\int_0^1 b_i(\omega(s)+\Phi(s))\, d\omega_i(s) = \int_0^1 b_i(\omega(s)+\Phi(s))\circ d\omega_i(s) - \frac{1}{2}\int_0^1 \frac{\partial b_i}{\partial x_i}(\omega(s)+\Phi(s))ds.$$

Now, developping $\frac{\partial b_i}{\partial x_i}(\omega(s) + \Phi(s))$ at the point $\Phi(s)$ up to the order 1, we get

$$\int_0^1 \frac{\partial b_i}{\partial x_i}(\omega(s) + \Phi(s))ds = \int_0^1 \frac{\partial b_i}{\partial x_i}(\Phi(s))ds + \int_0^1 \Psi_i(s,\omega)ds.$$

Since the derivatives of order 2 of b_i are bounded and since the norm $\|\cdot\|$ dominates the L^1-norm, there is $C > 0$ such that

$$\int_0^1 \Psi_i(s,\omega)ds \leq C\epsilon$$

on the event $\{\|\omega\| < \epsilon\}$. Thus we get for every $1 \leq i \leq d$ and for every real number c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{c\left(-\frac{1}{2}\int_0^1 \frac{\partial b_i}{\partial x_i}(\omega(s) + \Phi(s)))ds\right)\right\}\right) \leq \exp\left\{c\left(-\frac{1}{2}\int_0^1 \frac{\partial b_i}{\partial x_i}(\Phi(s))ds\right)\right\}.$$

Let us now analyze $\int_0^1 b_i(\omega(s) + \Phi(s)) \circ d\omega_i(s)$. By using an approximation of Φ by smooth functions on $[0,1]$, one can easily check that Lemma 2 is still valid for $\beta(s,x) = b(x + \Phi(s))$. So, we get

$$\begin{aligned}\int_0^1 b_i(\omega(s) + \Phi(s)) \circ d\omega_i(s) &= \sum_{j=1}^d \int_0^1 \frac{\partial \bar{b}_i}{\partial x_j}(s,\omega(s)) \circ dA^{ji}(s) \\ &\quad - \int_0^1 \frac{\partial \bar{b}_i}{\partial s}(s,\omega(s))\,\omega_i(s)ds + \bar{b}_i(1,\omega(1))\,\omega_i(1),\end{aligned}$$

where

$$\bar{b}_i(s,x) = \int_0^1 b_i(ux + \Phi(s))du,$$

$$A^{ji}(s) = \int_0^s \omega_j(u)d\omega_i(u) - \omega_i(u)d\omega_j(u).$$

Since b is bounded, we get by Lemma 4 that for every $1 \leq i \leq d$ and every real number c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{c\bar{b}_i(1,\omega(1))\,\omega_i(1)\right\}\Big|\|\omega\| < \epsilon\right) \leq 1.$$

Since the derivatives of b are bounded and since the norm $\|\cdot\|$ dominates the L^2-norm, we easily get for every $1 \leq i \leq d$ and every real number c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{-c\int_0^1 \frac{\partial \bar{b}_i}{\partial s}(s,\omega(s))\,\omega_i(s)ds\right\}\Big|\|\omega\| < \epsilon\right) \leq 1.$$

Now, rewrite

$$\begin{aligned}\int_0^1 \frac{\partial \bar{b}_i}{\partial x_j}(s,\omega(s)) \circ dA^{ji}(s) &= \int_0^1 \frac{\partial \bar{b}_i}{\partial x_j}(s,\omega(s))\,dA^{ji}(s) - \frac{1}{2}\int_0^1 \frac{\partial^2 \bar{b}_i}{\partial x_j^2}(s,\omega(s))\,\omega_i(s)ds \\ &\quad + \frac{1}{2}\int_0^1 \frac{\partial^2 \bar{b}_i}{\partial x_i \partial x_j}(s,\omega(s))\,\omega_j(s)ds.\end{aligned}$$

Similarly, for every $1 \leq i \leq d$ and every real number c,

$$\limsup_{\epsilon\to 0} E\left(\exp\left\{\frac{c}{2}\left(\int_0^1 \frac{\partial^2 \bar{b}_i}{\partial x_i \partial x_j}(s,\omega(s))\,\omega_j(s)ds - \int_0^1 \frac{\partial^2 \bar{b}_i}{\partial x_j^2}(s,\omega(s))\,\omega_i(s)ds\right)\right\}\middle| \|\omega\| < \epsilon\right) \leq 1.$$

Define for every $0 \leq s \leq 1$, the Brownian motion $Z_s = \sum_{k=1}^{d} \int_0^s \frac{\omega_k(t)}{|\omega(t)|} d\omega_k(t)$. A^{ji} and Z are orthogonal since $\langle A^{ji}, Z\rangle_t = 0$, for every $0 \leq t \leq 1$. The filtration generated by Z, $\mathcal{G} = \sigma\{Z_s, 0 \leq s \leq 1\}$, is also the one generated by $|\omega|$, $\sigma\{|\omega(s)|, 0 \leq s \leq 1\}$. Therefore, provided that the norm under consideration is a measurable variable with respect to the radial process, the event $\{\|\omega\| < \epsilon\}$ belongs to $\mathcal{G}$ for every $0 < \epsilon < 1$, and by Lemma 3, we get for every real number c and all $1 \leq i, j \leq d$,

$$E\left[\exp\left\{c\int_0^1 \frac{\partial \bar{b}_i}{\partial x_j}(s,\omega(s))\,dA^{ji}(s)\right\}\middle| \|\omega\| < \epsilon\right]$$
$$\leq \left\{E\left[\exp\left\{2c^2\int_0^1\left(\frac{\partial \bar{b}_i}{\partial x_j}(s,\omega(s))\right)^2\left(\omega_i^2(s)+\omega_j^2(s)\right)ds\right\}\middle| \|\omega\| < \epsilon\right]\right\}^{\frac{1}{2}}.$$

Therefore, for every real number c and all $1 \leq i, j \leq d$,

$$\limsup_{\epsilon\to 0} E\left[\exp\left\{c\int_0^1 \frac{\partial \bar{b}_i}{\partial x_j}(s,\omega(s))\,dA^{ji}(s)\right\}\middle| \|\omega\| < \epsilon\right] \leq 1$$

Now, Lemma 1 allows us to conclude that

$$\lim_{\epsilon\to 0} \frac{P(\|X-\Phi\| < \epsilon)}{P(\|\omega\| < \epsilon)} = \exp\left(-\frac{1}{2}\int_0^1 \left|\dot{\Phi}(s) - b(\Phi(s))\right|^2 ds - \frac{1}{2}\sum_{i=1}^{d}\int_0^1 \frac{\partial b_i}{\partial x_i}(\Phi(s))ds\right),$$

and the proof of Theorem 2 is complete.

In the last section, we briefly compare what we obtained to earlier, as well as more recent, results in this framework.

4.3 Previous results

It was proved in [8] that, if one chooses on $\mathcal{C}_0\left([0,1];\mathbb{R}^d\right)$ the supremum norm

$$\|\omega\|_\infty = \sup_{t\in[0,1]} |\omega(t)|, \tag{2}$$

and if Φ is of class $\mathcal{C}^2$, the Onsager-Machlup functional is given by

$$L\left(\Phi,\dot{\Phi}\right) = -\frac{1}{2}\sum_{i=1}^{d}\left|\dot{\Phi}_i - b_i(\Phi)\right|^2 - \frac{1}{2}\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i}(\Phi).$$

This case enters the setting of Theorem 1 or Theorem 2. L.A. Shepp and O. Zeitouni proved in [13] that the result still holds for every norm which is equivalent to (2) and if $\Phi - x_0$ only belongs to $\mathcal{H}^d$. Moreover they showed in [14] that this expression is still valid for other norms, in particular for L^p-norms, $p \geq 4$, and Hölder norms $\|\cdot\|_\alpha$ with $0 < \alpha < \frac{1}{3}$ in the case $d \geq 2$ and with $0 < \alpha < \frac{1}{2}$ in the case $d = 1$. They deal more generally with a class of completely convex norms and obtain the Onsager-Machlup functional for some norms which do not enter the context of Theorem 1 or Theorem 2. In [3], we extended this result to a large class of natural norms on Wiener space, including in particular Hölder norms for every $0 < \alpha < \frac{1}{2}$; our approach closely follows [14] but we have to use versions of the norms which are rotationaly invariant on the range of the Brownian paths. Nevertheless, the context in [3] is still a bit more general than the one of Theorem 2. Recently, O. Zeitouni and T. Lyons showed in [10] that this geometric property may actually be relaxed in the case of Hölder norms. On the other hand, using different approaches, O. Zeitouni in [16] and E. Mayer-Wolf and O. Zeitouni in [11], obtained the Onsager-Machlup functional for the L^2-norm.

In [8], [13], [14], [3], [10] the authors use Girsanov's transformation to come down to Brownian small balls (as it was done in the proof of Theorem 2). The difficulty is then to evaluate conditional exponential moments of the stochastic integral $\int_0^1 \langle b(\omega(s) + \Phi(s)), d\omega(s)\rangle$. The usual method consists in a Taylor development of $b(\omega(s) + \Phi(s))$ at the point $\Phi(s)$; the minimum order of this development is fixed by the probability of the small balls relative to the norm under consideration. The computation of the Onsager-Machlup functional therefore consists in the asymptotic evaluation of conditional exponential moments of the stochastic integrals appearing in the development.

Now, as we saw in the proof of Theorem 2, the stochastic Stokes theorem and the Kunita-Watanabe theorem (techniques which were naturally used in the case of diffusion processes on manifolds) give immediately the asymptotic evaluation of the exponential moments of $\int_0^1 \langle b(\omega(s) + \Phi(s)), d\omega(s)\rangle$, provided that the norm under consideration on $\mathcal{C}_0\left([0,1];\mathbb{R}^d\right)$ is such that the random variable $\|\omega\|$ is measurable with respect to the σ-field $\sigma\{|\omega(s)|, 0 \le s \le 1\}$. Using this approach, we have to assume that b is of class C^2, bounded, such that all its derivatives are bounded and its second derivatives are Lipschitz continuous. This smoothness assumption on b is independent of the norm under consideration whereas in the previous methods it is imposed by the order of the requisite Taylor development and thus depends on the norm. At last, this new proof has the noteworthy advantage of including the case of the L^2-norm as well as the case of the L^p-norms with $2 < p < 4$ (which did not seem to have been considered up to now). Indeed, in the previous methods, the techniques used to handle some stochastic integrals in which appear first derivatives of b, only hold in the case of a norm which dominates the L^4-norm.

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