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TIME DEPENDENT SUBORDINATION AND MARKOV PROCESSES WITH JUMPS

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Abstract

Bochner's subordination is extended to time-inhomogeneous Markov processes and the Feynman-Kac formula is generalized to the time-dependent subordination. As an application it is shown that stochastic differential equations with jumps can be directly solved with the help of the time-dependent subordination and consequently that the equation of motion for relativistic quantum particles is solved.

1. Introduction

For a prescribed drift coefficient $\boldsymbol{b}(s, x)$ and a potential function $c(s, x)$, we prepare a pair of operators

$$A_s^c = \frac{1}{2}(\sigma\nabla + \boldsymbol{b}(s, x))^2 + c(s, x)\,\mathrm{I},$$

and

$$\widehat{A}_t^c = \frac{1}{2}(\sigma\nabla - \boldsymbol{b}(t, x))^2 + c(t, x)\,\mathrm{I},$$

which are formal adjoint of each other, and set

$$M_s^c = -\sqrt{-A_s^c + \kappa^2\mathrm{I}} + \kappa\mathrm{I}, \quad \widehat{M}_t^c = -\sqrt{-\widehat{A}_t^c + \kappa^2\mathrm{I}} + \kappa\mathrm{I}.$$

We then consider

$$\frac{\partial\varphi}{\partial s} + M_s^c\varphi = 0, \tag{1.1}$$

$$-\frac{\partial\widehat{\varphi}}{\partial t} + \widehat{M}_t^c\widehat{\varphi} = 0, \tag{1.2}$$

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which is the equation of motion of Nagasawa (1996, 1997) for a relativistic (spinless) quantum particle(s). The movement of a relativistic quantum particle is described by Markov processes of pure-jumps $\{Y(t), t \in [a, b], \mathrm{Q}\}$ such that the distribution of $Y(t)$ is given by

$$\mathrm{Q}[Y(t) \in dx] = \widehat{\varphi}(t, x)\varphi(t, x)dx,$$

where $\varphi(t, x)$ and $\widehat{\varphi}(t, x)$ are solutions of equations (1.1) and (1.2), respectively. In the Schrödinger representation we have

$$\mathrm{Q}[f(Y(t))] = \int \widehat{\varphi}_a(x)dxq(a, x; t, y)dyf(y)q(t, y; b, z)\varphi_b(z)dz,$$

where $q(s, x; t, y)$ is the fundamental solution of the pair of equations (1.1) and (1.2) which are in duality with respect to $dtdx$, and $\{\widehat{\varphi}_a, \varphi_b\}$ is a prescribed entrance-exit law, for details cf. Nagasawa (1996, 1997). Nagasawa-Tanaka (1998, 1999) discussed the existence and uniqueness of solutions of equation (1.1) in terms of stochastic differential equations of pure-jumps. The objective of the present article is to solve equation (1.1) more directly, through extending Bochner's subordination to temporally inhomogeneous diffusion processes and generalizing the Feynman-Kac formula to the time-dependent subordination.

2. Time-Dependent Subordination

2.1 *Bochner's Subordination*

We begin with a remark that it is immediate to construct a pure-jump Markov process $\{Y_t, t \in [a, b], \mathrm{P}\}$ with the fractional power generator

$$M = -\sqrt{-A + \kappa^2 \mathrm{I}} + \kappa \mathrm{I},$$

where

$$A = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(x)\frac{\partial}{\partial x_i}, \tag{2.1}$$

which does not depend on time. In fact, we apply the subordination of Bochner (1949) to the semi-group P_t of the temporally homogeneous diffusion process $\{X(t), t \geq 0, \mathrm{P}\}$ with the generator A in (2.1), i.e., we set

$$Y_t = X(Z(t)), \quad t \in [0, \infty), \tag{2.2}$$

where $\{Z(t), t \in [0, \infty), \mathrm{P}\}$ is the subordinator of Sato (1990), which is independent of the diffusion process $X(t)$ (cf. also Vershik-Yor (1995)). Then the subordinate process

$\{Y_t = X(Z(t)), t \geq 0, \mathrm{P}_x\}$ is a temporally homogeneous Markov process of pure-jumps with the transition probability

$$\mathrm{Q}_t f(x) = \mathrm{P}_x[f(X(Z(t)))] = \int_0^\infty \mathrm{P}_s f(x) \mathrm{P}[Z(t) \in ds],$$

and the generator M of the semi-group Q_t has the expression

$$Mf(x) = \int_0^\infty \{\mathrm{P}_r f(x) - f(x)\} \nu(dr),$$

where $\nu(dr)$ is the Lévy measure of $Z(t)$. However, if the coefficients of the operator in (2.1) depend on time, Bochner's subordination in (2.2) is no longer applicable.

2.2. *Time Dependent Subordination*

A typical example of time-dependent coefficients appears in the equation of motion in (1.1). We consider a stochastic process governed by

$$M_s = -\sqrt{-A_s + \kappa^2 \mathrm{I}} + \kappa \mathrm{I},$$

with

$$A_s = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)^{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(s, x) \frac{\partial}{\partial x_i}. \tag{2.3}$$

Let $B(t)$ be a d-dimensional Brownian motion and $Z(t)$ be a subordinator which is independent of the Brownian motion, and define the inverse function of $Z(t)$ by

$$Z^{-1}(t) = \inf\{s : Z(s) > t\}, \tag{2.4}$$

which is right-continuous in t. We denote by $X_{t_0,x}(t)$ the unique solution of a stochastic differential equation

$$X(t) = x + \int_{t_0}^t \sigma(t_0 + Z^{-1}(s - t_0), X(s))dB(s) + \int_{t_0}^t b(t_0 + Z^{-1}(s - t_0), X(s))ds. \tag{2.5}$$

The key point in equation (2.5) is the inverse function $Z^{-1}(s - t_0)$ in the time parameter of the coefficients $\sigma(s, x)$ and $b(s, x)$. We assume that the entries of the matrix $\sigma(s, x)$ and vector $b(s, x)$ are bounded and continuous in (s, x), Lipschitz continuous in x for each fixed s, and the Lipschitz constants are bounded in s, so that equation (2.5) has a unique solution. We then set

$$Y_{t_0,x}(t) = X_{t_0,x}(t_0 + Z(t - t_0)),$$

which will be called *time-dependent subordination of the solution* $X_{t_0,x}(t)$ of equation (2.5). It is clear that $Y_{t_0,x}(t)$ satisfies

$$Y_{t_0,x}(t) = x + \int_{t_0}^{t_0 + Z(t - t_0)} \sigma(t_0 + Z^{-1}(s - t_0), X(s))dB(s)$$

$$+ \int_{t_0}^{t_0 + Z(t - t_0)} b(t_0 + Z^{-1}(s - t_0), X(s))ds.$$

To avoid notational complexity, let us set $t_0 = 0$, and denote $Y(t) = X(Z(t))$, where $X(t)$ is a solution of equation (2.5) with $t_0 = 0$, that is,

$$X(t) = x + \int_0^t \sigma(Z^{-1}(s), X(s))dB(s) + \int_0^t b(Z^{-1}(s), X(s))ds.$$

Then $Y(t) = X(Z(t))$ satisfies

$$Y(t) = x + \int_0^{Z(t)} \sigma(Z^{-1}(s), X(s))dB(s) + \int_0^{Z(t)} b(Z^{-1}(s), X(s))ds.$$

Putting $Z^{-1}(s) = u$ formally, we obtain a stochastic differential equation for $Y(t)$

$$Y(t) = x + \int_0^t \sigma(u, Y(u))dB(Z(u)) + \int_0^t b(u, Y(u))dZ(u),$$

which, however, does not give the right expression, but a more careful treatment of jumps of the subordinator $Z(t)$ will prove that $X(Z(t))$ satisfies

$$X(Z(t)) = x + \sum_{0<s\leq t} \int_{Z(s-)}^{Z(s)} \sigma(s, X(u))dB(u) + \sum_{0<s\leq t} \int_{Z(s-)}^{Z(s)} b(s, X(u))du, \tag{2.6}$$

where we assume $Z(t)$ is a pure-jump process.

On the other hand, let $W(dw)$ be the Wiener measure on the space Ω_c of all continuous paths, and $\xi_t(s, x, w)$ be the unique solution of

$$\xi(t) = x + \int_0^t \sigma(s, \xi(u))dw(u) + \int_0^t b(s, \xi(u))du, \tag{2.7}$$

where s is fixed. Then, as will be shown in Section 4, equation (2.6) is equivalent to the stochastic differential equation of pure-jumps

$$Y(t) = x + \int_{(0,t]\times(0,\infty)\times\Omega_c} \{\xi_\theta(s, Y(s\text{-}), w) - Y(s\text{-})\}N(dsd\theta dw),$$

that was discussed in Nagasawa-Tanaka (1998), where $N(dsd\theta dw)$ is a Poisson random measure with the mean measure $ds\,\nu(d\theta)W(dw)$, and $\nu(d\theta)$ is the Lévy measure of the subordinator $Z(t)$.

3. Lemmas

Let $Z(t)$ be a right continuous non-decreasing function on $[0, \infty)$ such that

$$Z(t) = \beta t + \sum_{0<u\le t} \{Z(u) - Z(u\text{-})\}, \tag{3.1}$$

where $\beta \ge 0$ is a constant.

Lemma 3.1. *Let* $Z(t)$ *be given in* (3.1), *and define* $Z^{-1}(t)$ *by* (2.4). *Then*

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))ds$$

$$= \beta\int_0^t f(u, g(Z(u)))du + \sum_{0<s\le t}\int_{Z(s\text{-})}^{Z(s)} f(s, g(u))du, \tag{3.2}$$

for any $\mathbf{R}^d$*-valued continuous function* $g(s)$ *on* $[0, \infty)$ *and real-valued continuous function* $f(s, x)$ *on* $[0, \infty)\times\mathbf{R}^d$. *In applications, equation* (3.2) *is often expressed in another form as*

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))ds = \beta \int_0^t f(u, g(Z(u)))du$$

$$+ \sum_{0<s\le t} \int_0^{\Delta Z(s)} f(s, g(Z(s-) + u))du,$$

where $\Delta Z(s) = Z(s) - Z(s-)$.

Proof. (i) At the first step, we assume that the set of jump times of $Z(t)$ has no finite accumulation point, and denote them by

$$0 < s_1 < s_2 < s_3 < \dots,$$

in natural order. Let us assume $\beta > 0$. The case $\beta = 0$ is simpler, and can be handled in the same way. If $0 \le t < s_1$, then $Z(t) = \beta t$ and $Z^{-1}(s) = s/\beta$ for $0 \le s < Z(t)$. Therefore, we have

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))ds = \beta \int_0^t f(u, g(Z(u)))du. \tag{3.3}$$

When $s_1 \le t < s_2$, we have

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))ds = \int_0^{Z(s_1-)} f(Z^{-1}(s), g(s))ds + \int_{Z(s_1-)}^{Z(s_1)} f(s_1, g(s))ds$$

$$+ \int_{Z(s_1)}^{Z(s_1)+\beta(t-s_1)} f(s_1 + \frac{s - Z(s_1)}{\beta}, g(s))ds,$$

on the right-hand side of which the first integral is equal to

$$\beta \int_0^{s_1} f(u, g(Z(u)))du,$$

in view of equation (3.3), in the second integral we have applied the property that $Z^{-1}(s)$ remains constant in the interval $(Z(s_1-), Z(s_1)]$ and hence $Z^{-1}(s) = Z^{-1}(Z(s_1)) = s_1$, and the third integral is equal to

$$\beta \int_{s_1}^{t} f(u, g(Z(u)))du,$$

for which we put $u = s_1 + (s - Z(s_1))/\beta$ and $s = Z(s_1) + \beta(u - s_1) = Z(u)$. Therefore,

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))ds = \beta \int_0^t f(u, g(Z(u)))du + \int_{Z(s_1-)}^{Z(s_1)} f(s_1, g(u))du.$$

Repeating the same argument, we have, for $s_n \le t < s_{n+1}$,

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))ds = \beta \int_0^t f(u, g(Z(u)))du + \sum_{k=1}^{n} \int_{Z(s_k-)}^{Z(s_k)} f(s_k, g(u))du, \tag{3.4}$$

which yields equation (3.2) in the special case of no accumulation point, and also

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))ds$$

$$= \beta \int_0^t f(u, g(Z(u)))du + \sum_{k=1}^{n} \int_0^{\Delta Z(s_k)} f(s_k, g(Z(s_k-) + u))du.$$

(ii) In the general case, we set

$$Z_\varepsilon(t) = \beta t + \sum_{0<s\le t} \{Z(s) - Z(s-)\} 1_{(\varepsilon,\infty)}(Z(s) - Z(s-)),$$

for $\varepsilon > 0$. Then $Z_\varepsilon(t)$ satisfies the condition of the first case, and hence

$$\int_0^{Z_\varepsilon(t)} f(Z_\varepsilon^{-1}(s), g(s))ds$$

$$= \beta \int_0^t f(u, g(Z_\varepsilon(u)))du + \sum_{0<s\le t} \int_{Z_\varepsilon(s-)}^{Z_\varepsilon(s)} f(s_k, g(u))du,$$

in view of equation (3.4). Since $Z_\varepsilon(t) \uparrow Z(t)$ and $Z_\varepsilon^{-1}(t) \downarrow Z^{-1}(t)$ as $\varepsilon \downarrow 0$, we have equation (3.2). This completes the proof.

Let $B(t)$ be a d-dimensional Brownian motion, and $Z(t)$ be a subordinator which is independent of the Brownian motion. By definition the subordinator $Z(t)$ is expressed as in equation (3.1) and its Lévy measure $\nu(d\theta)$ satisfies

$$\int_{(0,\infty)} (1 \wedge \theta)\nu(d\theta) < \infty.$$

We define

$$\mathcal{F}_t(B) = \sigma\{B(s)\,;\, 0 \le s \le t\}, \mathcal{F}(Z) = \sigma\{Z(t)\,;\, t \ge 0\}, \textit{and } \mathcal{F}_t = \mathcal{F}_t(B) \vee \mathcal{F}(Z).$$

Then $B(t)$ is an $\{\mathcal{F}_t\}$-Brownian motion.

Let $g(s)$ be an $\mathbf{R}^d$-valued continuous $\{\mathcal{F}_t\}$-adapted process, and $f(s, x)$ be a real-valued continuous function on $[0, \infty) \times \mathbf{R}^d$. Then $f(Z^{-1}(t), g(t))$ is a right-continuous $\{\mathcal{F}_t\}$-adapted process. Therefore, the Itô integral

$$\int_0^t f(Z^{-1}(s), g(s))dB(s)$$

is well-defined.

Lemma 3.2. *Let $g(s)$ be a $\mathbf{R}^d$-valued continuous $\{\mathcal{F}_t\}$-adapted process, and $f(s, x)$ be a real-valued continuous function on $[0, \infty) \times \mathbf{R}^d$. Let $Z(t)$ be a subordinator of the form in equation* (3.1) *with the Lévy measure $\nu(dr)$. Then*

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))dB(s) = \int_0^t f(u, g(Z(u)))d\tilde{B}(u)$$

$$+ \sum_{0<s\le t} \int_{Z(s-)}^{Z(s)} f(s, g(u))dB(u), \tag{3.5}$$

where $\tilde{B}(t)$ is the continuous part of the Lévy process $B(Z(t))$, which is equal to $\sqrt{\beta}B(t)$ in law.

Proof. We can and will proceed as in the proof of Lemma 3.1. (i) We first assume $\nu((0, \infty)) < \infty$, and denote the jump times of $Z(t)$ as $0 < s_1 < s_2 < s_3 < \dots$. The equation (3.5) then turns out to be

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))dB(s) = \int_0^t f(u, g(Z(u)))d\widetilde{B}(u)$$

$$+ \sum_{0<s_k\leq t} \int_{Z(s_k-)}^{Z(s_k)} f(s_k, g(u))dB(u). \qquad (3.6)$$

Let us prove equation (3.6).

If $0 \leq t < s_1$, then $Z(t) = \beta t$ and $Z^{-1}(s) = s/\beta$ for $0 \leq s < \beta s_1$. Therefore, we have

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))dB(s) = \int_0^{\beta t} f(\frac{s}{\beta}, g(s))dB(s) = \int_0^t f(u, g(\beta u)))dB(\beta u).$$

Hence, defining

$$\widetilde{B}_0(u) = B(\beta u), \ \ for \ \ u \leq s_1,$$

we have, for $0 \leq t < s_1$,

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))dB(s) = \int_0^t f(u, g(Z(u)))d\widetilde{B}_0(u). \qquad (3.7)$$

Let $t \geq 0$ and $s_1 \leq s_1 + t < s_2$. Then $Z(s_1 + t) = Z(s_1) + \beta t$, and

$$\int_0^{Z(s_1+t)} f(Z^{-1}(s), g(s))dB(s) = \int_0^{Z(s_1-)} f(Z^{-1}(s), g(s))dB(s)$$

$$+ \int_{Z(s_1-)}^{Z(s_1)} f(s_1, g(s))dB(s) + \int_{Z(s_1)}^{Z(s_1+t)} f(Z^{-1}(s), g(s))dB(s), \qquad (3.8)$$

where, in view of equation (3.7), the first integral is equal to

$$\int_0^{s_1} f(u, g(Z(u)))d\widetilde{B}_0(u),$$

and to the second integral we have applied that $Z^{-1}(s) = Z^{-1}(Z(s_1)) = s_1$ for $s \in (Z(s_1-), Z(s_1)]$. In the third integral on the right-hand side of (3.8), we have $Z^{-1}(s) = s_1 + (s - Z(s_1))/\beta$, and hence it is equal to

$$\int_{Z(s_1)}^{Z(s_1)+\beta t} f(s_1 + \frac{s - Z(s_1)}{\beta}, g(s))dB(s) = \int_{s_1}^{s_1+t} f(u, g(Z(u))d\widetilde{B}_1(u), \quad (3.9)$$

where we set, for $s_1 \leq u \leq s_2$,

$$\widetilde{B}_1(u) = B(Z(s_1) + \beta(u - s_1)) - B(Z(s_1)). \quad (3.10)$$

We can verify equation (3.9), going back to the definition of the stochastic integral. In fact, let

$$Z(s_1) = t_0 < t_1 < \dots < t_n = Z(s_1) + \beta t,$$

and set

$$u_k = s_1 + \frac{t_k - Z(s_1)}{\beta}.$$

Then

$$t_k = Z(s_1) + \beta(u_k - s_1),$$

and

$$s_1 = u_0 < u_1 < \dots < u_n = s_1 + t.$$

By definition, the left-hand side of equation (3.9) is the limit of

$$\sum_k f(s_1 + \frac{t_{k-1} - Z(s_1)}{\beta}, g(t_{k-1}))\{B(t_k) - B(t_{k-1})\}$$

$$= \sum_k f(u_{k-1}, g(Z(s_1) + \beta(u_{k-1} - s_1)))$$

$$\times\{B(Z(s_1) + \beta(u_k - s_1)) - B(Z(s_1) + \beta(u_{k-1} - s_1))\}$$

where $Z(s_1) + \beta(u_k - s_1) = Z(u_k)$, if $s_1 + t < s_2$, and hence,

$$= \sum_k f(u_{k-1}, g(Z(u_{k-1})))\{B(Z(u_k)) - B(Z(u_{k-1}))\},$$

from which we get the right-hand side of equation (3.9) in view of (3.10). Thus we have, for $s_1 + t < s_2$,

$$\int_0^{Z(s_1+t)} f(Z^{-1}(s), g(s))dB(s) = \int_0^{s_1} f(u, g(Z(u)))d\widetilde{B}_0(u)$$

$$+ \int_{Z(s_1-)}^{Z(s_1)} f(s_1, g(s))dB(s) + \int_{s_1}^{s_1+t} f(u, g(Z(u)))d\widetilde{B}_1(u). \tag{3.11}$$

Moreover, defining $\widetilde{B}(u)$ by

$$\begin{aligned} \widetilde{B}(u) &= \widetilde{B}_0(u), && \text{for } 0 \le u \le s_1, \\ &= \widetilde{B}(s_1) + \widetilde{B}_1(u), && \text{for } s_1 \le u \le s_2, \end{aligned}$$

we have, for $s_1 + t < s_2$,

$$\int_0^{Z(s_1+t)} f(Z^{-1}(s), g(s))dB(s)$$

$$= \int_0^{s_1+t} f(u, g(Z(u)))d\widetilde{B}(u) + \int_{Z(s_1-)}^{Z(s_1)} f(s_1, g(s))dB(s).$$

Applying the same argument, we have, for $s_2 \le t < s_3$,

$$\int_0^{Z(t)} f(Z^{-1}(s), g(s))dB(s)$$

$$= \int_0^{t} f(u, g(Z(u)))d\widetilde{B}(u) + \sum_{k=1}^{2} \int_{Z(s_k-)}^{Z(s_k)} f(s_k, g(u))dB(u), \tag{3.12}$$

where we set

$$\widetilde{B}_2(u) = B(Z(s_2) + \beta(u - s_2)) - B(Z(s_2)), \quad \text{for } s_2 \le u \le s_3 ,$$

and then define $\widetilde{B}(u)$ by

$$
\begin{aligned}
\widetilde{B}(u) &= \widetilde{B}_0(u), \qquad && for\ \ 0 \le u \le s_1\,, \\
&= \widetilde{B}(s_1) + \widetilde{B}_1(u), && for\ \ s_1 \le u \le s_2\,, \\
&= \widetilde{B}(s_2) + \widetilde{B}_2(u), && for\ \ s_2 \le u \le s_3\,.
\end{aligned} \tag{3.13}
$$

It is clear that $\widetilde{B}(u)$ is the continuous part of $B(Z(u))$ and equal to $\sqrt{\beta}B(u)$ in law. Repeating this procedure we obtain equation (3.6).

(ii) For the case of $\nu((0,\,\infty)) = \infty$. Let $N(dtd\theta)$ be a Poison random measure which is independent of $B(t)$, with the mean measure $dt\nu(d\theta)$, and set

$$
\begin{aligned}
Z(t) &= \beta t + \int_{(0,\,t]\times(0,\,\infty)} \theta N(dsd\theta), \\
Z_\varepsilon(t) &= \beta t + \int_{(0,\,t]\times(\varepsilon,\,\infty)} \theta N(dsd\theta).
\end{aligned} \tag{3.14}
$$

Then we have

$$Z_\varepsilon(t) \Rightarrow Z(t), \ \ as\ \ \varepsilon \downarrow 0, \tag{3.15}$$

$$Z_\varepsilon^{-1}(t) \Rightarrow Z^{-1}(t), \ \ as\ \ \varepsilon \downarrow 0, \tag{3.16}$$

and

$$B(Z_\varepsilon(t)) \Rightarrow B(Z(t)), \ \ as\ \ \varepsilon \downarrow 0, \tag{3.17}$$

where we may take $\varepsilon = 1/n$ and "$\Rightarrow$" denotes the uniform convergence on each finite time-interval almost surely. Let us denote by ν_ε the Lévy measure of $Z_\varepsilon(t)$. Then $\nu_\varepsilon((0,\,\infty)) < \infty$. Therefore, in view of equation (3.6), we have

$$
\begin{aligned}
\int_0^{Z_\varepsilon(t)} f(Z_\varepsilon^{-1}(s), g(s))dB(s) &= \int_0^t f(u, g(Z_\varepsilon(u)))d\widetilde{B}_\varepsilon(u) \\
&\quad + \sum_{0<s\le t} \int_{Z_\varepsilon(s-)}^{Z_\varepsilon(s)} f(s, g(u))dB(u),
\end{aligned} \tag{3.18}
$$

where $\widetilde{B}_\varepsilon(u)$ is defined by (3.13) with $Z_\varepsilon(t)$ in place of $Z(t)$. We notice that $\widetilde{B}_\varepsilon(t)$ and $\widetilde{B}(t)$ are the continuous parts of $B(Z_\varepsilon(t))$ and $B(Z(t))$, respectively. Let us define

$$J(t) = \lim_{n\to\infty} \sum_{\substack{0<s\le t \\ \Delta Z(s)>1/n}} \{B(Z(s)) - B(Z(s-))\},$$

where $\Delta Z(s) = Z(s) - Z(s-)$, and set

$$J_\varepsilon(t) = \sum_{0<s\le t} \{B(Z_\varepsilon(s)) - B(Z_\varepsilon(s-))\}.$$

Then

$$B(Z(t)) = \widetilde{B}(t) + J(t), \quad \textit{and} \quad B(Z_\varepsilon(t)) = \widetilde{B}_\varepsilon(t) + J_\varepsilon(t).$$

Moreover,

$$\begin{aligned} &\widetilde{B}_\varepsilon(t) \Rightarrow \widetilde{B}(t), \quad \textit{as} \quad \varepsilon \downarrow 0, \\ &J_\varepsilon(t) \Rightarrow J(t), \quad \textit{as} \quad \varepsilon \downarrow 0. \end{aligned} \tag{3.19}$$

Therefore, we have equation (3.5), making $\varepsilon \downarrow 0$ in equation (3.18), because of (3.15), (3.16), (3.17) and (3.19). This completes the proof.

4. Stochastic Differential Equations with Jumps

Let $X_{t_0,x}(t)$ be the solution of a stochastic differential equation

$$X(t) = x + \int_{t_0}^{t} \sigma(t_0 + Z^{-1}(u - t_0), X(u))dB(u)$$

$$+ \int_{t_0}^{t} b(t_0 + Z^{-1}(u - t_0), X(u))du, \tag{4.1}$$

where $B(t)$ is a d-dimensional Brownian motion and $Z(t)$ is a subordinator which is independent of the Brownian motion. Generalizing the subordination in (2.2), we set

$$Y_{t_0,x}(t) = X_{t_0,x}(t_0 + Z(t - t_0)), \tag{4.2}$$

which will be called *time-dependent* (or *time-inhomogeneous*) *subordination of the solution* $X_{t_0,x}(t)$ of equation (4.1). Then $X_{t_0,x}(t_0 + Z(t - t_0))$ satisfies

$$X_{t_0,x}(t_0 + Z(t - t_0)) = x + \int_{t_0}^{t_0+Z(t-t_0)} \sigma(t_0 + Z^{-1}(u - t_0), X_{t_0,x}(u))dB(u)$$

$$+ \int_{t_0}^{t_0+Z(t-t_0)} b(t_0 + Z^{-1}(u - t_0), X_{t_0,x}(u))du. \quad (4.3)$$

We first consider the case $t_0 = 0$, to avoid notational complexity. Let $X(t)$ be the unique solution of equation

$$X(t) = x + \int_0^t \sigma(Z^{-1}(u), X(u))dB(u) + \int_0^t b(Z^{-1}(u), X(u))du. \quad (4.4)$$

Then $X(Z(t))$ satisfies

$$X(Z(t)) = x + \int_0^{Z(t)} \sigma(Z^{-1}(u), X(u))dB(u) + \int_0^{Z(t)} b(Z^{-1}(u), X(u))du,$$

and hence by Lemmas 3.1 and 3.2

$$X(Z(t)) = x + \int_0^t \sigma(u, X(Z(u)))d\widetilde{B}(u) + \sum_{0<s\leq t} \int_{Z(s-)}^{Z(s)} \sigma(s, X(u))dB(u)$$

$$+ \beta\int_0^t b(u, X(Z(u)))du + \sum_{0<s\leq t} \int_0^{\Delta Z(s)} b(s, X(Z(s-) + u))du. \quad (4.5)$$

We first treat the case that $\nu((0, \infty)) < \infty$, and denote the jump times of $Z(t)$ by $D(\omega) = \{0 < \tau_1 < \tau_2 < \tau_3 < \dots, \textit{where } \tau_i = \tau_i(\omega), i = 1, 2, \dots\}$. We decompose Brownian paths $B(t, \omega)$ depending on jump times of $Z(t)$. Let us denote

$$\Delta Z(\tau_i) = Z(\tau_i) - Z(\tau_i-) > 0,$$

and set

$$\widehat{B}^{(\tau_i)}(u) = B(Z(\tau_i-) + u \wedge \Delta Z(\tau_i)) - B(Z(\tau_i-)).$$

Then

$$\widehat{B}^{(\tau_i)} = \{\widehat{B}^{(\tau_i)}(u), 0 \leq u < \Delta Z(\tau_i)\} \quad (4.6)$$

are Brownian motions with the life-times $\Delta Z(\tau_i)$. Enlarging the basic probability space, if necessary, we can introduce independent Brownian motions $B^{(\tau)}$, $\tau = \tau_1, \tau_2, \tau_3, \ldots$, with infinite life time such that (i) $\widehat{B}^{(\tau)}(u) = B^{(\tau)}(u)$ for $0 \le u \le \Delta Z(\tau)$, and (ii) the family $\{B^{(\tau)}, \tau = \tau_1, \tau_2, \tau_3, \ldots\}$ is independent of $\widetilde{B}(t)$ which is the continuous part of the Lévy process $B(Z(t))$ and equal to $\sqrt{\beta}B(t)$ in law. Then

$$p(\tau, \omega) = (\Delta Z(\tau), B^{(\tau)}), \ \tau \in D(\omega). \tag{4.7}$$

is a stationary Poisson point process with the characteristic measure $\nu(d\theta)\mathrm{W}(dw)$. Let

$$N((0, s] \times d\theta dw) = \#\{\tau \in D(\omega) : \tau \le s, p(\tau, \omega) \in d\theta dw\}$$

be the counting measure of the point process $p(\tau, \omega)$. Then it is a Poisson random measure with the mean measure $ds\,\nu(d\theta)\mathrm{W}(dw)$, where $\mathrm{W}(dw)$ is the Wiener measure on the space Ω_c of all continuous paths. We will sometimes write $\tau = \tau_i$, for simplicity.

We rewrite equation (4.5) as

$$X(Z(t)) = x + \int_0^t \sigma(u, X(Z(u)))d\widetilde{B}(u) + \beta\int_0^t b(u, X(Z(u)))du$$

$$+ \sum_{0<\tau\le t} \{\int_0^{\Delta Z(\tau)} \sigma(\tau, X(Z(\tau\text{-}) + u))dB^{(\tau)}(u) + \int_0^{\Delta Z(\tau)} b(\tau, X(Z(\tau\text{-}) + u))du\}. \tag{4.8}$$

Let (Ω, P) be a probability space, and W be the Wiener measure on the space Ω_c of all continuous paths on $\mathbf{R}^d$, and consider a stochastic differential equation

$$\xi(t) = x + \int_0^t \sigma(s, \xi(u))dw(u) + \int_0^t b(s, \xi(u))du, \tag{4.9}$$

where s is fixed. For fixed s and x, we denote by $\xi_t(s, x, w)$ the unique solution of equation (4.9), and set

$$\overline{\xi}_t(s, x, w) = \xi_t(s, x, w) - x. \tag{4.10}$$

Lemma 4.1. *Let $X(t)$ be the unique solution of equation* (4.4), *and define $X^{(\tau)}(t), t \le \Delta Z(\tau)$, by*

$$X^{(\tau)}(t) = X(Z(\tau\text{-}) + t). \tag{4.11}$$

Then $X^{(\tau)}(t)$ *satisfies equation* (4.9) *with* $x = X^{(\tau)}(0) = X(Z(\tau-))$ *and* $w = B^{(\tau)}$.

Proof. In view of equation (4.4), we have, for $t \le \Delta Z(\tau)$,

$$X^{(\tau)}(t) = X(Z(\tau-) + t)$$

$$= x + \int_0^{Z(\tau-)+t} \sigma(Z^{-1}(u), X(u))dB(u) + \int_0^{Z(\tau-)+t} b(Z^{-1}(u), X(u))du,$$

and

$$X^{(\tau)}(0) = X(Z(\tau-))$$

$$= x + \int_0^{Z(\tau-)} \sigma(Z^{-1}(u), X(u))dB(u) + \int_0^{Z(\tau-)} b(Z^{-1}(u), X(u))du.$$

Subtracting, we obtain

$$X^{(\tau)}(t) - X^{(\tau)}(0) = \int_{Z(\tau-)}^{Z(\tau-)+t} \sigma(Z^{-1}(u), X(u))dB(u)$$

$$+ \int_{Z(\tau-)}^{Z(\tau-)+t} b(Z^{-1}(u), X(u))du.$$

Since $Z^{-1}(u) = \tau$, for $u \in (Z(\tau-), Z(\tau)]$, we have, for $t \le \Delta Z(\tau)$,

$$X^{(\tau)}(t) - X^{(\tau)}(0) = \int_0^t \sigma(\tau, X^{(\tau)}(u))dB^{(\tau)}(u) + \int_0^t b(\tau, X^{(\tau)}(u))du, \quad (4.12)$$

on the right-hand of which we have used equation (4.11). This completes the proof.

In particular, equation (4.12) yields

$$X^{(\tau)}(\Delta Z(s)) - X^{(\tau)}(0) = \int_0^{\Delta Z(\tau)} \sigma(\tau, X^{(\tau)}(u))dB^{(\tau)}(u)$$

$$+ \int_0^{\Delta Z(\tau)} b(\tau, X^{(\tau)}(u))du,$$

which implies

$$\bar{\xi}_{\Delta Z(\tau)}(\tau, X^{(\tau)}(0), B^{(\tau)}) = \int_0^{\Delta Z(\tau)} \sigma(\tau, X^{(\tau)}(u))dB^{(\tau)}(u)$$
$$+ \int_0^{\Delta Z(\tau)} b(\tau, X^{(\tau)}(u))du,$$

where $\bar{\xi}_t(s, x, w)$ is defined by (4.10). Therefore, equation (4.8) can be expressed in terms of $\bar{\xi}_{\Delta Z(\tau)}(\tau, X^{(\tau)}(0), B^{(\tau)})$ as

$$X(Z(t)) = x + \int_0^t \sigma(r, X(Z(u)))d\tilde{B}(u) + \beta\int_0^t b(u, X(u))du$$
$$+ \sum_{0<\tau\leq t} \bar{\xi}_{\Delta Z(\tau)}(\tau, X(Z(\tau\text{-})), B^{(\tau)}), \tag{4.13}$$

where

$$\sum_{0<\tau\leq t} \bar{\xi}_{\Delta Z(\tau)}(\tau, X(Z(\tau\text{-})), B^{(\tau)})$$
$$= \int_{(0,t]\times(0,\infty)\times\Omega_C} \bar{\xi}_\theta(s, X(Z(s\text{-})), w)N(dsd\theta dw), \tag{4.14}$$

with the counting measure $N(dsd\theta dw)$ of the Poisson point process $p(\tau, \omega) = (\Delta Z(\tau), B^{(\tau)})$ given in (4.7). Equation (4.13) together with equation (4.14) implies that $Y(t) = X(Z(t))$ satisfies

$$Y(t) = x + \int_0^t \sigma(u, Y(u))\sqrt{\beta}dB(u) + \beta\int_0^t b(u, Y(u))du$$
$$+ \int_{(0,t]\times(0,\infty)\times\Omega_C} \{\xi_\theta(s, Y(s\text{-}), w) - Y(s\text{-})\}N(dsd\theta dw). \tag{4.15}$$

In general, for the case that $\nu((0, \infty)) = \infty$, let $Z(t)$ and $Z_\varepsilon(t)$ be defined by (3.14). Then (3.15), (3.16) and (3.17) hold. Let $X_\varepsilon(t)$ be the unique solution of equation (4.4) with $Z_\varepsilon(t)$. Then $Y_\varepsilon(t) = X_\varepsilon(Z_\varepsilon(t))$ satisfies equation (4.15) with $N_\varepsilon(dsd\theta dw)$, whose mean measure is $ds\nu((\varepsilon, \infty)\cap d\theta)\mathrm{W}(dw)$, that is,

$$Y_\varepsilon(t) = x + \int_0^t \sigma(u, Y_\varepsilon(u))\sqrt{\beta}dB(u) + \beta\int_0^t b(u, Y_\varepsilon(u))du$$

$$+ \int_{(0,t]\times(\varepsilon,\infty)\times\Omega_c} \{\xi_\theta(s, Y_\varepsilon(s-), w) - Y_\varepsilon(s-)\}N(dsd\theta dw). \quad (4.16)$$

Letting $\varepsilon \downarrow 0$, since

$$Y_\varepsilon(t) = X_\varepsilon(Z_\varepsilon(t)) \Rightarrow Y(t) = X(Z(t)), \quad as \ \ \varepsilon \downarrow 0, \quad (4.17)$$

we have

Theorem 4.1. *Let* $Z(t)$ *be a subordinator of the form as in equation* (3.1). *Then the time-dependent subordination* $Y(t) = X(Z(t))$ *of the unique solution* $X(t)$ *of equation* (4.4) *satisfies*

$$Y(t) = x + \int_0^t \sigma(u, Y(u))\sqrt{\beta}dB(u) + \beta\int_0^t b(u, Y(u))du$$

$$+ \int_{(0,t]\times(0,\infty)\times\Omega_c} \{\xi_\theta(s, Y(s-), w) - Y(s-)\}N(dsd\theta dw), \quad (4.18)$$

where $\xi_t(s, x, w)$ *is the unique solution of equation* (4.9).

For the case $t_0 \neq 0$, we apply, instead of Lemmas 3.1 and 3.2,

Lemma 4.2. *Let* $Z(t)$ *be given in* (3.1), *define* $Z^{-1}(t)$ *by* (2.4), *and denote* $\Delta Z(s) = Z(s) - Z(s\text{-})$. *Then*

$$\int_{t_0}^{t_0+Z(t-t_0)} f(t_0 + Z^{-1}(u - t_0), g(u))du$$

$$= \beta\int_{t_0}^t f(u, g(t_0 + Z(u - t_0)))du$$

$$+ \sum_{t_0<s\le t}\int_0^{\Delta Z(s-t_0)} f(s, g(t_0 + Z((s - t_0)-) + u))du,$$

for any $\mathbf{R}^d$*-valued continuous function* $g(s)$ *on* $[t_0, \infty)$ *and real-valued continuous function* $f(s, x)$ *on* $[t_0, \infty)\times\mathbf{R}^d$.

Lemma 4.3. *Let* $B(t)$ *be a d-dimensional Brownian motion, and let* $f(s, x)$ *and* $g(s)$, $s \in [t_0, \infty)$, *be as in Lemma* 3.2, *and* $Z(t)$ *be a subordinator of the form as in equation* (3.1). *Then*

$$\int_{t_0}^{t_0 + Z(t - t_0)} f(t_0 + Z^{-1}(u - t_0), g(u))dB(u - t_0)$$

$$= \int_{t_0}^{t} f(u, g(t_0 + Z(u - t_0)))d\widetilde{B}(u - t_0) + \sum_{t_0<s\leq t} \int_{Z((s - t_0)-)}^{Z(s - t_0)} f(s, g(t_0 + u))dB(u),$$

where $\widetilde{B}(u)$ *is defined by* (3.13) *and equal to* $\sqrt{\beta}\,B(u)$ *in law.*

Proofs of the lemmas can be carried over in the same way as for Lemmas 3.1 and 3.2. Then, applying Lemmas 4.2 and 4.3, we obtain the general forms of equation (4.8) and equation (4.18) on a time interval $(t_0, t]$. Hence we have

Theorem 4.2. *The time-dependent subordination* $Y_{t_0,x}(t) = X_{t_0,x}(t_0 + Z(t - t_0))$ *of the unique solution* $X_{t_0,x}(t)$ *of equation* (4.1) *satisfies a stochastic differential equation with jumps*

$$Y(t) = x + \int_{t_0}^{t} \sigma(u, Y(u))\sqrt{\beta}dB(u - t_0) + \beta\int_{t_0}^{t} b(u, Y(u))du$$

$$+ \int_{(t_0, t]\times(0,\infty)\times\Omega_c} \{\xi_\theta(s, Y(s-), w) - Y(s-)\}N(dsd\theta dw),$$

where $\{\sqrt{\beta}B(t) : t \geq 0\} = \{\widetilde{B}(t) : t \geq 0\}$, *in law, which is the continuous part of the Lévy process* $B(Z(t))$, $\xi_t(s, x, w)$ *is the unique solution of equation* (4.9), *and* $N(dsd\theta dw)$ *is a Poisson random measure with the mean measure* $ds\,\nu(d\theta)\mathrm{W}(dw)$, *where* $\nu(d\theta)$ *is the Lévy measure of* $Z(t)$, *and* $\mathrm{W}(dw)$ *is the Wiener measure defined on the space* Ω_c *of all continuous sample paths.*

Theorem 4.2 solves the problem of constructing Markov processes with jumps in the case of no scalar potential. To solve the case with potential functions, we shall generalize the method of Kac to the time-dependent subordination.

5. A Formula of Feynman-Kac Type

Let $X_{s,x}(t)$, $a \le s \le t \le b$, be the unique solution of the stochastic differential equation in (4.1) with $t_0 = s$, defined on a probability space $\{\Omega, \mathrm{P}\}$. Let $c(t, x)$ be continuous *a.e.* and bounded above

$$c(t, x) \le c_0 < \infty. \tag{5.1}$$

We define

$$M(s, t) = \exp\Big(\int_s^{s+Z(t-s)} c(s + Z^{-1}(u-s), X_{s,x}(u))du\Big),$$

taking it for granted that the right-hand side is well-defined, where $Z(t)$ is the same subordinator adopted to define $X_{s,x}(t)$ in (4.1). We set

$$c(t, x) = c^+(t, x) + c^-(t, x),$$

with $c^+ = (c) \vee 0$ and $c^- = (c) \wedge 0$. Then

$$M(s, t) = M^{(+)}(s, t) M^{(-)}(s, t),$$

where

$$M^{(+)}(s, t) = \exp\Big(\int_s^{s+Z(t-s)} c^+(s + Z^{-1}(u-s), X_{s,x}(u))du\Big),$$

and

$$M^{(-)}(s, t) = \exp\Big(\int_s^{s+Z(t-s)} c^-(s + Z^{-1}(u-s), X_{s,x}(u))du\Big).$$

Moreover, by Lemma 4.2

$$M^{(+)}(s, t) = M^{(1,+)}(s, t) M^{(2,+)}(s, t), \quad M^{(-)}(s, t) = M^{(1,-)}(s, t) M^{(2,-)}(s, t),$$

with

$$M^{(1,\pm)}(s, t) = \exp\Big(\beta \int_s^t c^\pm(r, X_{s,x}(s + Z(r-s)))dr\Big),$$

and

$$M^{(2,\pm)}(s, t) = \exp\Big(\sum_{s<r\le t} \int_{Z((r-s)-)}^{Z(r-s)} c^\pm(s, X_{s,x}(s+u))du\Big),$$

where

$$0 \le M^{(1,-)}(s, t),\ M^{(2,-)}(s, t) \le 1,$$

and, by (5.1),

$$P[M^{(1,+)}(s, t)] \le e^{\beta c_0(t-s)}.$$

We then assume

$$P[M^{(2,+)}(s, t)] \le e^{c_1(t-s)}, \tag{5.2}$$

so that $M^{(2,+)}$ is well-defined. We notice that

$$M^{(2,+)}(s, t) \le \exp(c_0 \sum_{0 < r \le t-s} (Z(r) - Z(r-))) = \exp(c_0 Z(t - s)),$$

and hence

$$P[M^{(2,+)}(s, t)] \le \exp((t - s)\int_0^\infty (e^{c_0\theta} - 1)\nu(d\theta)).$$

Therefore, a sufficient condition for (5.2) is

$$\int_0^\infty (e^{c_0\theta} - 1)\nu(d\theta) = c_1 < \infty.$$

In this case we have (5.2) with this constant $c_1 < \infty$.

Lemma 5.1. (i) *Let* $Y_{s,x}(t) = X_{s,x}(s + Z(t - s))$. *Then*

$$Y_{r,x}(t) = Y_{s, Y_{r,x}(s)}(t), \quad r \le s \le t.$$

(ii) *Denote the functional* $M(s, t)$ *as* $M(s, x, t, \omega)$. *Then*

$$M(r, x, t, \omega) = M(r, x, s, \omega)M(s, Y_{r,x}(s), t, \omega), \quad r \le s \le t.$$

The proof of the lemma is routine and omitted.

For fixed s we define a semi-group $P_t^{(s)}$, $t \ge 0$, by

$$P_t^{(s)} f(x) = W[f(\xi_t(s, x)) \exp(\int_0^t c(s, \xi_r(s, x))dr)], \tag{5.3}$$

with the unique solution $\xi_t(s, x)$ of

$$\xi(t) = x + \int_0^t \sigma(s, \xi(u))dw(u) + \int_0^t b(s, \xi(u))du. \tag{5.4}$$

We assume that the semi-group $P_t^{(s)}$ is a strongly continuous on $C_0(\mathbf{R}^d)$, whose generator is

$$A_s^c = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)^{ij}(s,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(s,x)\frac{\partial}{\partial x_i} + c(s,x)\mathrm{I}, \tag{5.5}$$

with a core $C_K^\infty(\mathbf{R}^d)$. Moreover, we apply Bochner's subordination with

$$Z^{(d)}(t) = \int_{(0,t]\times(0,\infty)} \theta N(dr d\theta),$$

(see (3.14)) to the semi-group $P_t^{(s)}$. Then the subordinate generator is given by

$$\mathbf{M}^{(s)}f(x) = \int_0^\infty \{P_\theta^{(s)}f(x) - f(x)\}\,\nu(d\theta), \tag{5.6}$$

where $\nu(d\theta)$ is the Lévy measure of $Z^{(d)}(t)$, and $C_K^\infty(\mathbf{R}^d)$ is a core of $\mathbf{M}^{(s)}$.

Let us define

$$T(s,x) = \sup\{t > s : \int_s^{s+Z(t-s)} c^-(s + Z^{-1}(u-s), X_{s,x}(u))du) > -\infty\},$$
$$(\sup \phi = s),$$

and set

$$D = \{(s,x) : \mathrm{P}[s < T(s,x)] = 1\}.$$

We define $\mathrm{Q}_{s,t}^c$ by

$$\mathrm{Q}_{s,t}^c f(x) = \mathrm{P}[f(Y_{s,x}(t))M(s,t)], \ \ for\ \ (s,x) \in D. \tag{5.7}$$

We set $\mathrm{Q}_{s,t}^c f(x) = 0$, for $(s,x) \notin D$. Then we have

Theorem 5.1. *Let $c(s,x)$ be any potential function being continuous a.e., and satisfying the conditions in* (5.1) *and* (5.2), *and let $\mathrm{Q}_{s,t}^c f$ be defined by the formula in* (5.7). *Assume $\mathbf{M}^{(s)}f(x)$ is continuous in (s,x). Then*

$$\lim_{t\downarrow s}\frac{\mathrm{Q}_{s,t}^c f(x) - f(x)}{t-s} = \beta A_s^c f(x) + \mathbf{M}^{(s)}f(x), \ \ (s,x) \in D,$$

$$for\ any\ f \in C_K^\infty(\mathbf{R}^d).$$

Proof. To avoid notational complexity we set $s = 0$ and $(0, x) \in D$, and consider

$$Q^c_{0,t}f(x) = P[f(X(Z(t)))\exp(\int_0^{Z(t)} c(Z^{-1}(u), X(u))du)], \tag{5.8}$$

where $X(t)$ is the unique solution of equation (4.4). Let us denote

$$F(t) = f(X(Z(t))),$$

and

$$M(t) = \exp(\int_0^{Z(t)} c(Z^{-1}(u), X(u))du).$$

Let $0 = t_0 < t_1 < t_2 < \dots < t_n = b$ be a partition of $[0, b]$, and $\delta = \max | t_i - t_{i-1} |$. Then

$$F(b)M(b) - f(x) = \sum_{i=1}^{n} \{F(t_i)M(t_i) - F(t_{i-1})M(t_{i-1})\}$$

$$= \sum_{i=1}^{n} F(t_{i-1})\{M(t_i) - M(t_{i-1})\} + \sum_{i=1}^{n} M(t_i)\{F(t_i) - F(t_{i-1})\}. \tag{5.9}$$

We set $D_\varepsilon(\omega) = \{\tau : \Delta Z(\tau) > \varepsilon\} = \{0 < \tau_1 < \tau_2 < \tau_3 < \dots \}$, since $\nu((0, \infty))$ may be infinite, and denote by $p_\varepsilon(\tau, \omega) = (\Delta Z_\varepsilon(\tau), B^{(\tau)})$ the point process defined on $D_\varepsilon(\omega)$, where $Z_\varepsilon(t)$ is given by (3.14). Let $X_\varepsilon(t)$ be the solution of equation (4.4) with $Z_\varepsilon(t)$. Let us set

$$F_\varepsilon(t) = f(X_\varepsilon(Z_\varepsilon(t))),$$

and

$$M_\varepsilon(t) = \exp(\int_0^{Z_\varepsilon(t)} c(Z_\varepsilon^{-1}(u), X_\varepsilon(u))du).$$

We have, by Lemma 3.1,

$$\int_0^{Z_\varepsilon(t)} c(Z_\varepsilon^{-1}(u), X_\varepsilon(u))du$$

$$= \beta\int_0^t c(u, Y_\varepsilon(u))du + \sum_{0 < s \le t} \int_{Z_\varepsilon(s-)}^{Z_\varepsilon(s)} c(s, X_\varepsilon(u))du,$$

where $s = \tau_k$ in the summation and $Y_\varepsilon(u) = X_\varepsilon(Z_\varepsilon(u))$. Hence,

$$M_\varepsilon(t) = M_1(t)M_2(t),$$

with

$$M_1(t) = \exp(\beta\int_0^t c(u, Y_\varepsilon(u))du),$$

and

$$M_2(t) = \exp(\sum_{0<s\le t}\int_{Z_\varepsilon(s-)}^{Z_\varepsilon(s)} c(s, X_\varepsilon(u))du).$$

Then the first summation on the right-hand side of (5.9) with F_ε and M_ε is equal to

$$\sum_{i=1}^n F_\varepsilon(t_{i-1})M_2(t_i)\{M_1(t_i) - M_1(t_{i-1})\}$$
$$+ \sum_{i=1}^n F_\varepsilon(t_{i-1})M_1(t_{i-1})\{M_2(t_i) - M_2(t_{i-1})\},$$

where the first summation converges, as $\delta \downarrow 0$, to

$$\beta\int_0^t c(u, Y_\varepsilon(u)) f(Y_\varepsilon(u-))M_\varepsilon(u)du.$$

The second summation is equal to

$$\sum_{i=1}^n F_\varepsilon(t_{i-1})M_1(t_{i-1})M_2(t_{i-1})\{M_2(t_i)/M_2(t_{i-1}) - 1\},$$

and it converges, as $\delta \downarrow 0$, to

$$\sum_{0<s\le t} f(Y_\varepsilon(s-))M_\varepsilon(s-)\{m(\Delta Z_\varepsilon(s)) - 1\},$$

with

$$m(\Delta Z_\varepsilon(s)) = M_\varepsilon(s)/M_\varepsilon(s-), \tag{5.10}$$

where

$$m(\Delta Z_\varepsilon(s)) = \exp(\int_0^{\Delta Z_\varepsilon(s)} c(s, X_\varepsilon(Z_\varepsilon(s-) + u))du). \tag{5.11}$$

Moreover, in view of (4.7), setting $S_\varepsilon = (0, t]\times(\varepsilon, \infty)\times\Omega_c$, we have

$$\sum_{0<s\leq t} f(Y_\varepsilon(s-))M_\varepsilon(s-)\{m(\Delta Z_\varepsilon(s)) - 1\}$$

$$= \int_{S_\varepsilon} f(Y_\varepsilon(s-))M_\varepsilon(s-)\{m_s^\theta(Y_\varepsilon(s-)) - 1\}N(dsd\theta dw), \quad (5.12)$$

where

$$m_s^\theta(x) = \exp(\int_0^\theta c(s, \xi_u(s, x, w))du). \quad (5.13)$$

Therefore, we have

$$\lim_{\delta\downarrow 0} \mathrm{P}[\sum_{i=1}^{n} F_\varepsilon(t_{i-1})\{M_\varepsilon(t_i) - M_\varepsilon(t_{i-1})\}]$$

$$= \beta\mathrm{P}[\int_0^t c(u, Y_\varepsilon(u))f(Y_\varepsilon(u-))M_\varepsilon(u)du]$$

$$+ \mathrm{P}[\int_{S_\varepsilon} f(Y_\varepsilon(s-))M_\varepsilon(s-)\{m_s^\theta(Y_\varepsilon(s-)) - 1\}N(dsd\theta dw)]. \quad (5.14)$$

We now compute the second summation on the right-hand side of (5.9) with F_ε and M_ε.

We first notice that we have, by Theorem 4.1,

$$Y_\varepsilon(t) = X_\varepsilon(Z_\varepsilon(t)) = x + \int_0^t \sigma(u, Y_\varepsilon(u))\sqrt{\beta}dB(u) + \beta\int_0^t b(u, Y_\varepsilon(u))du$$

$$+ \int_{(0,\, t]\times(\varepsilon,\, \infty)\times\Omega_c} \overline{\xi}_\theta(s, Y_\varepsilon(s-), w)N(dsd\theta dw),$$

where $\overline{\xi}_r(s, x, w) = \xi_r(s, x, w) - x$.

Then, by Itô (1951) and Kunita-Watanabe (1967),

$$f(Y_\varepsilon(t)) - f(Y_\varepsilon(0)) = \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(Y_\varepsilon(u))\sigma^{ij}(u, Y_\varepsilon(u))\sqrt{\beta}dB^j(u)$$

$$+ \beta \sum_{i=1}^{d} \int_0^t b^i(u, Y_\varepsilon(u)) \frac{\partial f}{\partial x^i}(Y_\varepsilon(u))du$$

$$+ \beta \sum_{i,j=1}^{d} \int_0^t (\sigma\sigma^T)^{ij}(u, Y_\varepsilon(u)) \frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j}(Y_\varepsilon(u))du$$

$$+ \int_{(0,t]\times(\varepsilon,\infty)\times\Omega_c} \{f(Y_\varepsilon(s-) + \overline{\xi}_\theta(s, Y_\varepsilon(s-), w)) - f(Y_\varepsilon(s-))\}N(dsd\theta dw),$$

in the last integral of which we apply $Y_\varepsilon(s-) + \overline{\xi}_r(s, Y_\varepsilon(s-), w) = \xi_r(s, Y_\varepsilon(s-), w)$. Therefore, we have

$$\lim_{\delta\downarrow 0} \mathrm{P}[\sum_{i=1}^{n} M_\varepsilon(t_i)\{F_\varepsilon(t_i) - F_\varepsilon(t_{i-1})\}] = \mathrm{P}[\int_0^t M_\varepsilon(s)\beta A_s f(Y_\varepsilon(s))ds]$$

$$+ \mathrm{P}[\int_{S_\varepsilon} M_\varepsilon(s-)m_s^\theta(Y_\varepsilon(s-))\{f(\xi_\theta(s, Y_\varepsilon(s-), w)) - f(Y_\varepsilon(s-))\}N(dsd\theta dw)], \tag{5.15}$$

where we have applied equations (5.10), (5.11) and (5.13).

Combining equations (5.9), (5.14) and (5.15), we have

$$\mathrm{P}[f(X_\varepsilon(Z_\varepsilon(t)))M_\varepsilon(t)] - f(x)$$

$$= \mathrm{P}[\int_0^t M_\varepsilon(s)\beta\{A_s f(Y_\varepsilon(s)) + c(s, Y_\varepsilon(s))f(Y_\varepsilon(s-))\}ds]$$

$$+ \mathrm{P}[\int_0^t dsM_\varepsilon(s-)\int_\varepsilon^\infty \mathrm{W}[f(\xi_\theta(s, Y_\varepsilon(s-))m_s^\theta(Y_\varepsilon(s-)) - f(Y_\varepsilon(s-))]\nu(d\theta)], \tag{5.16}$$

where the second integral is equal to

$$\mathrm{P}[\int_0^t dsM_\varepsilon(s-)\int_0^\infty 1_{(\varepsilon,\infty)}(\theta)\{P_\theta^{(s)}f(Y_\varepsilon(s-)) - f(Y_\varepsilon(s-))\}\nu(d\theta)],$$

with $P_\theta^{(s)} f(x)$ given in (5.3). We notice that $(1 \wedge \theta)\nu(d\theta)$ is a finite measure and

$$\frac{1}{\theta}\{P_\theta^{(s)} f(Y_\varepsilon(s\text{-})) - f(Y_\varepsilon(s\text{-}))\}, \quad s < t,$$

is bounded, since it converges, as $\theta \downarrow 0$, to the generator of $P_\theta^{(s)}$. Therefore, making $\varepsilon \downarrow 0$ in equation (5.16), we have, by the dominated convergence theorem,

$$\mathrm{P}[f(X(Z(t)))M(t)] - f(x) = \mathrm{P}[\int_0^t \{\{M(s)\beta A_s^c f(Y(s)) + M(s\text{-})\mathbf{M}^{(s)} f(Y(s\text{-})\} ds], \tag{5.17}$$

where A_s^c is given by (5.5) and $\mathbf{M}^{(s)}$ by (5.6). Equation (5.17) implies

$$\lim_{t \downarrow 0} \frac{Q_{0,t}^c f(x) - f(x)}{t} = \beta A_0^c f(x) + \mathbf{M}^{(0)} f(x), \quad (0, x) \in D,$$

for $f \in C_K^\infty(\mathbf{R}^d)$, by the dominated convergence theorem.

We can prove the case $s \neq 0$ in the same way. Let $X_{s,x}(t)$ be the unique solution of equation (4.1) with $t_0 = s$, and set

$$F(t) = f(X_{s,x}(s + Z(t - s))), \text{ and } M(t) = M(s, t).$$

Then we can repeat the same argument that we have adopted in the case $s = 0$. However, to make it simpler, we can apply the integration by parts formula

$$F(t)M(t) - F(s)M(s) = \int_s^t F(r\text{-})dM(r) + \int_s^t M(r)dF(r), \tag{5.18}$$

and the same approximation argument. We remark that, by Theorem 4.2,

$$Y(t) = Y_{s,x}(t) = x + \int_s^t \sigma(r, Y(r))d\widetilde{B}(r) + \beta \int_s^t b(r, Y(r))dr$$

$$+ \int_{(s,t] \times (0,\infty) \times \Omega_c} \overline{\xi}_\theta(r, Y(r\text{-}), w) N(dr d\theta dw),$$

where $\overline{\xi}_\theta(r, x, w) = \xi_\theta(r, x, w) - x$. Moreover, we have, by Lemma 4.2,

$$\int_s^{s+Z(t-s)} c(s+Z^{-1}(r-s), X_{s,x}(r))dr$$

$$= \beta\int_s^t c(u, X_{s,x}(s+Z(u-s)))du + \sum_{s<r\le t}\int_{Z((r-s)-)}^{Z(r-s)} c(r, X_{s,x}(s+u))du$$

$$= \beta\int_s^t c(r, Y(r))dr + \sum_{s<r\le t}\int_{Z((r-s)-)}^{Z(r-s)} c(r, X_{s,x}(s+u))du.$$

Hence, the first integral on the right-hand side of (5.18) is

$$\int_s^t F(r\text{-})dM(r)$$

$$= \int_s^t f(Y(r\text{-}))\, d\{\exp(\int_s^{s+Z(r-s)} c(s+Z^{-1}(u-s),\, X_{s,x}(u))du)\}$$

$$= \beta\int_s^t c(r, Y(r))f(Y(r\text{-}))M(r)dr$$

$$+ \sum_{s<r\le t} f(Y(r\text{-}))M(r\text{-})\{m(\Delta Z(r-s)) - 1\}$$

where

$$m(\Delta Z(r-s)) = \exp(\int_0^{\Delta Z(r-s)} c(r, X_{s,x}(s+Z((r-s)\text{-})+u))du\ ,$$

and hence

$$= \beta\int_s^t c(r, Y(r))f(Y(r\text{-}))M(r)dr$$

$$+ \int_{(s,t]\times(0,\infty)\times\Omega_C} M(r\text{-})f(Y(r\text{-}))\{m_r^\theta(Y(r\text{-})) - 1\}N(drd\theta dw), \qquad (5.19)$$

where $m_r^\theta(x)$ is defined by (5.13). The second integral of the right-hand side of (5.18) is

$$\int_s^t M(r)dF(r) = \int_s^t M(r)\beta A_r f(Y(r))dr$$

$$+ \int_{(s,t]\times(0,\infty)\times\Omega_c} M(r-)m_r^{\theta}(Y(r-))\{f(\xi_\theta(r, Y(r-), w)) - f(Y(r-))\}N(drd\theta dw)$$

$$+ \int_s^t M(r) \sum_{i,j=1}^{d} \frac{\partial f}{\partial x^i}(Y(r))\sigma^{ij}(r, Y(r))\sqrt{\beta}dB^j(r), \tag{5.20}$$

where $\xi_t(s, x, w)$ is the unique solution of equation (5.4) and $Y(t) = Y_{s,x}(t)$.

Combining (5.19) and (5.20) we have

$$\mathrm{P}[f(X(Z(t)))M(t)] - f(x)$$

$$= \mathrm{P}[\int_s^t M(r)\beta\{A_r f(Y(r)) + c(r, Y(r))f(Y(r-))\}dr]$$

$$+ \mathrm{P}[\int_{(s,t]\times(0,\infty)\times\Omega_c} M(r-)\{f(\xi_\theta^r(w))\, m_r^{\theta}(Y(r-)) - f(Y(r-))\}N(drd\theta dw)],$$

$$= \mathrm{P}[\int_s^t M(r)\beta A_r^c f(Y(r))dr] + \mathrm{P}[\int_s^t M(r-)\mathbf{M}^{(r)}f(Y(r-))\,dr],$$

where $Y(t) = Y_{s,x}(t)$. We have thus obtained the general form of equation (5.17) on $(s, t]$, and we can complete the proof.

Let us consider a special case that a subordinator $Z(t)$ has the Lévy measure

$$\nu^{(\kappa)}(d\theta) = \frac{1}{2\sqrt{\pi}} e^{-\kappa^2\theta} \frac{1}{\theta^{3/2}} d\theta, \tag{5.21}$$

with a parameter κ, and potential functions satisfying

$$c(t, x) \le \kappa^2 < \infty. \tag{5.22}$$

Then the condition (5.2) is satisfied. We can then generalize the results in Nagasawa-Tanaka (1998, 1999) as follows.

Theorem 5.2. *Let $X_{s,x}(t)$ be the solution of the stochastic differential equation in* (4.1) *($t_0 = s$) with the subordinator $Z(t)$ which is independent of $B(t)$ and has the Lévy measure $\nu^{(\kappa)}(d\theta)$ given in* (5.21), *and let $c(t, x)$ be any potential function being continuous a.e., and satisfying* (5.22). *Define evolution operators $Q^c_{s,t}$ by the formula in* (5.7). *Then it solves*

$$\frac{\partial u}{\partial s} + \beta A^c_s u + M^c_s u = 0, \tag{5.23}$$

where

$$A^c_s = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)^{ij}(s, x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(s, x)\frac{\partial}{\partial x_i} + c(s, x)\mathrm{I},$$

and

$$M^c_s = -\sqrt{-A^c_s + \kappa^2 \mathrm{I}} + \kappa \mathrm{I}.$$

Remark. If $\beta = 0$, equation (5.23) reduces to

$$\frac{\partial u}{\partial s} + M^c_s u = 0. \tag{5.24}$$

This generalizes the results on stochastic differential equations of pure-jumps in Nagasawa-Tanaka (1998, 1999)). Equation (5.24) is the equation of motion for a relativistic spinless quantum particle in an electromagnetic field mentioned in Introduction, cf. Nagasawa (1997) for details.

6. Markov Processes with (Pure) Jumps

Markov processes with jumps determined by the evolution operator $Q^c_{s,t}$ in equation (5.7) can be constructed with the help of the Schrödinger representation. In this section we assume that A_s in (2.3) and A^c_s in (5.5) are given by

$$A_s = \frac{1}{2}\Delta + \sum_{i=1}^{d} b^i(s, x)\frac{\partial}{\partial x_i}, \tag{6.1}$$

and

$$A^c_s = \frac{1}{2}\Delta + \sum_{i=1}^{d} b^i(s, x)\frac{\partial}{\partial x_i} + c(s, x)\mathrm{I}, \tag{6.2}$$

respectively, where Δ is the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{\sigma_2(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{\sigma_2(x)}\,(\sigma\sigma^T(x))^{ij}\frac{\partial}{\partial x^j}\right),$$

with a positive definite diffusion matrix $\sigma\sigma^T(x)$, where we denote $\sigma_2(t,x) = det(\sigma\sigma^T(x))$. To adopt the operators A_s and A_s^c in (6.1) and (6.2), respectively, we need to replace the drift coefficient $\boldsymbol{b}(t,x)$ in the preceding sections by

$$\boldsymbol{b}^\circ(t,x) = \boldsymbol{b}(t,x) + \boldsymbol{b}_\sigma(t,x),$$

with a correction term

$$\boldsymbol{b}_\sigma(x)^j = \frac{1}{2}\frac{1}{\sqrt{\sigma_2(x)}}\frac{\partial}{\partial x^i}(\sqrt{\sigma_2(x)}\,(\sigma\sigma^T(x))^{ij}).$$

If σ is independent of the space variables, then the correction is not necessary and $\boldsymbol{b}^\circ(t,x) = \boldsymbol{b}(t,x)$.

Let $Q^c_{s,t}$ be the operator defined by (5.7). Then there exists a transition kernel $q^c(s,x\,;t,dy)$ such that

$$Q^c_{s,t}f(x) = \int q^c(s,x\,;t,dy)f(y), \tag{6.3}$$

which obeys the Chapman-Kolmogorov equation but $Q^c_{s,t}1 \neq 1$, namely, the normality condition is not satisfied. To construct the Schrödinger process with jumps we need a prescribed entrance-exit law $\{\widehat{\phi}_a(x),\ \phi_b(x)\}$ satisfying a normality condition

$$\int dx\,\widehat{\phi}_a(x)q^c(a,x\,;b,dy)\phi_b(y) = 1.$$

With the triplet $\{q^c(s,x\,;t,dy),\ \widehat{\phi}_a(x),\ \phi_b(x)\}$ we define a probability measure Q by the Schrödinger representation (cf. Nagasawa (1993, 1997)):

$$\begin{aligned} &Q[f(X_a, X_{t_1}, \dots, X_{t_{n-1}}, X_b)] \\ &\quad= \int dx_0\widehat{\phi}_a(x_0)q^c(a,x_0\,;t_1,dx_1)q^c(t_1,x_1\,;t_2,dx_2)\cdots \\ &\qquad\qquad \cdots q^c(t_{n-1},x_{n-1}\,;b,dx_n)\phi_b(x_n)f(x_0,x_1,\dots,x_n), \end{aligned}$$

where $a < t_1 < \cdots < t_{n-1} < b$ and $f(x_0, x_1, \dots, x_n)$ is any bounded measurable function on the product space $(\mathbf{R}^d)^{n+1}$, $n = 1, 2, \dots$. We thus obtain a Schrödinger process $\{X_t, \mathcal{F}^r_a \vee \mathcal{F}^b_s, a \le r < t < s \le b, Q\}$, where $\mathcal{F}^t_a = \sigma\{X_r\,; r \in [a,t]\}$, and $\{X_t, \mathcal{F}^t_a, a \le t \le b, Q\}$ is a Markov process with jumps by Theorem 4 of Nagasawa (1997) (or Theorem 3.6 of Nagasawa (1993)).

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References

Bochner, S., (1949): Diffusion equations and stochastic processes. Proc. Nat. Acad. Sci. USA, **35**, 368-370.

Itô, K., (1951): *On stochastic differential equations.* Memoirs of the AMS, **4**. American Math. Soc.

Kunita, H. & Watanabe, S., (1967) On square integrable martingales. Nagoya Math. J. **30**, 209-245.

Nagasawa, M., (1993): *Schrödinger equations and Diffusion Theory.* Birkhäuser Verlag, Basel Boston Berlin.

Nagasawa, M., (1996): Quantum theory, theory of Brownian motions, and relativity theory. Chaos, Solitons and Fractals, **7**, 631-643.

Nagasawa, M., (1997): Time reversal of Markov processes and relativistic quantum theory. Chaos, Solitons and Fractals, **8**, 1711-1772.

Nagasawa, M., Tanaka, H., (1998): Stochastic differential equations of pure-jumps in relativistic quantum theory. Chaos, Solitons and Fractals, 10, No 8, 1265-1280.

Nagasawa, M., Tanaka, H., (1999): The principle of variation for relativistic quantum particles. Preprint.

Sato, K., (1990): Subordination depending on a parameter. Probability Theory and Mathematical Statistics, Proc. Fifth Vilnius Conf. Vol. **2**, 372-382.

Vershik, A., Yor, M., (1995): Multiplicativité du processus gamma et étude asymptotique des lois stables d'indice α, lorsque α tend vers 0. Prepublications du Lab. de probab. de l'université Paris VI, 284 (1995).