Séminaire de probabilités (Strasbourg)

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Séminaire de probabilités (Strasbourg), tome 34 (2000), p. 218-238
<http://www.numdam.org/item?id=SPS_2000__34__218_0>

Saturations of Gambling Houses

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Abstract

Suppose that X is a Borel subset of a Polish space. Let $\mathbb{P}(X)$ be the set of probability measures on the Borel σ-field of X. We equip $\mathbb{P}(X)$ with the weak topology. A *gambling house* Γ on X is a subset of $X \times \mathbb{P}(X)$ such for each $x \in X$, the section $\Gamma(x)$ of Γ at x is nonempty. Assume moreover that Γ is an analytic subset of $X \times \mathbb{P}(X)$. Then we can associate with Γ optimal reward operators G_Γ, R_Γ, and M_Γ as follows:

$$(G_\Gamma u)(x) = \sup\{\textstyle\int u\, d\gamma : \gamma \in \Gamma(x)\}, \qquad x \in X,$$
$$(R_\Gamma u)(x) = \sup \int u(x_t)\, dP_\sigma, \qquad x \in X,$$

where u is a bounded, Borel measurable function on X, the sup in the definition of R_Γ is over all measurable strategies σ available in Γ at x and Borel measurable stop rules t (including $t \equiv 0$), x_t is the terminal state and P_σ the probability measure on H, the space of infinite histories, induced by σ;

$$(M_\Gamma g)(x) = \sup \int g\, dP_\sigma, \qquad x \in X,$$

where g is a bounded, Borel measurable function on H and the sup is over all measurable strategies σ available in Γ at x. The aim of this article is to describe the “largest” houses or "saturations" for which the associated operators are the same as the corresponding operators for the original house. Our methods are constructive and will show that the saturations are again analytic gambling houses.

1 INTRODUCTION

The point of departure of this article is a beautiful result of Dellacherie and Meyer [5, 38] in gambling theory. We will describe this result in the framework of the Dubins-Savage ([6]) theory.

Let X be a Borel subset of a Polish space, and let $\mathbb{P}(X)$ be the set of probability measures on the Borel σ-field of X. Give $\mathbb{P}(X)$ the topology of weak convergence, so $\mathbb{P}(X)$ is again a Borel subset of a Polish space (see ([10], $17E$) for details). A *gambling house* on X is a subset Γ of $X \times \mathbb{P}(X)$ such that each section $\Gamma(x)$ of Γ at x is nonempty. A strategy σ *available in* Γ *at* x is a sequence $\sigma_o, \sigma_1, \dots$ such that $\sigma_o \in \Gamma(x)$ and, for $n \geq 1, \sigma_n$ is a universally measurable function on X^n into $\mathbb{P}(X)$ such that $\sigma_n(x_1, x_2, \dots, x_n) \in \Gamma(x_n)$ for every $x_1, x_2, \dots, x_n \in X$. Such a σ defines a unique probability measure on the Borel subsets of the history space $H = X^N$, where N is the set of positive integers and H is given the product topology. We will use the same symbol σ for this probability measure. (See ([3], 7.45) for the existence of

[1] Research supported by National Science Foundation Grant DMS-9703285.

this measure.) If σ is a strategy available in Γ at x and $x' \in X$, then the *conditional strategy* $\sigma[x']$ is the strategy defined as follows:

$$(\sigma[x'])_0 = \sigma_1(x')$$

and, for $n \geq 1$,

$$(\sigma[x'])_n(x_1, x_2, \dots, x_n) = \sigma_{n+1}(x', x_1, x_2, \dots, x_n)$$

for $x_1, x_2, \dots, x_n \in X$. Note that $\sigma[x']$ is available in Γ at x'. The set of strategies (and also the measures induced on H by these strategies) available in Γ at x will be denoted by $\Sigma_\Gamma(x)$.

A *stop rule* is a universally measurable function t on H into $\omega = N \cup \{0\}$ such that $t(h) = k$ and $h \equiv_k h'$ imply $t(h') = k$, where $h \equiv_k h'$ means that h and h' agree through the first k coordinates. In particular, if $t(h) = 0$ for some h, then t is identically zero. If t is a stop rule such that $t \geq 1$ and $x \in X$, then the *conditional stop rule* $t[x]$ is defined by

$$t[x](h) = t(xh) - 1, \; h \in H,$$

where xh is the history obtained by catenating x and h. Note that $t[x]$ is again a stop rule. A pair $\pi = (\sigma, t)$ where $\sigma \in \Sigma_\Gamma(x)$ and t is a stop rule is said to be a *policy* available at x.

In the sequel, none of the results would be affected if we had restricted ourselves to Borel measurable stop rules.

A *measurable leavable gambling problem* is a triple (X, Γ, u), where X is a Borel subset of a Polish space, Γ is a gambling house which is an analytic subset of $X \times \mathbb{P}(X)$, and u is a bounded, upper analytic function on X, that is, $[u > a]$ is an analytic subset of X for every real a. Such structures with Γ and u both Borel measurable were introduced by Strauch ([19]); the extension to analytic gambling house and upper analytic utility functions is due to Meyer and Traki ([17]).

If Γ is an analytic gambling house on X, then $\Sigma_\Gamma(x) \neq \phi$ for each x, courtesy of the von-Neumann selection theorem ([10], 29.9). Furthermore the set $\Sigma_\Gamma = \bigcup_{x \in X} \{x\} \times \Sigma_\Gamma(x)$ is analytic in $X \times \mathbb{P}(H)$, as was established by Dellacherie ([4], Theorem 3).

The *optimal reward operator* for a measurable leavable gambling problem (X, Γ, u) is defined by

$$(R_\Gamma u)(x) = \sup_\pi \int u(h_t) \, d\sigma, \qquad x \in X, \tag{1.1}$$

where h_t abbreviates $h_{t(h)}$ and the sup is taken over all policies $\pi = (\sigma, t)$ available in Γ at x.

The Fundamental Theorem of Gambling (see([17]) or ([14], Theorem 4.8) provides another description of R_Γ as follows. First we need a definition. We say that a bounded function g on X is $\Gamma - excessive$ if it is upper analytic and $\int g \, d\gamma \leq g(x)$ for every $\gamma \in \Gamma(x)$ and $x \in X$.

Theorem 1.1. *(Fundamental Theorem of Gambling) If Γ is an analytic gambling house on X and u is a bounded, upper analytic function on X, then $R_\Gamma u$ is the least Γ-excessive function g such that $g \geq u$.*

Note that the function $R_\Gamma u$ can be defined for *every* house Γ for which $\Sigma_\Gamma(x) \neq \phi$ for each x and any bounded, universally measurable function u on X by using (1.1).

We associate with each analytic house Γ on X a house Γ^c as follows:

$$\Gamma^c(x) = \{\, \sigma\psi_t^{-1} : \sigma \in \Sigma_\Gamma(x) \text{ and } t \text{ is a bounded stop rule} \,\}, \qquad x \in X,$$

where $\psi_t(h) = h_t$, $h \in H$. (For $t = 0$, $\sigma\psi_t^{-1}$ is defined to be $\delta(x)$ if $\sigma \in \Sigma_\Gamma(x)$.) In other words, $\Gamma^c(x)$ is the set of distributions of the terminal state induced by policies $\pi = (\sigma, t)$ available in Γ at x for which t is bounded. As we will prove in section 3, Γ^c is an analytic subset of $X \times \mathbb{P}(X)$.

If L_1 and L_2 are operators that map bounded functions on X (respectively H) to bounded functions on X, then we will write $L_1 \sim L_2$ if $L_1 = L_2$ on bounded, Borel measurable functions on X (respectively H); and we will write $L_1 \approx L_2$ if $L_1 = L_2$ on bounded, upper analytic functions on X (respectively H).

We are now ready to state the result of Dellacherie and Meyer which was mentioned in the first paragraph.

Theorem 1.2. *Suppose that Γ is an analytic gambling house on X. Then the largest gambling house Γ' such that $R_{\Gamma'} \sim R_\Gamma$ is defined by*

$$\Gamma^s(x) = \overline{\text{sco}}\, \Gamma^c(x), \qquad x \in X, \tag{1.2}$$

where $\overline{\text{sco}}\, \Gamma^c(x)$ denotes the (total variation) norm closure of the strong convex hull of $\Gamma^c(x)$. In particular, Γ^s is an analytic gambling house.

Recall that if $M \subseteq \mathbb{P}(X)$, then the strong convex hull of M,written $\text{sco}\, M$, is the set of all $\nu \in \mathbb{P}(X)$ such that there is $\mu \in \mathbb{P}(\mathbb{P}(X))$ with $\mu^*(M) = 1$ and

$$\nu(B) = \int_{\mathbb{P}(X)} \eta(B)\, \mu(d\eta)$$

for every Borel subset B of X,where μ^* is the outer measure induced by μ. For $M \subseteq \mathbb{P}(X)$, we say that M is *strongly convex* if $M = \text{sco}\, M$.

The gambling house Γ^s is called the saturation of the gambling house Γ. Dellacherie and Meyer [5] define the saturation of Γ to be the largest house having the same excessive functions as Γ. It is easy to see that their definition is equivalent to the one given above. The statement of Theorem 1.2 differs slightly from the formulation of Dellacherie and Meyer. In place of the gambling house Γ^c they have a "house" consisting of sub-probability measures and they remark ([5], p.192) that their proof depends critically on allowing sub-probability measures in their construction. They pay a price for this: they need to perform the operation of hereditary closure on the strong convex hull before taking the norm-closure and then intersect the result with $\mathbb{P}(X)$. In fairness, we must point out that they are aware, as they remark ([5], p.183), that they could have worked with Γ^c but chose not to do so as the proof that Γ^c is analytic is laborious. It turns out that proving the analyticity of Γ^c is not so hard after all, as we shall see presently.

Given a gambling house Γ, there are other optimal reward operators of interest. The aim of this article is to construct "largest" houses or "saturations" keeping those operators invariant in the spirit of Theorem 1.2. We will now define two such operators.

Dubins et al. ([8]) define a *measurable non-leavable gambling problem* to be a triple (X, Γ, u^*), where X is a Borel subset of a Polish space, Γ an analytic gambling house on X, u a bounded, upper analytic function on X and

$$u^*(h) = \limsup_n u(h_n), \qquad h \in H.$$

The *optimal reward operator* for the nonleavable gambling problem is defined by

$$(V_\Gamma u)(x) = \sup_{\sigma \in \Sigma_\Gamma(x)} \int u^* \, d\sigma, \qquad x \in X.$$

Note that V_Γ is defined even when Γ is not analytic just so long as $\Sigma_\Gamma(x)$ is nonempty for each $x \in X$.

For any set $\Sigma \subseteq X \times \mathbb{P}(H)$ such that $\Sigma(x)$ is nonempty for each x and any bounded, upper analytic function g on H, we define

$$(M_\Sigma g)(x) = \sup_{\sigma \in \Sigma(x)} \int g \, d\sigma, \qquad x \in X.$$

We will also write M_Γ for M_Σ in case $\Sigma = \Sigma_\Gamma$ for a gambling house Γ on X. In this case, we will say that Σ is a *global gambling house* on X.

Here are the main results of the paper.

Theorem 1.3. *Suppose that Γ is an analytic gambling house on X. Then the largest gambling house Γ' such that $M_\Gamma \sim M_{\Gamma'}$ is*

$$\Gamma^{s_1}(x) = \overline{\text{sco}}\,\Gamma(x), \qquad x \in X.$$

In consequence, Γ^{s_1} is analytic.

In the sequel, we will write $\overline{\text{sco}}\,\Gamma$ for Γ^{s_1}.

Theorem 1.4. *Let Γ be an analytic gambling house on X. Then the largest global gambling house Σ such that $M_\Sigma \sim M_{\Sigma_\Gamma}$ is $\Sigma_{\overline{\text{sco}}\,\Gamma}$.*

Theorem 1.5. *Suppose that Γ is an analytic gambling house on X. Then the largest set $\Sigma \subseteq X \times \mathbb{P}(H)$ such that $\Sigma(x)$ is nonempty for each x and $M_\Sigma \sim M_{\Sigma_\Gamma}$ is $\Sigma_{\overline{\text{sco}}\,\Gamma}$.*

Theorem 1.6. *For $X = \{0, 1\}$, there is a Borel gambling house Γ on X such that there is no largest gambling house Γ' such that $V_\Gamma \sim V_{\Gamma'}$.*

A word about notation. Throughout the paper, the operations of forming the strong convex hull and the (variation) norm closure will be performed (vertical) sectionwise on subsets of $X \times \mathbb{P}(X)$ or $X \times \mathbb{P}(H)$. Thus if Γ is a gambling house on X, then $\text{sco}\,\Gamma$ is the gambling house whose x-section is the strong convex hull of $\Gamma(x)$; or if Σ is a subset of $X \times \mathbb{P}(H)$ then $\overline{\text{sco}}\,\Sigma$ is the subset of $X \times \mathbb{P}(H)$ whose x-section is the norm-closure of the strong convex hull of $\Sigma(x)$ and $\overline{\Sigma}$ is the subset of $X \times \mathbb{P}(H)$ whose x-section is the norm closure of $\Sigma(x)$.

For X countable, there are versions of Theorems 1.2 and 1.4 in Maitra and Sudderth ([13], 6.8.16 and 6.8.21). A finitely additive version of Theorem 1.2 is in Armstrong ([1]).

The article is organized as follows. Section 2 contains a summary of the properties of the Mokobodzki capacity and related results. In section 3 we prove that Γ^c is an analytic gambling house. Section 4 is devoted to results in gambling theory. The proofs of the theorems stated in this section are in section 5.

2 THE MOKOBODZKI CAPACITY

As in the Dellacherie-Meyer proof of Theorem 1.2, the ad hoc capacity of Mokobodzki will play a crucial role in our proofs. We will also need the effective analogue of the Mokobodzki capacity as defined by Louveau ([12]). In this section we summarize the properties of the Mokobodzki capacity and prove some consequences of these properties and other related results which will be used in the sequel.

Let X be a compact metric space and let λ be a probability measure on the Borel σ-field of X. For $A \subseteq \mathbb{P}(X)$, define

$$I(A;\lambda) = I(A) = \inf\{\sup_{\eta\in A} \eta(f) + \lambda(1-f) : f \in \Phi\},$$

where Φ is the set of Borel measurable functions on X into $[\,0,\,1]$ and we write $\mu(f)$ for $\int f\,d\mu$ when $f \in \Phi$ and $\mu \in \mathbb{P}(X)$.

Theorem 2.1. *([5], 35) For fixed $\lambda \in \mathbb{P}(X)$,*
(a) $I(\cdot;\lambda) = I$ is a capacity on $\mathbb{P}(X)$.
(b) If A is an analytic, strongly convex subset of $\mathbb{P}(X)$, then

$$I(A) = \sup_{\eta\in A}(\eta \wedge \lambda)(1),$$

where 1 is the function that is identically equal to one on X and $\wedge$ is the minimum operation in the lattice of bounded, signed measures.

Corollary 2.2. *([5], 34). If A is a strongly convex, analytic subset of $\mathbb{P}(X)$, then*

$$\lambda \in norm - cl(A) \iff (\forall f \in \Phi)(\lambda(f) \leq \sup_{\eta\in A} \eta(f)),$$

where norm stands for the total variation norm.

Proof. The 'only if' part is easy. For the 'if' part, the hypothesis is equivalent to the statement that $I(A) \geq 1$. Hence, for each n, there is $\eta_n \in A$ such that $\eta_n \wedge \lambda(1) > 1 - \frac{1}{n}$ by virtue of Theorem 2.1. Now

$$\eta_n \wedge \lambda = \lambda - (\eta_n - \lambda)^-,$$

hence

$$\lambda(1) - (\eta_n - \lambda)^-(1) > 1 - \frac{1}{n}$$

so that, since $\lambda(1) = 1$,

$$(\eta_n - \lambda)^-(1) < \frac{1}{n}.$$

Also

$$\eta_n \wedge \lambda = \eta_n - (\lambda - \eta_n)^-,$$

from which it follows that, since $(\eta_n - \lambda)^+ = (\lambda - \eta_n)^-$,

$$(\eta_n - \lambda)^+(1) < \frac{1}{n}.$$

Hence,

$$\| \eta_n - \lambda \| = (\eta_n - \lambda)^+(1) + (\eta_n - \lambda)^-(1) < \frac{2}{n},$$

so $\| \eta_n - \lambda \| \to 0$ as $n \to \infty$. Consequently, $\lambda \in norm - cl(A)$. $\square$

Lemma 2.3. *If* Γ *is an analytic gambling house on* X, *then* $\operatorname{sco}\Gamma$ *and* $\overline{\operatorname{sco}}\,\Gamma$ *are both analytic.*

Proof. First note that the set

$$E = \{(x,\mu) \in X \times \mathbb{P}(\mathbb{P}(X)) : \mu(\Gamma(x)) = 1\}$$

is analytic ([10], 29.26). Consequently, the set $\operatorname{sco}\Gamma$ is analytic, since it is the projection to the first two coordinates of the analytic set

$$\{(x,\nu,\mu) \in X \times \mathbb{P}(X) \times \mathbb{P}(\mathbb{P}(X)) : (x,\mu) \in E \\ and\, \nu(\cdot) = \int \eta(\cdot)\mu\,(d\eta)\}.$$

The set $\overline{\operatorname{sco}}\,\Gamma$ is analytic, since it is the projection to the first two coordinates of the analytic set

$$\{(x,\nu,(\mu_n)) \in X \times \mathbb{P}(X) \times \mathbb{P}(X)^N : \mu_n \in \operatorname{sco}\Gamma(x),\ n = 1,2,\ldots \\ and \parallel \mu_n - \nu \parallel \to 0\, as\, n \to \infty\}.$$

To see that the above set is analytic, use the fact from ([7]) that the map $\mu \to \parallel \mu \parallel$ is Borel measurable on the space of bounded signed measures (on X) equipped with the weak topology. $\square$

Lemma 2.4. *Let* $\mu_n, \mu \in \mathbb{P}(X)$ *be such that* $\mu_n \to \mu$ *in norm. Then, for any bounded, upper analytic function* g *on* X, $\int g\,d\mu_n \to \int g\,d\mu$.

Proof. Choose $\nu \in \mathbb{P}(X)$ such that μ_n, μ are absolutely continuous with respect to ν. Then there is a bounded, Borel measurable function f on X such that $f = g\,a.s(\nu)$. Hence

$$\int g\,d\mu_n = \int f\,d\mu_n$$

and

$$\int g\,d\mu = \int f\,d\mu.$$

The conclusion now follows from the fact that $\int f\,d\mu_n \to \int f\,d\mu$, since $\mu_n \to \mu$ in norm. $\square$

We now turn to the effective analogue of the capacity I. Effective descriptive set theory takes place in recursively presented Polish spaces (see [18] for details). We will take X to be the recursively presented compact metric space 2^ω and λ to be a Δ^1_1 probability measure. For $A \subseteq \mathbb{P}(X)$, let

$$J(A;\lambda) = J(A) = inf\{\sup_{\eta\in A} \eta(f) + \lambda(1-f) : f \text{ is } \Delta^1_1 - \text{ recursive on } X \text{ into } [0,1]\}.$$

Theorem 2.5. *([12], 2.4, 2.12(a), 2.14(a)) If* A *is a* Σ^1_1 *subset of* $\mathbb{P}(X)$, *then*

$$I(A) = \inf\{J(C) : C \textit{ is } \Delta^1_1 - \textit{ recursive and } A \subseteq C\}.$$

An immediate consequence is

Corollary 2.6. *If A is a Σ_1^1 subset of $\mathbb{P}(X)$, $\lambda \in \mathbb{P}(X)$ is a Δ_1^1 measure and $I(A) < 1$, then there is a Δ_1^1-recursive function f on X into [0, 1] such that*

$$\sup_{\eta \in A} \eta(f) + \lambda(1-f) < 1.$$

For the next result, we need a coding of $\Delta_1^1(\alpha)$-recursive functions on $X = 2^\omega$ into [0, 1], that is, a set W and a function U with the following properties:
(i) W is a Π_1^1 subset of ω^ω,
(ii) U is a Π_1^1-recursive partial function on $\omega^\omega \times \omega \times X$ into [0, 1],
(iii) if $(\alpha, n) \in W$ and $(\forall x)(U(\alpha, n, x)$ is defined), then $U(\alpha, n, .)$ is a $\Delta_1^1(\alpha)$-recursive function on X into [0, 1] , and
(iv) if g is a $\Delta_1^1(\alpha)$-recursive function on X into [0, 1], then there is n such that $(\alpha, n) \in W$ and $(\forall x)(g(x) = U(\alpha, n, x))$.

Such a coding is easy to construct from the coding of $\Delta_1^1(\alpha)$ subsets of $X\times$ [0, 1] (see ([11], p.13)). For the next result, regard 2^ω as a Π_1^0 subset of ω^ω.

Theorem 2.7. *Let Γ be a Σ_1^1 gambling house on $X = 2^\omega$. Suppose that $x \to \mu_x$ is a Δ_1^1-recursive function on a Δ_1^1 set $E \subseteq X$ into $\mathbb{P}(X)$. Assume that $I(\Gamma(x); \mu_x) = I_x(\Gamma(x)) < 1$ for all $x \in E$. Then there is a Δ_1^1-recursive function $f : E \times X \to [0,1]$ such that*

$$\sup_{\eta \in \Gamma(x)} \int f(x,y)\,\eta(dy) + \int (1 - f(x,y))\,\mu_x(dy) < 1$$

for each $x \in E$.

Proof. Let

$$P(x,n) \leftrightarrow x \in E \,\&(x,n) \in W \,\&(\forall y)(U(x,n,y) \text{ is defined})$$
$$\& \left(\sup_{\eta \in \Gamma(x)} \int U(x,n,\,\cdot\,)\,d\eta + \int (1 - U(x,n,\,\cdot\,))\,d\mu_x < 1 \right).$$

It is easy to check that P is Π_1^1. It follows by relativizing Corollary 2.6 that $(\forall x \in E)(\exists n)\, P(x,n)$. So, by Kreisel's selection theorem ([18], 4B.5), there is a Δ_1^1-recursive function $\varphi : E \to \omega$ such that $(\forall x \in E)\, P(x, \varphi(x))$. Set

$$f(x,y) = U(x, \varphi(x), y), \qquad x \in E,\, y \in X.$$

It is now easily verified that f satisfies the assertions of the theorem. □

The bold-face version of this theorem is obtained by replacing Σ_1^1 by analytic and Δ_1^1-recursive by Borel measurable.

3 THE GAMBLING HOUSE Γ^c

The present section contains the proof that Γ^c is analytic whenever Γ is an analytic gambling house on X. We start with a technical result.

Lemma 3.1. *([14], Lemma 2.2) Suppose that X and Y are Borel subsets of Polish spaces. Then there is a Borel measurable mapping $(x, \mu) \to \mu[x]$ from $X \times \mathbb{P}(X \times Y)$ to $\mathbb{P}(Y)$ such that $\mu[x]$ is a version of the μ-regular conditional distribution on Y given x.*

For $\mu \in \mathbb{P}(X \times X)$ or $\mu \in \mathbb{P}(H)$, $\mu\pi_i^{-1}$ will denote the μ-distribution of the i-th coordinate, $i \geq 1$; $\mu[x]$ is a version of the μ-regular conditional distribution of the remaining coordinates given that the first coordinate is x such that $\mu[x]$ is jointly Borel measurable in μ and x, as is guaranteed by Lemma 3.1.

Suppose now that Γ is an analytic gambling house on X. If $\Delta \subseteq X \times \mathbb{P}(X)$, denote by Δ^* the subset of $X \times \mathbb{P}(X)$ whose x-section $\Delta^*(x)$ is $\Delta(x) \bigcup \{\delta(x)\}$. Define next an operator χ that takes subsets of $X \times \mathbb{P}(X)$ to subsets of $X \times \mathbb{P}(X)$ as follows:for $\Delta \subseteq X \times \mathbb{P}(X)$and $x \in X$, the x-section of $\chi(\Delta)$,namely, $\chi(\Delta)(x)$, is defined as the set of all $\gamma \in \mathbb{P}(X)$ such that $(\exists \mu \in \mathbb{P}(X \times X))$ satisfying these three conditions

(i) $\mu\pi_1^{-1} \in \Gamma(x)$,

(ii) $\mu\pi_2^{-1} = \gamma$, and

(iii) $(\mu\pi_1^{-1})^*(\{ x' \in X : \mu[x'] \in \Delta^*(x') \}) = 1$.

Here $(\mu\pi_1^{-1})^*$ is the outer measure induced by $\mu\pi_1^{-1}$.

We also define an operator ψ that takes subsets of $X \times \mathbb{P}(X)$ to subsets of $X \times \mathbb{P}(X) \times \mathbb{P}(X \times X)$ by letting the x-section of $\psi(\Delta)$, namely, $\psi(\Delta)(x)$, be the set of all pairs (γ, μ) in $\mathbb{P}(X) \times \mathbb{P}(X \times X)$ satisfying conditions (i), (ii), and (iii) above.

Lemma 3.2. *If Δ is an analytic subset of $X \times \mathbb{P}(X)$, then $\chi(\Delta)$ is an analytic gambling house on X.*

Proof. First observe that $\psi(\Delta)$ is the intersection of three sets, the first of which is clearly analytic and the second Borel. The third is analytic by virtue of the fact that Δ^* is analytic and ([10], 29.26). So $\psi(\Delta)$ is analytic. Since $\chi(\Delta)$ is the projection to the first two coordinates of $\psi(\Delta)$, it follows that $\chi(\Delta)$ is analytic. To see that $\chi(\Delta)$ is a gambling house, note that for each $x \in X$, $\chi(\Delta)(x) \supseteq \chi(\emptyset) = \Gamma(x)$ and so $\chi(\Delta)(x)$ is nonempty. This completes the proof. □

Define by induction on n subsets Γ_n of $X \times \mathbb{P}(X)$ as follows:

$$\Gamma_0 = \emptyset, \qquad \text{and} \qquad \Gamma_{n+1} = \chi(\Gamma_n), \qquad n \geq 0.$$

It is easy to see that $\Gamma_n \subseteq \Gamma_{n+1}$ since χ is monotone. Also, by Lemma 3.2, the gambling houses Γ_n are analytic. Finally, set

$$\Gamma_\infty = \bigcup_{n \geq 0} \Gamma_n.$$

Then Γ_∞ is an analytic house on X.

Here is the main result of this section.

Theorem 3.3. *If Γ is an analytic gambling house on X, then*

$$\Gamma^c(x) = \Gamma_\infty(x) \cup \{\delta(x)\}$$

for each $x \in X$. Consequently, Γ^c is analytic.

Proof. For a policy $\pi = (\sigma, t)$ available in Γ at x, denote by $\bar{\gamma}(\pi)$ the distribution of the terminal state h_t under σ.

To start with, recall that if $\pi = (\sigma, t)$ is available in Γ at x and $t \equiv 0$ then $\bar{\gamma}(\pi) = \delta(x)$. So to prove the inclusion $\subseteq$, it will suffice to show that if $\pi = (\sigma, t)$ is available in Γ at x and $1 \le t \le n$, then $\bar{\gamma}(\pi) \in \Gamma_n(x)$. The proof is by induction on n. For $n = 1$ the assertion is clear. So suppose the assertion is true for $n = m$. Let $\pi = (\sigma, t)$ be available in Γ at x_0 and suppose that $1 \le t \le m+1$. It is easy to verify that $x \to \bar{\gamma}(\sigma[x], t[x])$ is σ_0 - measurable. Furthermore, $\bar{\gamma}(\sigma[x], t[x]) = \delta(x)$ if $t[x] \equiv 0$ while if $t[x] \not\equiv 0$, then $1 \le t[x] \le m$, so that $\bar{\gamma}(\sigma[x], t[x]) \in \Gamma_m(x)$ by virtue of the inductive hypothesis. It follows that $\bar{\gamma}(\pi) \in \Gamma_{m+1}(x_0)$, since $\sigma_0 \in \Gamma(x_0)$ and

$$\bar{\gamma}(\pi)(B) = \int \bar{\gamma}(\sigma[x], t[x])(B)\, \sigma_0(dx)$$

for every Borel subset B of X.

For the reverse inclusion $\supseteq$, we will prove, again by induction, that there is a universally measurable function $\phi_n : \Gamma_n \to \mathbb{P}(X)$ such that $\phi_n(x,\gamma) \in \Sigma_\Gamma(x)$ for all $(x,\gamma) \in \Gamma_n$, and a universally measurable function $t_n : \Gamma_n \times H \to N$ with $t_n(x,\gamma,\cdot)$ a stop rule on H and $t_n \le n$ for all $(x,\gamma) \in \Gamma_n$ such that whenever $(x,\gamma) \in \Gamma_n$, $\gamma = \bar{\gamma}(\phi_n(x,\gamma), t_n(x,\gamma,\cdot))$.

For $n \ge 1$, fix a universally measurable function f_n from $\chi(\Gamma_n)$ to $\mathbb{P}(X \times X)$ such that $(x, \gamma, f_n(x,\gamma)) \in \psi(\Gamma_n)$ for every $(x,\gamma) \in \chi(\Gamma_n)$. The existence of f_n is guaranteed by the von Neumann selection theorem ([10], 29.9). Use the theorem one more time to fix a universally measurable selector f^* for Γ. For each $x \in X$, let $\sigma^*(x)$ be the strategy that uses as initial gamble $f^*(x)$ and thereafter uses $f^*(y)$ when the current state is y. Note that $\sigma^*(x) \in \Sigma_\Gamma(x)$.

To start the induction, set

$$(\phi_1(x,\gamma))_0 = \gamma$$

and the conditional strategy

$$(\phi_1(x,\gamma))[x_1] = \sigma^*(x_1)$$

for $(x,\gamma) \in \Gamma_1 = \Gamma$. (Note that a strategy σ is completely determined by the specification of σ_0 and the collection of conditional strategies $\sigma[x], x \in X$.) Let

$$t_1(x,\gamma,h) = 1 \qquad \text{for } (x,\gamma) \in \Gamma_1 \text{ and } h \in H.$$

Suppose now that ϕ_n, t_n have been defined. Set $\phi_{n+1} = \phi_n$ on Γ_n and $t_{n+1} = t_n$ on $\Gamma_n \times H$. We will next define $\phi_{n+1}(x,\gamma)$ and $t_{n+1}(x,\gamma,\cdot)$ for $(x,\gamma) \in \Gamma_{n+1} - \Gamma_n$. First let

$$(\phi_{n+1}(x,\gamma))_0 = f_n(x,\gamma)\pi_1^{-1}$$

and define the conditional strategies by

$$(\phi_{n+1}(x,\gamma))[x_1] = \begin{cases} \phi_n(x_1, (f_n(x,\gamma))[x_1]), & \text{if } (f_n(x,\gamma))[x_1] \in \Gamma_n(x_1), \\ \sigma^*(x_1), \text{ otherwise.} \end{cases}$$

Note that $\phi_{n+1}(x,\gamma) \in \Sigma_\Gamma(x)$ for all $(x,\gamma) \in \Gamma_{n+1}$. Define $t_{n+1}(x,\gamma,\cdot)$ to be the stop rule whose conditional stop rules are given by

$$(t_{n+1}(x,\gamma,\cdot))[x_1] = \begin{cases} t_n(x_1, (f_n(x,\gamma))[x_1]), \cdot), & \text{if } (f_n(x,\gamma))[x_1] \in \Gamma_n(x_1), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that ϕ_{n+1} and t_{n+1} are universally measurable and that $t_{n+1}(x,\gamma,\cdot)$ is a stop rule with $t_{n+1} \le n+1$ for every $(x,\gamma) \in \Gamma_{n+1}$.

Finally, let $\gamma \in \Gamma_{n+1}(x)$. If $\gamma \in \Gamma_n(x)$, then

$$\bar{\gamma}(\phi_{n+1}(x,\gamma), t_{n+1}(x,\gamma,\cdot)) = \bar{\gamma}(\phi_n(x,\gamma), t_n(x,\gamma,\cdot)) = \gamma,$$

by the inductive hypothesis. So suppose that $\gamma \in \Gamma_{n+1}(x) - \Gamma_n(x)$. Let B be a Borel subset of X and define $A = \{x_1 : (f_n(x,\gamma))[x_1] \in \Gamma_n(x_1)\}$. Then

$$\begin{aligned}
&\bar{\gamma}(\phi_{n+1}(x,\gamma), t_{n+1}(x,\gamma,\cdot))(B)\\
&\quad = \int \bar{\gamma}((\phi_{n+1}(x,\gamma))[x_1], t_{n+1}(x,\gamma,\cdot))[x_1])(B)\,(\phi_{n+1}(x,\gamma))_0(dx_1)\\
&\quad = \int \bar{\gamma}((\phi_{n+1}(x,\gamma))[x_1], t_{n+1}(x,\gamma,\cdot))[x_1])(B)\,(f_n(x,\gamma)\pi_1^{-1})(dx_1)\\
&\quad = \int_A \bar{\gamma}((\phi_{n+1}(x,\gamma))[x_1], t_{n+1}(x,\gamma,\cdot))[x_1])(B)\,(f_n(x,\gamma)\pi_1^{-1})(dx_1)\\
&\qquad + \int_{A^c} \bar{\gamma}(\sigma^*(x_1), t^*)(B)\,(f_n(x,\gamma)\pi_1^{-1})(dx_1)\\
&\quad = \int_A ((f_n(x,\gamma))[x_1])(B)\,(f_n(x,\gamma)\pi_1^{-1})(dx_1)\\
&\qquad + \int_{A^c} \delta(x_1)(B)\,(f_n(x,\gamma)\pi_1^{-1})(dx_1)\\
&\quad = \int ((f_n(x,\gamma))[x_1])(B)(f_n(x,\gamma)\pi_1^{-1})(dx_1)\\
&\quad = (f_n(x,\gamma)\pi_2^{-1})(B)\\
&\quad = \gamma(B),
\end{aligned}$$

where t^* is the identically zero stop rule and the fourth equality is by virtue of the fact that $t^* \equiv 0$. Consequently, $\bar{\gamma}(\phi_{n+1}(x,\gamma), t_{n+1}(x,\gamma,\cdot)) = \gamma$. This completes the proof. □

4 Gambling

We turn to gambling theory in this section. Let Γ be a gambling house on X. We associate with Γ an operator G_Γ as follows: for any bounded, upper analytic function f on X

$$(G_\Gamma f)(x) = \sup_{\gamma \in \Gamma(x)} \int f\,d\gamma, \qquad x \in X.$$

It is easy to verify that if Γ is analytic and f is upper analytic, then $G_\Gamma f$ is upper analytic.

Lemma 4.1. *If Γ is an analytic gambling house on X, then $R_\Gamma \approx G_{\Gamma^c}$. Consequently, $R_\Gamma f$ is upper analytic if f is a bounded, upper analytic function on X.(Recall that $R_\Gamma \approx G_{\Gamma^c}$ means that R_Γ and G_{Γ^c} agree on bounded, upper analytic functions on X.)*

Proof. Let f be a bounded, upper analytic function on X. Then, by the change of variable theorem, for any $x \in X$,

$$(G_{\Gamma^c} f)(x) = \sup\left\{\int f(h_t)\, d\sigma : \sigma \in \Sigma(x)\ \&\ t \text{ a bounded stop rule}\right\} \leq (R_\Gamma f)(x).$$

For the reverse inequality, let $\pi = (\sigma, t)$ be a policy available in Γ at x. Then, by the Dominated Convergence Theorem,

$$\lim_n \int f(h_{t\wedge n})\, d\sigma = \int f(h_t)\, d\sigma$$

where $(t \wedge n)(h) = \min\{t(h), n\}$, $h \in H$. It follows that

$$\int f(h_t)\, d\sigma \leq (G_{\Gamma^c} f)(x).$$

Now take the sup over all policies π available at x to get

$$(R_\Gamma f)(x) \leq (G_{\Gamma^c} f)(x).$$

The second assertion is now a consequence of Theorem 3.3. $\square$

Lemma 4.2. *Let Γ be an analytic gambling house on X. If g is a bounded Γ-excessive function on X, then g is Γ^s-excessive.*

Proof. We first prove that g is Γ^c-excesssive. Fix $x_0 \in X$ and $\gamma \in \Gamma^c(x_0)$. Choose a policy $\pi = (\sigma, t)$ available in Γ at x with t bounded such that $\gamma = \sigma\psi_t^{-1}$. Since g is Γ-excessive, it follows that, under σ, $g(x_0)$, $g(h_1)$, $g(h_2)$, $\ldots$ is a supermartingale. So, by the optional sampling theorem ([6], 2.12.2),

$$\int g(h_t)\, d\sigma \leq g(x_0)$$

Hence, by the change of variable theorem,

$$\int g\, d\gamma \leq g(x_0)$$

This shows that g is Γ^c-excessive. It is now easy to verify first that g is $sco\,\Gamma^c$-excessive and then that g is $\overline{sco}\,\Gamma^c$-excessive by using Lemma 2.4. So, by (1.2), g is Γ^s-excessive. $\square$

Lemma 4.3. *If Γ is an analytic gambling house on X, then $R_\Gamma \approx R_{\Gamma^s}$.*

Proof. Let f be a bounded, upper analytic function on X. Note that Γ^s is analytic by Theorem 3.3 and Lemma 2.3. Moreover, $\Gamma \subseteq \Gamma^c \subseteq \Gamma^s$, so $R_\Gamma f \leq R_{\Gamma^s} f$. For the reverse inequality, recall that $R_\Gamma f$ is Γ-excessive by virtue of the Fundamental Theorem applied to Γ. Hence, by Lemma 4.2, $R_\Gamma f$ is Γ^s-excessive. Also $R_\Gamma f \geq f$. So, by the Fundamental Theorem applied to Γ^s, $R_\Gamma f \geq R_{\Gamma^s} f$. $\square$

The next theorem states that the operator M_Γ is determined by G_Γ for analytic gambling houses. The proof is based on a number of results proved elsewhere. We will now summarize these results.

Lemma 4.4. *If* Γ *and* Γ' *are analytic gambling houses on* X *and* $G_\Gamma \approx G_{\Gamma'}$, *then* $R_\Gamma \approx R_{\Gamma'}$

Proof. Since $G_\Gamma \approx G_{\Gamma'}$, it follows that the class of Γ-excessive functions is exactly the same as the class of Γ'-excessive functions. Now use the Fundamental Theorem to see that $R_\Gamma \approx R_{\Gamma'}$. □

With each analytic gambling house Γ on X, we associate the operator T_Γ as follows:

$$(T_\Gamma u)(x) = \sup_\pi \int u(h_t)\, d\sigma, \qquad x \in X,$$

where u is a bounded, upper analytic function on X and the sup is taken over all policies $\pi = (\sigma, t)$ available in Γ at x such that $t \geq 1$.

The next lemma shows that T_Γ is closely related to R_Γ.

Lemma 4.5. *If* Γ *is an analytic gambling house on* X, *then* $T_\Gamma \approx G_\Gamma \circ R_\Gamma$.

Proof. Let g be a bounded, upper analytic function on X. Fix $x_0 \in X$. Let $\pi = (\sigma, t)$ be a policy available in Γ at x_0 with $t \geq 1$. Then

$$\int g(h_t)\, d\sigma = \int [\int g(h'_{t[x]})\sigma[x](dh')]\sigma_0(dx) \quad \leq \int (R_\Gamma g)(x)\, \sigma_0(dx) \leq (G_\Gamma(R_\Gamma g))(x_0),$$

where the last inequality is by virtue of the fact that $\sigma_0 \in \Gamma(x_0)$. Now take the sup of the left side over all $\pi = (\sigma, t)$ available in Γ at x_0 with $t \geq 1$ to get

$$(T_\Gamma g)(x_0) \leq (G_\Gamma(R_\Gamma g))(x_0).$$

For the reverse inequality, fix $\epsilon > 0$. Choose $\gamma_0 \in \Gamma(x_0)$ such that

$$\int (R_\Gamma g)\, d\gamma_0 \geq (G_\Gamma(R_\Gamma g))(x_0) - \epsilon/2.$$

Next use a selection theorem (see, for example, [15], Lemma 2.1) to choose $\gamma_x \in \Gamma^c(x)$ such that $x \to \gamma_x$ is universally measurable and

$$\int g\, d\gamma_x > (R_\Gamma g)(x) - \epsilon/2$$

for each $x \in X$.

Now define a policy $\pi = (\sigma, t)$ available in Γ atx_0 as follows:

$$\sigma_0 = \gamma_0$$

and

$$\sigma[x] = \begin{cases} \phi_n(x, \gamma_x), & \text{if } (x, \gamma_x) \in \Gamma_n - \Gamma_{n-1}, n \geq 1, \\ \sigma^*(x), & \text{otherwise,} \end{cases}$$

where $\phi_n, \sigma^*(x)$ are defined in the proof of Theorem 3.3 and Γ_n is defined just before the statement of Theorem 3.3; t is defined so that

$$t[x] = \begin{cases} t_n(x, \gamma_x, \cdot), & \text{if } (x, \gamma_x) \in \Gamma_n - \Gamma_{n-1}, n \geq 1 \\ 0, & \text{otherwise,} \end{cases}$$

where t_n is defined in the proof of Theorem 3.3. Note that $t \geq 1$. Consequently,

$$\begin{aligned}(T_\Gamma g)(x_0) &\geq \int g(h_t)\, d\sigma \\ &= \int [\int g(h'_{t[x]})\, \sigma[x](dh')]\, \sigma_0(dx) \\ &= \int [\int g(y)\, \gamma_x(dy)]\, \gamma_0(dx) \\ &\geq \int (R_\Gamma g)(x)\, \gamma_0(dx) - \epsilon/2 \\ &\geq (G_\Gamma(R_\Gamma))(x_0) - \epsilon,\end{aligned}$$

where the second equality holds because $\bar{\gamma}(\phi_n(x,\gamma_x), t_n(x,\gamma_x,\cdot)) = \gamma_x$ if $\gamma_x \in \Gamma_n - \Gamma_{n-1}$ for some $n \geq 1$ and $t[x] = 0$ otherwise. Since ϵ is arbitrary, we have: $(T_\Gamma g)(x_0) \geq (G_\Gamma(R_\Gamma g))(x_0)$. This completes the proof.

□

The next result is a characterization of $V_\Gamma u$ for bounded, upper analytic u.

Lemma 4.6. *([8], Theorem 7.1) If Γ is an analytic gambling house on X and u is a bounded, upper analytic function on X, then $V_\Gamma u$ is the largest bounded, upper analytic function v on X such that $T_\Gamma(u \wedge v) = v$, where $u \wedge v$ is the pointwise minimum of u and v.*

An immediate consequence of Lemmas 4.4-4.6 is

Lemma 4.7. *If Γ and Γ' are analytic gambling houses on X such that $G_\Gamma \approx G'_\Gamma$, then $V_\Gamma \approx V_{\Gamma'}$.*

We now define a class $\mathcal{F}$ of relatively simple functions on H, which will be used to approximate bounded, upper analytic functions. Let $\mathcal{F}$ be the set of all $f : H \to [0,1]$ such that f takes on finitely many values and $[f \geq c]$ is a countable intersection of Borel, open sets for each $c \in [0,1]$, where "open" refers to the product topology on H when X is given the discrete topology.

Let $X^* = \bigcup_{n \geq 0} X^n$ and $H^* = (X^*)^N$. For $h \in H$, we will write $p_n(h)$ for $(h_1, h_2, \cdots, h_n)$. Let $\phi : H \to H^*$ be defined by setting

$$\phi(h) = (p_1(h), p_2(h), \cdots).$$

Set $H' = \phi(H)$. The next lemma gives a representation for elements of $\mathcal{F}$.

Lemma 4.8. *If $f \in \mathcal{F}$, then there is a Borel measurable function $u : X^* \to [0,1]$ such that*

$$f(h) = \limsup_n u(p_n(h))$$

for every $h \in H$; that is, $f = u^* \circ \phi$.

Proof. Suppose that f assumes the values $a_1, a_2, \cdots, a_m$ with $a_1 < a_2 < \cdots < a_m$. By Lemma 6.6 in ([15]), we can choose, for each $i = 1, 2, \cdots, m$, a Borel subset S_i of X^* such that

$$[f \geq a_i] = \{h \in H : p_n(h) \in S_i \text{ for infinitely many } n\}, \tag{4.1}$$

and $X^* = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_m$.

Write $[S_i\, i.o.]$ for the set on the right side of (4.1) and e for the empty sequence in X^*. Define u on X^* as follows:

$$u(e) = 0$$

and, for $p \in X^* - \{e\}$,

$$u(p) = a_i \qquad \text{if } p \in S_i - S_{i+1}\,, i = 1, 2, \cdots, m,$$

where $S_{m+1} = \emptyset$.

To complete the proof, suppose that $f(h) = a_i$. Then $h \in [S_i\, i.o.]$ and $h \notin [S_{i+1}\, i.o.]$. It follows that $h \in [S_i - S_{i+1}\, i.o.]$, so that $u^*(\phi(h)) \geq a_i$. But $u^*(\phi(h)) \leq a_i$ because $h \notin [S_{i+1}\, i.o.]$. Hence $u^*(\phi(h)) = a_i$. □

The next result implies that the operator M_Γ is determined by its values on the class $\mathcal{F}$.

Lemma 4.9. *([16], Theorem 10.1) If Γ is an analytic house on X and $g : H \to [0, 1]$ is upper analytic, then*

$$(Mg)(x) = \inf\{(Mf)(x) \,:\, f \in \mathcal{F} \text{ and } f \geq g\}$$

for every $x \in X$.

Theorem 4.10. *If Γ and Γ' are analytic gambling houses on X such that $G_\Gamma \approx G_{\Gamma'}$, then $M_\Gamma \approx M_{\Gamma'}$.*

Proof. By virtue of Lemma 4.9, it suffices to prove that $M_\Gamma = M_{\Gamma'}$ on the class $\mathcal{F}$. So fix $x_0 \in X$ and $f \in \mathcal{F}$. By Lemma 4.8, choose a Borel measurable function $u : X^* \to [0, 1]$ such that $f = u^* \circ \varphi$.

Now consider the nonleavable gambling problems (X^*, Γ^*, u^*) and (X^*, Γ'^*, u^*), where

$$\Gamma^*(p) = \{\gamma\varphi_p^{-1} : \gamma \in \Gamma(l(p))\},\ p \in X^*,$$

$$\varphi_p(x) = px,\ x \in X,\ p \in X^*,$$

and $l(p)$ is the last coordinate of p if $p \neq e$, while $l(e) = x_0$.
Similarly, define Γ'^* from Γ'. Observe that, if σ is a strategy available in Γ^* or Γ'^* at e, then $\sigma(H') = 1$. It is then easy to verify that

$$(M_\Gamma f)(x_0) = (V_{\Gamma^*} u)(e)$$

and

$$(M_{\Gamma'} f)(x_0) = (V_{\Gamma'^*} u)(e).$$

Consequently, the proof will be complete as soon as we establish that $V_{\Gamma^*} \approx V_{\Gamma'^*}$. This in turn will be proved, courtesy of Lemma 4.7, if we can show that $G_{\Gamma^*} \approx G_{\Gamma'^*}$. So let g be a bounded, upper analytic function on X^*. Then, for any $p \in X^*$,

$$(G_{\Gamma^*} g)(p) = (G_\Gamma g_p)(l(p))$$

and

$$(G_{\Gamma'^*} g)(p) = (G_{\Gamma'} g_p)(l(p))$$

where $g_p(x) = g(px)$, $x \in X$. It follows that $G_{\Gamma^*} g = G_{\Gamma'^*} g$. This completes the proof. □

The converse of Theorem 4.10 holds in a very strong sense.

Lemma 4.11. *Suppose that Γ, Γ' are gambling houses on X such that $\Sigma_{\Gamma(x)}$ and $\Sigma_{\Gamma'}(x)$ are nonempty for each $x \in X$ (so M_Γ and $M_{\Gamma'}$ are defined on bounded, universally measurable functions on H). If $M_\Gamma \sim M_{\Gamma'}$, then $G_\Gamma \sim G_{\Gamma'}$.*

Proof. Suppose that g is a bounded, Borel measurable function on X. Define $\tilde{g}$ on H by

$$\tilde{g}(h) = g(h_1).$$

Then, for any $x \in X$,

$$(G_\Gamma g)(x) = (M_\Gamma \tilde{g})(x) = (M_{\Gamma'} \tilde{g})(x) = (G_{\Gamma'} g)(x).$$

□

For the next theorem, recall that Φ is the set of Borel measurable functions from X into $[0, 1]$.

Theorem 4.12. *Suppose that Γ is an analytic gambling house on X. If f is an upper analytic function on X into [0, 1], then*

$$(G_\Gamma f)(x) = \inf\{(G_\Gamma g)(x) : g \in \Phi \text{ and} g \geq f\}$$

for each $x \in X$.

Proof. Fix $x_0 \in X$ and $\epsilon > 0$. Let

$$E = \{(x, a) \in X \times [0, 1] : f(x) \geq a\}.$$

Then E is analytic. By Corollary 4.4 in ([15]), there is a Borel subset B of $X\times$ [0, 1] such that $B \supseteq E$ and

$$\sup_{\gamma \in \Gamma(x_0)} (\gamma \times \lambda)(B) \leq \sup_{\gamma \in \Gamma(x_0)} (\gamma \times \lambda)(E) + \epsilon, \tag{4.2}$$

where λ is Lebesgue measure on [0, 1]. Define g on X into [0, 1] by

$$g(x) = \lambda(B_x).$$

Then g is Borel measurable and $g \geq f$. It follows from (4.2) that

$$(Gg)(x_0) \leq (Gf)(x_0) + \epsilon.$$

This completes the proof. □

An immediate consequence of the previous result is:

Corollary 4.13. *Suppose that Γ, Γ' are analytic gambling houses on X such that $G_\Gamma \sim G_{\Gamma'}$. Then $G_\Gamma \approx G_{\Gamma'}$.*

We conclude this section with a result on the randomization of strategies. First we introduce some notation.

For $\mu \in \mathbb{P}(H)$ and $n \geq 0$, $(\mu)^n$ will denote the μ-probability distribution of the first $(n + 1)$ coordinates and $\mu(x_1, x_2, \dots, x_n)$ will denote a version of the μ-conditional distribution of x_{n+1} given $x_1, x_2, \dots, x_n$ which is jointly Borel measurable in $\mu, x_1, x_2, \dots, x_n$, as guaranteed by Lemma 3.1.

Theorem 4.14. *Let Γ be an analytic gambling house on X. Then*

$$\operatorname{sco}\Sigma_\Gamma \subseteq \Sigma_{\operatorname{sco}\Gamma}.$$

Proof. Fix $x_0 \in X$ and $\nu \in \operatorname{sco}\Sigma_\Gamma(x_0)$. So there is a probability measure m on the Borel subsets of $\Sigma_\Gamma(x_0)$ such that

$$\nu(E) = \int_{\Sigma_\Gamma(x_0)} \mu(E)\, m(d\mu)$$

for Borel subsets E of H. We have to define a strategy σ available in $\operatorname{sco}\Gamma$ at x_0 such that $\nu = \sigma$.

Let

$$\sigma_0(B) = \int_{\Sigma_\Gamma(x_0)} (\mu)^0(B)\, m(d\mu)$$

for Borel subsets B of X. Plainly, $\sigma_0 \in \operatorname{sco}\Gamma(x_0)$ and $\sigma_0 = (\nu)^0$.

Suppose that σ_0, σ_1, $\dots$, σ_{n-1} have been defined so that $\sigma_i(x_1, x_2, \dots, x_i) \in \operatorname{sco}\Gamma(x_i)$ for all x_1, x_2, $\dots$, $x_i \in X$ and $(\sigma)^i =$ the σ-distribution of the first $(i+1)$ coordinates $= (\nu)^i$, $i = 0, 1, 2, \dots, n-1$. We will now define σ_n. By von Neumann's selection theorem ([10], 29.9), fix an analytically measurable selector ψ for Γ. Let

$$\varphi(\mu, x_1, x_2, \dots, x_n) = \begin{cases} \mu(x_1, x_2, \dots, x_n) & \text{if } \mu \in \Sigma_\Gamma(x_0)\,\&\,\mu(x_1, x_2, \dots, x_n) \in \Gamma(x_n) \\ \psi(x_n) & \text{if } \mu \in \Sigma_\Gamma(x_0)\,\&\,\mu(x_1, x_2, \dots, x_n) \notin \Gamma(x_n). \end{cases}$$

Then φ is universally measurable. Next define a probability measure P on the Borel subsets of $\Sigma(x_0) \times X^n$ such that

$$P(S \times B) = \int_S (\mu)^{n-1}(B)\, m(d\mu),$$

for Borel subsets S of $\Sigma(x_0)$ and B of X^n. Fix a version $P(\cdot \mid x_1, x_2, \dots, x_n)$ of the P-regular conditional probability on the Borel subsets of $\Sigma(x_0)$ given $x_1, x_2, \dots, x_n$. Finally set

$$\sigma_n(x_1, x_2, \dots, x_n)(B) = \int_{\Sigma(x_0)} \varphi(\mu, x_1, x_2, \dots, x_n)(B)\, P(d\mu \mid x_1, x_2, \dots, x_n)$$

for Borel subsets B of X and $x_1, x_2, \dots, x_n \in X$. Then σ_n is universally measurable and $\sigma_n(x_1, x_2, \dots, x_n) \in sco\Gamma(x_n)$.

Now let A be a Borel subset of X^n and B be a Borel subset of X. Abbreviate $(x_1, x_2, \dots, x_n)$ by $\vec{x}$ in the following calculation:

$$\begin{aligned}
(\nu)^n(A \times B) &= \int_{\Sigma(x_0)} (\mu)^n(A \times B)\, m(d\mu) \\
&= \int_{\Sigma(x_0)} \left[\int_A \varphi(\mu, \vec{x})(B)\, (\mu)^{n-1}(d\vec{x})\right] m(d\mu) \\
&= \int_{\Sigma(x_0) \times A} \varphi(\mu, \vec{x})(B)\, dP \\
&= \int_A \left[\int_{\Sigma(x_0)} \varphi(\mu, \vec{x})(B)\, P(d\mu \mid \vec{x})\right] (\nu)^{n-1}(d\vec{x}) \\
&= \int_A \sigma_n(\vec{x})(B)\, (\sigma)^{n-1}(d\vec{x}) \\
&= (\sigma)^n(A \times B),
\end{aligned}$$

where the fourth equality is by virtue of the fact that the marginal of P on X^n is $(\nu)^{n-1}$ and the penultimate equality is by the inductive hypothesis. It follows that $(\nu)^n = (\sigma)^n$ and the induction is complete. So $\nu = \sigma$, hence $\nu \in \Sigma_{\text{sco}\,\Gamma}(x_0)$. $\square$

The reverse inclusion $\Sigma_{sco\Gamma} \subseteq sco\Sigma_\Gamma$ is also true. See Aumann ([2]) and especially, Feinberg ([9], Theorem 5.2), who proved the inclusion in the context of dynamic programming. But we will have no use for the result in this article.

5 Proofs of Theorems in Section 1

Lemma 5.1. *Suppose that Γ is an analytic gambling house on X. Then the largest gambling house Γ' such that $G_\Gamma \sim G_{\Gamma'}$ is $\overline{sco}\Gamma$.*

Proof. It is easy to verify that $G_\Gamma f = G_{sco\Gamma} f$ for $f \in \Phi$. Now $sco\Gamma$ is analytic by Lemma 2.3 and, obviously, $sco\Gamma(x)$ is strongly convex for each $x \in X$. So it follows from the 'only if' part of Corollary 2.2 that $G_{sco\Gamma} f = G_{\overline{\text{sco}}\,\Gamma} f$ for $f \in \Phi$. Consequently, $G_\Gamma \sim G_{\overline{\text{sco}}\,\Gamma}$. Suppose next that Γ' is a gambling house on X such that $G_{\Gamma'} \sim G_\Gamma$. Then $G_{\Gamma'} = G_{\text{sco}\,\Gamma}$ on Φ. It nows follows from the 'if' part of Corollary 2.2 that $\Gamma' \subseteq \overline{sco}\Gamma$. $\square$

Proof of Theorem 1.2. The final assertion of Theorem 1.2 follows from Theorem 3.3 and Lemma 2.3. By Lemma 4.3, $R_\Gamma \sim R_{\Gamma^s}$. Suppose then that Γ' is a gambling house on X such that $R_{\Gamma'} \sim R_\Gamma$. Let $x_0 \in X$ and $\gamma \in \Gamma'(x_0)$. Then, for any $g \in \Phi$,

$$\int g\, d\gamma \leq (R_{\Gamma'} g)(x_0) = (R_\Gamma g)(x_0) = (G_{\Gamma^c} g)(x_0)$$

where the last equality is by virtue of Lemma 4.1. Consequently,

$$(\forall g \in \Phi)(\gamma(g) \leq \sup_{\eta \in \Gamma^c(x_0)} \eta(g) \quad = \sup_{\eta \in sco\Gamma^c(x_0)} \eta(g)),$$

from which it follows by Corollary 2.2 that $\gamma \in \overline{\text{sco}}\,\Gamma^c(x_0) = \Gamma^s(x_0)$. Since $\gamma \in \Gamma'(x_0)$ and x_0 were arbitrary, we have : $\Gamma' \subseteq \Gamma^s$. This completes the proof. $\square$

Proof of Theorem 1.3. It follows from Lemma 5.1 and Corollary 4.13 that $G_\Gamma \approx G_{\overline{\text{sco}}\,\Gamma}$ since both Γ and $\overline{\text{sco}}\,\Gamma$ are analytic gambling houses. Hence, by Theorem 4.10, $M_\Gamma \sim M_{\overline{\text{sco}}\,\Gamma}$. Suppose next that Γ' is a gambling house on X such that $M_{\Gamma'} \sim M_\Gamma$. Then, by Lemma 4.11, $G_{\Gamma'} \sim G_\Gamma$, , so $\Gamma' \subseteq \overline{sco}\Gamma$ by Lemma 5.1. $\square$

Proof of Theorem 1.4. By Theorem 1.3, $M_{\Sigma_\Gamma} \sim M_{\Sigma_{\overline{\text{sco}}\,\Gamma}}$. Suppose then that Σ is a global gambling house on X such that $M_\Sigma \sim M_{\Sigma_\Gamma}$ Choose a gambling house Γ' on X such that $\Sigma = \Sigma_{\Gamma'}$. Then $M_\Gamma \sim M_{\Gamma'}$, so $\Gamma' \subseteq \overline{\text{sco}}\,\Gamma$ by Theorem 1.3. Hence, $\Sigma = \Sigma_{\Gamma'} \subseteq \Sigma_{\overline{\text{sco}}\,\Gamma}$. $\square$

Proof of Theorem 1.5. As observed above, $M_{\Sigma_\Gamma} \sim M_{\Sigma_{\overline{\text{sco}}\,\Gamma}}$ Suppose now that $\Sigma \subseteq X \times \mathbb{P}(H)$ such that $M_\Sigma \sim M_{\Sigma_\Gamma}$. By arguing as in the proof of Lemma 5.1, it is easy to show that $\Sigma \subseteq \overline{sco}\Sigma_\Gamma$. To complete the proof, it will suffice to show that $\overline{\text{sco}}\,\Sigma_\Gamma \subseteq \Sigma_{\overline{\text{sco}}\,\Gamma}$. By Theorem 4.14, $sco\Sigma_\Gamma \subseteq \Sigma_{sco\Gamma} \subseteq \Sigma_{\overline{\text{sco}}\,\Gamma}$. So we will be done as soon as we show that $\Sigma_{\overline{\text{sco}}\,\Gamma}(x)$ is norm-closed for each $x \in X$.

Fix $x_0 \in X$ and let $\mu_k \in \Sigma_{\overline{\text{sco}}\,\Gamma}(x_0), k \geq 1$, and assume that $\mu_k \to \mu$ in norm. In order to show that $\mu \in \Sigma_{\overline{\text{sco}}\,\Gamma}(x_0)$, we must prove that $(\mu)^0 \in \overline{sco}\Gamma(x_0)$ and for each $n \geq 1, \mu(x_1, x_2, \ldots, x_n) \in \overline{sco}\Gamma(x_n)$ almost surely $((\mu)^{n-1})$.

To see that $(\mu)^0 \in \overline{sco}\Gamma(x_0)$, note that $(\mu_k)^0 \in \overline{sco}\Gamma(x_0)$ and $(\mu^k)^0 \to (\mu)^0$ in norm. Suppose next that for some $n \geq 1$ there is a Borel set $E \subseteq X^n$ such that $(\mu)^{n-1}(E) > 0$ and $\mu(x_1, x_2, \ldots, x_n) \notin \overline{sco}\Gamma(x_n)$ for all $(x_1, x_2, \ldots, x_n) \in E$. Write $\vec{x}$ for $(x_1, x_2, \ldots, x_n)$ and, using the notation of sections 2 and 4, define

$$I_{\vec{x}}(A) = I(A; \mu(\vec{x}))$$

for $A \subseteq \mathbb{P}(X)$. Since $\mu(\vec{x}) \notin \overline{sco}\Gamma(x_n)$ for $\vec{x} \in E$, it follows that $I_{\vec{x}}(sco\Gamma(x_n)) < 1$ for all $\vec{x} \in E$ by virtue of Corollary 2.2. Hence, by the bold-face version of Theorem 2.7, there is a Borel measurable function $g : E \times X \to [0,1]$ such that

$$\sup_{\gamma \in \text{sco}\,\Gamma(x_n)} \int g(\vec{x}, y)\gamma(dy) < \int g(\vec{x}, y)\mu(\vec{x})(dy),$$

for $\vec{x} \in E$. Hence

$$\int_{E \times X} gd(\mu)^n = \int_E \left[\int g(\vec{x}, y)\mu(\vec{x})(dy)\right] d(\mu)^{n-1} > \int_E \left(G_{\text{sco}\,\Gamma} g_{\vec{x}}\right)(x_n) d(\mu)^{n-1}, \quad (5.1)$$

where $g_{\vec{x}}$ is the $\vec{x}$-section of g.

On the other hand, for $k \geq 1$,

$$\int_{E \times X} g\, d(\mu_k)^n = \int_E \left[\int g(\vec{x}, y)\mu_k(\vec{x})(dy)\right] d(\mu_k)^{n-1} \leq \int_E (G_{\text{sco}\,\Gamma} g_{\vec{x}})(x_n) d(\mu_k)^{n-1}, \quad (5.2)$$

since $\mu_k(\vec{x}) \in \overline{\text{sco}}\,\Gamma(x_n)$ almost surely $((\mu_k)^{n-1})$ and $G_{sco\Gamma} \sim G_{\overline{\text{sco}}\,\Gamma}$. Now, as is easily verified, the function $\vec{x} \to (G_{sco\Gamma} g_{\vec{x}})(x_n)$ is bounded and upper analytic. Since $(\mu_k)^i \to (\mu)^i$ in norm for each $i \geq 0$, by letting $k \to \infty$ in (5.2), we get, by virtue of Lemma 2.4,

$$\int_{E \times X} g\, d(\mu)^n \leq \int_E (G_{sco\Gamma} g_{\vec{x}})(x_n)\, d(\mu)^{n-1},$$

which contradicts (5.1). It follows that $\mu(\vec{x}) \in \overline{sco}\Gamma(x_n)$ almost surely $((\mu)^{n-1})$ This completes the proof. □

Corollary 5.2. *If Γ is an analytic gambling house on X, then*

$$\overline{\text{sco}}\,\Sigma_\Gamma = \Sigma_{\overline{\text{sco}}\,\Gamma} = \overline{\Sigma}_{sco\Gamma}.$$

In particular, $\overline{\text{sco}}\,\Sigma_\Gamma$ and $\overline{\Sigma}_{sco\Gamma}$ are global gambling houses on X.

Proof. The first equality is implicit in the proof of Theorem 1.5. The other equality is proved by observing that $\text{sco}\,\Sigma_\Gamma \subseteq \Sigma_{\text{sco}\,\Gamma}$ (Theorem 4.14), so $\overline{\text{sco}}\,\Sigma_\Gamma \subseteq \overline{\Sigma}_{\text{sco}\,\Gamma}$. On the other hand, $\Sigma_{\text{sco}\,\Gamma} \subseteq \Sigma_{\overline{\text{sco}}\,\Gamma}$ and hence $\overline{\Sigma}_{\text{sco}\,\Gamma} \subseteq \Sigma_{\overline{\text{sco}}\,\Gamma}$, since $\Sigma_{\overline{\text{sco}}\,\Gamma}(x)$ is norm-closed for each $x \in X$, as was observed in the course of proving Theorem 1.5. □

Proof of Theorem 1.6. Let $X = \{0, 1\}$ and define a gambling house Γ on X as follows:

$$\Gamma(0) = \{\delta(0)\} = \Gamma(1)$$

Then, for any real-valued function u on X,

$$(V_\Gamma u)(0) = u(0) = (V_\Gamma u)(1).$$

For each $a \in (0, 1)$, define a gambling house Γ^a thus:

$$\Gamma^a(0) = \{\delta(0)\}; \Gamma^a(1) = \{(1-b)\delta(0) + b\delta(1) : 0 \leq b \leq a\}.$$

It is easy to check that $V_{\Gamma^a} = V_\Gamma$ for every $a \in (0, 1)$. Towards a contradiction, assume that there is a largest house Γ^*. Then $\Gamma^* \supseteq \Gamma^a$ for each $a \in (0, 1)$. In particular,

$$\Gamma^*(1) \supseteq \{(1-b)\delta(0) + b\delta(1) : 0 \leq b < 1\}.$$

Now consider the following strategy σ available in Γ^* at 1 :

$$\sigma_0 = \frac{1}{2^2}\delta(0) + (1 - \frac{1}{2^2})\delta(1)$$

and, for $n \geq 1$,

$$\sigma_n(x_1, x_2, \ldots, x_n) = \begin{cases} \delta(0) & \text{if } x_n = 0 \\ \frac{1}{(n+2)^2}\delta(0) + \left(1 - \frac{1}{(n+2)^2}\right)\delta(1) & \text{if } x_n = 1. \end{cases}$$

Then

$$\sigma(\{h \in H : h_n = 1 \text{ for all } n \geq 1\}) = \prod_{n=0}^{\infty}(1 - \frac{1}{(n+2)^2}) = p(say) > 0.$$

Hence, for any $u : X \to \mathcal{R}$,

$$\int u^* d\sigma = (1-p)u(0) + pu(1).$$

Consequently,

$$(V_{\Gamma^*}u)(1) \geq (1-p)u(0) + pu(1),$$

so $V_{\Gamma^*} \not\sim V_\Gamma$, as can be seen by a suitable choice of u. This yields the desired contradiction. □

The following result is a close analogue of Theorem 1.2.

Theorem 5.3. *If Γ is an analytic gambling house on X, then the largest gambling house Γ' on X such that $T_{\Gamma'} \sim T_\Gamma$ is $\overline{\text{sco}}\,\Gamma_\infty$.*

We omit the proof.

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