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LARGE DEVIATIONS FOR SOME POISSON RANDOM INTEGRALS

by

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In the theory of large deviations one of the main examples is Schilder's theorem. It gives the large deviation estimates for the convergence $\sqrt{\epsilon}W \Rightarrow \delta_0$ on $C([0,\infty),\mathbf{R}^d)$ as $\epsilon \to 0$, for the Brownian Motion W. In this paper we investigate analogous problems for $\epsilon N(f) := \epsilon \int f dN$ or $\epsilon\widetilde{N}(f) := \epsilon \int f d\widetilde{N}$, where N (resp. $\widetilde{N}$) is a (resp. compensated) Poisson point process and f is a deterministic function. We find that this large deviation estimation depends strongly on the tail behavior of f. This differs from the Brownian Motion case where only the norm of f in L^2 is involved. In particular,we get the large deviations principle for the Lévy class L distributions (called also self-decomposable measures). The question about large deviations for the multiple Poisson integrals is not discussed here. (The case of Brownian Motion is solved by Ledoux [L].)

1. Notation and basic terminology. Let S be a metric separable and complete space (or Polish space) with the Borel σ-field $\mathcal{S}$. A function $I : S \to [0,\infty]$ such that $\{I \leq L\}$ are compact subsets of S for all $L > 0$, is called *good rate function.* We say that a family $\{\mu_\epsilon, \epsilon > 0\}$ of Borel probability measures on S satisfies *large deviations principle* with the good rate I and the *speed* $\lambda(\epsilon)$ provided

$$\overline{\lim_{\epsilon\to 0}} \frac{1}{\lambda(\epsilon)} \log \mu_\epsilon(F) \leq - \inf_{s\in F} I(s) \tag{1.1}$$

for all closed subsets F in S; and

$$- \inf_{s\in G} I(s) \leq \underline{\lim_{\epsilon\to 0}} \frac{1}{\lambda(\epsilon)} \log \mu_\epsilon(G) \tag{1.2}$$

for all open sets G in S.
[Here: $\lambda(\epsilon) > 0$ and $\lambda(\epsilon) \to +\infty$ as $\epsilon \to 0$. Also we adopt convention $\inf \phi = +\infty$ throughout this paper]. In short, we write that (μ_ϵ) satisfies *LDP*. Note that (1.1) and (1.2) roughly mean that $\mu_\epsilon(A) \simeq \exp[-\lambda(\epsilon)\inf_{s\in A} I(s)]$.

We require the following variant of comparison technique in large deviation theory (see e.g. [DS, Exercice 2.1.20, p.47-49] for other versions), whose proof is left to the reader.

Comparison Lemma : *Let $(X_\epsilon^n, X_\epsilon, n \in \mathbf{N}, \epsilon > 0)$ be a family of random variables valued in a Polish space S with metric $d(\cdot,\cdot)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Assume*

(i) for each $n \in \mathbf{N}$, $\mathbf{P}(X_\epsilon^n \in \cdot)$ satisfies as $\epsilon \to 0$, the LDP on S with speed $\lambda(\epsilon)$ and good rate function $I_n(x)$;

(ii) there is a good rate function I on S such that $\forall L \geq 0$,

$$\sup_{x\in[I\leq L]} |I_n(x) - I(x)| \longrightarrow 0, \quad \textit{as } n \to \infty \; ; \tag{1.3}$$

(iii) for every $\delta > 0$,

$$\lim_{n\to\infty} \overline{\lim_{\epsilon\to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}\left(d(X_\epsilon^n, X_\epsilon) > \delta\right) = -\infty \; . \tag{1.4}$$

Then $\mathbf{P}(X_\epsilon \in \cdot)$, as $\epsilon \to 0$, satisfies the LDP on S with speed $\lambda(\epsilon)$ and good rate function $I(x)$ given in (ii) above.

Let $(E, \mathcal{E}, \rho)$ be a σ-finite measure space and $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. A mapping

$$N : \Omega \to \left\{ \sum_i \delta_{x_i} \text{ (at most countable)} : x_i \in E \right\} , \tag{1.5}$$

where δ_x denotes Dirac measure, is called *Poisson point process with intensity measure ρ* if

(i) $$\mathbf{P}[N(A) = k] = e^{-\rho(A)} \frac{(\rho(A))^k}{k!} , \quad k = 0, 1, 2, \dots ,$$

for all $A \in \mathcal{E}$ such that $0 < \rho(A) < \infty$;

(ii) for $k \geq 1$ and $A_j \in \mathcal{E}$, $j = 1, 2, \dots, k$, pair-wise disjoint, and $0 < \rho(A_j) < \infty$, random variables $N(A_1)$, $N(A_2), \dots, N(A_k)$ are independent.

We shall often denote the integral $\int_E f dN$ also by $N(f)$, where f is a ρ-integrable function. If $\widetilde{N}$ is the compensated Poisson point process, i.e., $\widetilde{N} := N - \rho$, then the random integral $\widetilde{N}(f) := \int_E f d\widetilde{N}$ exists for $f \in L^2(\rho)$; cf, [JS].

2. Large deviations for integrals $N(f)$ on $\mathbf{R}^d$ for bounded f. The main results will be preceeded by two auxilliary steps.

Step 1. *If $0 < \rho(A) < \infty$ then the probability measures $\mu_\epsilon(\cdot) := \mathbf{P}[\epsilon N(A) \in \cdot]$ on $\mathbf{R}$, $\epsilon > 0$, satisfy LDP with the speed $\lambda(\epsilon) = \epsilon^{-1}|\log \epsilon|$ and the rate function*

$$I(x) = \begin{cases} +\infty , & \text{for } x < 0 \; ; \\ x , & \text{for } x \geq 0 \; . \end{cases}$$

P r o o f. Since for $x > 0$ we have

$$\lim_{\epsilon\to 0} \frac{\epsilon}{|\log \epsilon|} [\frac{x}{\epsilon}] \log[\frac{x}{\epsilon}] = x \; , \tag{2.1}$$

therefore the Stirling formula implies that

$$\lim_{\epsilon\to 0} \frac{\epsilon}{|\log \epsilon|} \log([\frac{x}{\epsilon}]!) = x \; , \tag{2.2}$$

where $[\cdot]$ denotes the integral part. From the inequalities

$$e^{-\rho(A)}\frac{(\rho(A))^{[\frac{x}{\epsilon}]+1}}{([\frac{x}{\epsilon}]+1)!}\le \mathbf{P}[\epsilon N(A)>x]=\sum_{j\ge[\frac{x}{\epsilon}]+1} e^{-\rho(A)}\frac{(\rho(A))^j}{j!}\le\frac{\rho(A)^{[\frac{x}{\epsilon}]+1}}{([\frac{x}{\epsilon}]+1)!}$$

and (2.1) with (2.2) we conclude that for all $x>0$

$$\lim_{\epsilon\to 0}\frac{\epsilon}{|\log\epsilon|}\log\mathbf{P}[\epsilon N(A)>x]=-x\ . \tag{2.3}$$

For a closed F in $\mathbf{R}$, with $\inf_{s\in F} I(s)=x>0$ and all $0<\delta<x$, one has

$$\mathbf{P}[\epsilon N(A)\in F]\le\mathbf{P}[\epsilon N(A)\ge x]\le P[\epsilon N(A)>x-\delta]\ .$$

Hence by (2.3) we conclude

$$\overline{\lim_{\epsilon\to 0}}\frac{\epsilon}{|\log\epsilon|}\log\mathbf{P}[\epsilon N(A)\in F]\le-\inf_{s\in F} I(s)+\delta\ ,$$

for all $0<\delta<x$, which proves the upper bound (1.1). If $\inf_{s\in F} I(s)=0$ then (1.1) holds automatically.

For an open set $G\ni s$, let us choose $\delta>0$ such that $(s-\delta,s+\delta)\subseteq G$. Then

$$\mathbf{P}[\epsilon N(A)\in G]\ge\mathbf{P}[N(A)\in\epsilon^{-1}(s-\delta,s+\delta)]\ge e^{-\rho(A)}\frac{\rho(A)^{[\frac{s}{\epsilon}]+1}}{([\frac{s}{\epsilon}]+1)!}\ ,$$
$$\text{whenever}\quad \epsilon([\frac{s}{\epsilon}]+1)\in(s-\delta,s+\delta)\ .$$

Of course, the last claim is true for all sufficiently small ϵ, and by (2.2) we get

$$\underline{\lim_{\epsilon\to 0}}\ \frac{\epsilon}{|\log\epsilon|}\ \log\mathbf{P}[\epsilon N(A)\in G]\ge -s$$

for any $s\in G$. Hence follows the lower bound (1.2) and the proof of Step 1 is completed.

Also note that the rate function does not depend on set A.

Step 2. *If $0<\rho(A_l)<\infty$, and $A_l's$ are pair-wise disjoint $l=1,2,...,k$, then the probability measures $\mathbf{P}[\epsilon(N(A_1),...,N(A_k))\in\cdot]$ on $\mathbf{R}^k$, satisfy LDP with the speed $|\log\epsilon|/\epsilon$ and the rate function*

$$I((x_1,...,x_k))=\begin{cases} x_1+x_2+...+x_k\ , & \text{if}\quad x_l\ge 0\ ,\ l=1,2,,...,k\ ;\\ +\infty\ , & \textit{otherwise}\ .\end{cases}$$

P r o o f. It follows from Step 1 and Lemma 2.8 from [LS].

Theorem 1. *Let $(E,\mathcal{E},\rho)$ be a finite measure space and $f:E\to\mathbf{R}^d$ be measurable and bounded function. Let $K_f:=$ conv(supp $\rho\circ f^{-1}$) be the convex hull spanned by the support of the measure $\rho\circ f^{-1}$ on $\mathbf{R}^d$, and*

$$q_{K_f}(x):=\inf\{c>0:\ c^{-1}x\in K_f\}\ ,\quad\forall x\in\mathbf{R}^d\setminus\{0\}\ ,\quad q_{K_f}(0):=0\ ,$$

be the Minkowski functional of the set K_f. Then the measures $\mathbf{P}[\epsilon N(f) \in \cdot]$ satisfy LDP on $\mathbf{R}^d$ with the speed $\lambda(\epsilon) = |\log \epsilon|/\epsilon$ and the rate function $I(x) = q_{K_f}(x)$.

P r o o f. Suppose first that f is a simple $\mathbf{R}^d$-valued function, i.e., for $e \in E$

$$f(e) = \sum_{j=1}^{m} x^j 1_{A_j}(e) \ , \quad x^j \in \mathbf{R}^d \ , \quad \rho(A_j) > 0 \ ,$$

and $A_j \in \mathcal{E}$ are pair-wise disjoint. From Step 2 and the contraction principle we infer that $\mathbf{P}[\epsilon N(f) \in \cdot]$ satisfy LDP on $\mathbf{R}^d$ with $\lambda(\epsilon)$ as before and the rate function

$$\begin{aligned} I_f(u) &:= \inf \left\{ y_1 + \dots + y_m : \ y_i \geq 0 \ , \sum_{j=1}^{m} x^j y_j = u \right\} \\ &= \inf \left\{ \nu(E) : \ \nu \in \mathcal{M}^+(E) \ \text{ and } \ \int_E f d\nu = u \right\} \end{aligned}$$

which is equal to zero for $u = 0$, and for $u \neq 0$,

$$\begin{aligned} &= \inf \left\{ c > 0 : \ \text{ there is } \ \nu \in \mathcal{M}_1^+(E) \ \text{ and } \ \int_E f d\nu = \frac{u}{c} \right\} \\ &= \inf \left\{ c > 0 : \ c^{-1} u \in K_f \right\} \ , \end{aligned}$$

because for simple f we have

$$K_f = \left\{ \sum_{j=1}^{m} x^j \lambda_j : \ \sum_{j=1}^{m} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \right\} = \left\{ \int_E f d\nu : \ \nu \in \mathcal{M}_1^+(E) \right\} \tag{2.4}$$

where $\mathcal{M}^+$ and $\mathcal{M}_1^+$ denote the sets of non-negative and probability measures, respectively.

For a general bounded and measurable function $f : \ E \to \mathbf{R}^d$, let us choose $f_n : \ E \to \mathbf{R}^d$, $n \geq 1$, simple, measurable such that $\|f - f_n\|_\infty := \sup_{s \in E} \|f(s) - f_n(s)\| \to 0$. Since for each $\delta > 0$

$$\begin{aligned} \overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon \|N(f_n) - N(f)\| \geq \delta] &\leq \overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon N(E) \|f_n - f\|_\infty \geq \delta] \\ &= - \frac{\delta}{\|f_n - f\|_\infty} \ , \end{aligned}$$

where the last equality follows from Step 1, we conclude that

$$\lim_{n \to \infty} \overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon \|N(f_n) - N(f)\| \geq \delta] = -\infty \ .$$

And it is easy to see that $I_n := q_{\mathbf{K}_{f_n}}$ converges to $I := q_{\mathbf{K}_f}$ in the sense of (1.3). Now the comparison lemma completes the proof of Theorem 1.

Remark 1. The assumption in Theorem 1, that ρ is finite can be replaced by $\rho(f \neq 0) < \infty$.

Corollary 1. *Let E be locally compact metric, separable space and ρ a Radon measure such that $\rho(U) > 0$ for all open sets $U \neq \emptyset$. Let $M_R^+(E)$ be the space of nonnegative Radon measures on E, equipped with the vague convergence topology. Then $\mathbf{P}[\epsilon N \in \cdot]$ satisfy LDP on $\mathcal{M}_R^+(E)$ with the speed $\lambda(\epsilon) = |\log \epsilon|/\epsilon$, $\epsilon > 0$, and the rate function*

$$I(\nu) := \nu(E) \ , \quad \nu \in \mathcal{M}_R^+(E) \ .$$

P r o o f. Let $K_n \uparrow E$, $K_n \subset K_{n+1}$ are compact such that $0 < \rho(K_n) < \infty$. We know that the space $\mathcal{M}_R^+(E)$ with *vague* topology is the projective limit space of sequences $\mathcal{M}_R^+(K_n)$ with the *weak* topology. By Dawson-Gärtner (1987) we can and do assume that E is *compact.* For every $\nu \in \mathcal{M}_R^+(E)$ fixed, the sets

$$U(\nu, f, \delta) := \left\{ \nu' \in \mathcal{M}_R^+(E) : \ | \int_E f d\nu - \int_E f d\nu' | < \delta \right\} \ ,$$

where $\delta > 0$, $f : \ E \to \mathbf{R}^d$ is continuous, $d \geq 1$, form a basis of neighbourhoods of ν. Since

$$\begin{aligned} q_{\mathbf{K}_f}(u) &= \inf \left\{ \nu(E) : \ \nu \in \mathcal{M}^+(E) \ \text{ and } \ \int_E f d\nu = u \right\} \\ &= \inf \left\{ I(\nu) : \ \nu \in \mathcal{M}^+(E) \ \text{ and } \ \int_E f d\nu = u \right\} \ , \end{aligned}$$

$\forall u \in \mathbf{R}^d$, by the proof of Theorem 1, we get by Theorem 1

$$\begin{aligned} \lim_{\epsilon \to 0} \binom{\inf}{\sup} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon N \in U(\nu, f, \delta)] &= \lim_{\epsilon \to 0} \binom{\inf}{\sup} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[|\epsilon N(f) - \nu(f)| < \delta] \\ &\in [- \inf_{u: \ |u - \nu(f)| < \delta} q_{\mathbf{K}_f}(u), - \inf_{u: \ |u - \nu(f)| \leq \delta} q_{\mathbf{K}_f}(u)] \\ &= [- \inf_{\nu' \in U(\nu, f, \delta)} I(\nu') \ , \ - \inf_{\nu' \in \overline{U(\nu, f, \delta)}} I(\nu')] \ . \end{aligned}$$

This implies the *weak LDP* by [DS, p. 46, (v)]. Furthermore, for any $L > 0$, $[I \leq L] =: K_L$ is compact in $\mathcal{M}_R^+(E)$ and by Step 1,

$$\limsup_{\epsilon \to 0} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}(\epsilon N \notin K_L) = \limsup_{\epsilon \to 0} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}(\epsilon N(E) > L) \leq -L \ .$$

This exponential tightness with the *weak LDP* shown before gives the desired *LDP.*

Remark 2. This corollary is one counterpart of the classical Schilder theorem about the Brownian Motion, and it complements a result in [GW], recalled below. In the setting of the Corollary 1, let N^ϵ be the Poisson point process with intensity measure $\epsilon\rho$. Then the laws of N^ϵ satisfy the LDP with the same rate function $I(\nu) = \nu(E)$, but with a different speed $\lambda(\epsilon) = |log \epsilon|$.

Theorem 2. *Assume that $(E,\mathcal{E},\rho)$ is an infinite measure space, i.e., $\rho(E)=\infty$, and $f: E\to\mathbf{R}^d$ be measurable bounded and square integrable, i.e., $f\in L^2(\rho)\cap L^\infty$. Then for the compensated Poisson point process $\widetilde{N}=N-\rho(dx)$, we have that $\mathbf{P}[\epsilon\widetilde{N}(f)\in\cdot]$, $\epsilon>0$, satisfy LDP on $\mathbf{R}^d$ with the speed $\lambda(\epsilon)=|\log\epsilon|/\epsilon$ and the rate $I(x)=q_{K_f}(x)$.*

P r o o f. Note that the integral $\widetilde{N}(f)$ is well defined for $f\in L^2(\rho)\cap L^\infty$, cf. [JS]. Taking $f_n:=1_{[\|f\|>n^{-1}]}f$, $n\geq 1$, we have $\rho(f_n\neq 0)<\infty$. Since $\epsilon\widetilde{N}(f_n):=\epsilon N(f_n)-\epsilon\int_E f_n d\rho$, $\epsilon>0$, and the last term goes to zero as $\epsilon\to 0$, therefore by Theorem 1 and Remark 1, we conclude that $\mathbf{P}[\epsilon\widetilde{N}(f_n)\in\cdot]$ satisfy *LDP* on $\mathbf{R}^d$ with the rate $q_{K_{f_n}}(\cdot)$. But $q_{K_{f_n}}(x)\to q_{K_f}(x)$ uniformly over compact sets.

By the comparison Lemma, in order to complete the proof it is enough to show that

$$\lim_{n\to\infty}\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\log\mathbf{P}[\epsilon\|\widetilde{N}(f_n)-\widetilde{N}(f)\|\geq\delta]=-\infty \tag{2.5}$$

for each $\delta>0$. Since (for Euclidean norm)

$$\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\log\mathbf{P}[\epsilon\|\widetilde{N}(f-f_n)\|\geq\delta]\leq\max_{1\leq i\leq d}\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\log\mathbf{P}[\epsilon|\widetilde{N}(f^i-f^i_n)|\geq\delta]$$

we can and do assume below that f is a real-valued function.

Let $F:=f-f_n=f1_{[|f|\leq n^{-1}]}$, so $\|F\|_\infty\leq n^{-1}$ and let

$$\wedge(t):=\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\log\mathbf{E}[e^{\lambda(\epsilon)\epsilon t\widetilde{N}(F)}]\ ,\quad t\in\mathbf{R}\,. \tag{2.6}$$

Then we have

$$\begin{aligned}
\wedge(t)&=\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\int_{[|f|\leq n^{-1}]}[e^{\epsilon\lambda(\epsilon)tf}-1-\epsilon\lambda(\epsilon)tf]\,d\rho\\
&\leq\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\int_{[|f|\leq n^{-1}]}e^{\epsilon\lambda(\epsilon)|tf|}\frac{1}{2}(\epsilon\lambda(\epsilon))^2t^2f^2d\rho\\
&\leq\overline{\lim_{\epsilon\to 0}}[\frac{1}{2}\epsilon|\log\epsilon|\,t^2\exp(|\log\epsilon|\,|t|n^{-1})\int_E f^2d\rho]\\
&=\frac{1}{2}t^2\int_E f^2d\rho\ \overline{\lim_{\epsilon\to 0}}|\log\epsilon|\epsilon^{1-|t|n^{-1}}=0\ ,\quad\text{for}\quad|t|<n\,.
\end{aligned}$$

Hence for $t=n-1$ we obtain

$$\begin{aligned}
&\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\log\mathbf{P}[\epsilon\widetilde{N}(f-f_n)\geq\delta]\\
&=\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\log\mathbf{P}[\exp(\lambda(\epsilon)t\epsilon\widetilde{N}(f-f_n)\geq e^{t\lambda(\epsilon)\delta}]\\
&\leq\overline{\lim_{\epsilon\to 0}}\frac{1}{\lambda(\epsilon)}\log(e^{-t\lambda(\epsilon)\delta}\mathbf{E}[e^{\lambda(\epsilon)t\epsilon\widetilde{N}(F)}])\\
&=-\delta t+\wedge(t)=-\delta(n-1)\,.
\end{aligned}$$

Similarly taking $t = -(n-1)$, we obtain

$$\overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}(\epsilon \widetilde{N}(f - f_n) \le -\delta) \le -\delta(n-1) .$$

These two estimations lead to

$$\lim_{n \to \infty} \overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[|\epsilon \widetilde{N}(f - f_n)| \ge \delta] = -\infty.$$

Thus we conclude (2.5) and the proof of Theorem 2 is complete.

Corollary 2. *Suppose $f: E \to \mathbf{R}^d$ belongs to $L^2(\rho)$ and for $\lambda(\epsilon) = |\log \epsilon|/\epsilon$, $\epsilon > 0$, one has*

$$\overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon \|\widetilde{N}(f)\| \ge a] < 0 , \tag{2.7}$$

for some $a > 0$. Then $f \in L^\infty$.

P r o o f. Without loss of generality we assume that f is a real-valued function. Furthermore note that

$$-I(a) := \overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon |\widetilde{N}(f)| \ge a] = -aI(1) ,$$

and therefore $I(a) > 0$ for some $a > 0$ is equivalent to $I(a) > 0$ for all $a > 0$. Let us assume that $f \notin L^\infty$ and choose $r > 2/I(1)$ such that $\rho(f_r := f 1_{[r \le |f| \le r+1]} \neq 0) > 0$. Observe

$$\mathbf{P}[\epsilon |\widetilde{N}(f - f_r)| \le 1] > \frac{1}{2} , \quad \text{for all sufficiently small } \epsilon > 0 . \tag{2.8}$$

By the independence of $\widetilde{N}(f_r)$ and $\widetilde{N}(f - f_r)$, (2.8) implies that

$$\mathbf{P}[\epsilon |\widetilde{N}(f)| \ge 1] \ge \mathbf{P}[\epsilon |\widetilde{N}(f_r)| \ge 2] \cdot \mathbf{P}[\epsilon |\widetilde{N}(f - f_r)| \le 1] \ge 2^{-1} \mathbf{P}[\epsilon |\widetilde{N}(f_r)| \ge 2] .$$

Hence with Theorem 2 we get

$$-I(1) \ge \underline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon |\widetilde{N}(f_r)| \ge 2] = - \inf_{|x| \ge 2} q_{K_{f_r}}(x) \ge -\frac{2}{r} ,$$

which contradicts the selection of r, and the proof is complete.

Corollary 3. *Let (E, ξ, ρ) be σ-finite measure space, a function $f \in L^2(E, \mathcal{E}, \rho; \mathbf{R}^d)$ and $\widetilde{N}$ be the compensated Poisson point process. For $\lambda(\epsilon) := |\log \epsilon|/\epsilon$, $\epsilon > 0$, define*

$$a := -\overline{\lim_{\epsilon \to 0}} \frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\epsilon \|\widetilde{N}(f)\| > 1]$$

$$b := \sup \left\{ \alpha > 0 : \mathbf{E}[\exp(\alpha \|\widetilde{N}(f)\| \log(1 + \|\widetilde{N}(f)\|))] < \infty \right\} .$$

Then $a = b = \|f\|_\infty^{-1}$.

P r o o f. For each $0 < \eta < a$ we have

$$\frac{1}{\lambda(\epsilon)} \log \mathbf{P}[\|\widetilde{N}(f)\| > \epsilon^{-1}] \leq -\eta ,$$

for all sufficiently small ϵ, i.e., $\mathbf{P}[\|\widetilde{N}(f)\| > s] \leq \exp(-\eta s \log s)$, for all sufficiently large s. Hence $\mathbf{E}[\exp(\eta - \delta)\widetilde{\mathbf{N}}(f)\log(1 + \|\widetilde{N}(f)\|)] < \infty$ for all $\delta > 0$ such that $\eta - \delta > 0$. Thus $\eta \leq b$ and hence $a \leq b$. One gets the converse inequality using Tschebyshev's inequality.

If $\|f\|_\infty < \infty$, then Theorem 2 gives $a = \|f\|_\infty^{-1}$. In fact, we have

$$a = \inf_{\|x\|>1} q_{K_f}(x) = \inf_{\|x\|\geq 1} q_{K_f}(x) = \|f\|_\infty^{-1} .$$

For $f \notin L_\infty$, Corollary 2 justifies $a = 0$. Thus the proof is completed.

3. Large deviations on R under exponential integrability. In this section we consider the case of $f \notin L^\infty(E, \mathcal{E}, \rho; \mathbf{R})$. Let us introduce parameters

$$\gamma^+ := \sup\left\{\alpha \geq 0 : \int_{[f\geq 1]} e^{\alpha f} d\rho < \infty\right\} \text{and}\, \gamma^- := \sup\left\{\alpha \geq 0 : \int_{[f\leq -1]} e^{-\alpha f} d\rho < \infty\right\}.$$

Arguing as in the proof of Corollary 3 we infer that

$$\gamma^+ = -\overline{\lim}_{k\to\infty} \frac{1}{k} \log \rho(f \geq k) \ , \quad \text{and} \tag{3.1}$$

$$\gamma^- = -\overline{\lim}_{k\to\infty} \frac{1}{k} \log \rho(f \leq -k) \ . \tag{3.2}$$

Theorem 3. *Assume that $f \in L^2(\rho)$, and $\gamma^+, \gamma^- > 0$. Then $\mathbf{P}[\epsilon\widetilde{N}(f) \in \cdot]$, $\epsilon > 0$, satisfy LDP on $\mathbf{R}$ with the speed $\lambda(\epsilon) = \epsilon^{-1}$ and the rate function*

$$I(x) = \begin{cases} \gamma^+ x \ , & for \quad x \geq 0 \ ; \\ -\gamma^- x \ , & for \quad x < 0 \ . \end{cases}$$

P r o o f. Let us write $f = f \cdot 1_{[|f|<1]} + f \cdot 1_{[f\geq 1]} + f \cdot 1_{[f\leq -1]} =: f_0 + f_1 + f_{-1}$. From Theorem 1, for each $\delta > 0$

$$\lim_{\epsilon\to 0} \frac{\epsilon}{|\log \epsilon|} \log \mathbf{P}[\epsilon|\widetilde{N}(f_0)| > \delta] = -\frac{1}{\|f_0\|_\infty} ,$$

and hence

$$\lim_{\epsilon\to 0} \epsilon \log \mathbf{P}[\epsilon|\widetilde{N}(f_0)| > \delta] = -\infty \ . \tag{3.3}$$

Case 1. Now let us assume that $f \geq 1$ on E, i.e., $f = f_1$. For the upper bound (in *LDP*) note

$$\begin{aligned}\Lambda_1(t) &:= \lim_{\epsilon\to 0} \epsilon \log \mathbf{E}[\exp t\widetilde{N}(\epsilon f_1)/\epsilon] \\ &= \lim_{\epsilon\to 0} \epsilon \int_{[f\geq 1]} (e^{tf_1} - 1 - tf_1)d\rho \\ &= \lim_{\epsilon\to 0} \epsilon \int_{[f\geq 1]} e^{tf} d\rho = \begin{cases} 0\,, & \text{for} \quad t < \gamma^+ \\ +\infty, & \text{for} \quad t > \gamma^+\,. \end{cases}\end{aligned}$$

Hence its Fenchel-Legendre transformation is given by

$$I_1(x) := \Lambda_1^*(x) = \sup_{t\in\mathbf{R}}[xt - \Lambda_1(t)] = \sup_{t<\gamma^+} xt = \begin{cases} \gamma^+ x\,, & \text{for} \quad x \geq 0\,, \\ +\infty\,, & \text{for} \quad x < 0\,. \end{cases} \tag{3.4}$$

By Ellis-Gärtner Theorem ([DS], Thm.2.2.4), the upper bound of large deviations holds for $\mathbf{P}[\epsilon\widetilde{N}(f_1) \in \cdot]$, $\epsilon > 0$, on $\mathbf{R}$ with the speed $\lambda(\epsilon) = \epsilon^{-1}$ and with the rate function $I_1(x)$.

For the lower bound observe that $f_1 \in L^1(\rho)$ and instead of $\widetilde{N}(f_1)$ we can consider $N(f_1)$. Furthermore, using inequality $N(f_1) \geq tN(f_1 > t)$ we get, for $a > 0$ and $t \geq 1$,

$$\begin{aligned}\mathbf{P}[\epsilon N(f_1) > a] &\geq \mathbf{P}[t\epsilon N(f_1 > t) > a] \\ &\geq \sum_{k>a/\epsilon t} e^{-\rho(f_1>t)}(\rho(f_1 > t))^k/k! \\ &\geq e^{-\rho(f_1>t)}(\rho(f_1 > t))^{[\frac{a}{\epsilon t}]+1}/([\frac{a}{\epsilon t}] + 1)!\end{aligned}$$

Choose now $t = t(\epsilon)$, a positive function of ϵ verifying $\lim_{\epsilon\to 0} \epsilon t(\epsilon) = +\infty$ and

$$\gamma^+ = -\lim_{\epsilon\to 0} \frac{1}{t(\epsilon)} \log \rho[f > t(\epsilon)]$$

(possible by (3.1)!). Hence using the above inequality and (2.2), we get

$$\varliminf_{\epsilon\to 0} \epsilon \log \mathbf{P}[\epsilon N(f_1) > a] \geq \lim_{\epsilon\to 0} \epsilon([\frac{a}{\epsilon t(\epsilon)}] + 1) \log \rho[f_1 > t(\epsilon)] = -\gamma^+ a\,.$$

Since we used only one term estimate we infer the lower bound for $\mathbf{P}[\epsilon N(f_1) \in (a, b)]$ as well. In other words, the measures $\mathbf{P}[\epsilon\widetilde{N}(f_1) \in \cdot]$, $\epsilon > 0$, satisfy *LDP* with the speed $\lambda(\epsilon) = \epsilon^{-1}$ and the rate function $I_1(x)$.

Case 2. Applying Case 1 for $-f$ and observing that $(-f)_1 = -f_{-1}$, $\widetilde{N}(f_{-1}) = -\widetilde{N}((-f)_1)$ we conclude *LDP* for $\mathbf{P}[\epsilon N(f_{-1}) \in \cdot]$, $\epsilon > 0$, with the speed $\lambda(\epsilon) = \epsilon^{-1}$, $\epsilon > 0$, and the rate function

$$I_{-1}(x) = \begin{cases} +\infty\,, & \text{for} \quad x > 0\,; \\ \gamma^-|x|\,, & \text{for} \quad x \leq 0\,. \end{cases} \tag{3.5}$$

Finally since $\widetilde{N}(f) = \widetilde{N}(f_0) + \widetilde{N}(f_1) + \widetilde{N}(f_{-1})$ is a sum of independent variables, (3.3),(3.4) and (3.5) imply the LDP in Theorem 3 by [LS, Lemma 2.8].

4. Applications to the class L distributions. For the basic information on the class L (or selfdecomposable) distributions cf. [JM] p. 177-182. For the purpose of this application, let us recall that

$$(4.1) \qquad \mu \in L \quad \text{iff} \quad \mu \stackrel{d}{=} Z(0) := \int_{(0,\infty)} e^{-s} dY(s) \ , \quad \mathbf{E} \log(1 + \|Y(1)\|) < \infty \ ,$$

and Y is a Lévy process. Of course,

$$(4.2) \qquad \mathbf{P}[Z(t) = \int_t^\infty e^{-s} dY(s) \in \cdot] \to \delta_0(\cdot) \ , \text{as} \ \ t \to \infty \ ,$$

and it is "natural" to ask for LDP for probability distributions in (4.2).

Let Y be without Gaussian component and shift, i.e., $Y(1) \stackrel{d}{=} [0,0,M]$ (these are the parameters in the Lévy-Khintchine formula of $Y(1)$; M is the spectral Lévy measure). Then

$$(4.3) \qquad Y(t) = \int_0^t \int_{\mathbf{R}^d \setminus \{0\}} x \widetilde{N}(dx, ds) \ , \quad t \geq 0 \ ,$$

where $\widetilde{N}$ is a Poisson point process with the compensator $\rho(dt, dx) = dt \times dM$ on $E := [0, \infty) \times \mathbf{R}^d \setminus \{0\}$ (dt= Lebesque measure); see [JS]. Thus

$$(4.4) \qquad Z(0) = \int_E e^{-t} x \widetilde{N}(dx, dt) = \widetilde{N}(f_0) \ , \quad \text{for} \quad f_0(t, x) := e^{-t} x \ .$$

¿From (4.1) we also have that

$$e^{-t} Z(0) = \int_0^\infty e^{-(t+s)} dY(s) = \int_t^\infty e^{-u} dY(u - t) \stackrel{d}{=} \int_t^\infty e^{-u} dY(u) = Z(t) \ .$$

Hence (4.2) with (4.4) is equivalent to

$$\mathbf{P}[e^{-t} \widetilde{N}(f_0) \in \cdot] \to \delta_0 \ , \quad \text{as} \quad t \to \infty \ .$$

All the above we can summarize in the following

Theorem 4. *Let* $Y(\cdot)$ *be a real Lévy process with* $Y(1) \stackrel{d}{=} [0,0,M]$. *Assume that suppM is compact. Then* $\mathbf{P}[\int_{(t,\infty)} e^{-s} dY(s) \in \cdot]$ *satisfy LDP on* $\mathbf{R}$, *as* $t \to \infty$ *with the speed* $\lambda(t) := te^t$ *and the rate function*

$$(4.5) \qquad I_M(x) := \begin{cases} x/b \ , & for \quad 0 < b := sup(supp M) \ , \quad x > 0 \ ; \\ x/a \ , & for \quad inf(supp M) =: a < 0 \ , x < 0 \ ; \\ +\infty \ , & otherwise \ . \end{cases}$$

P r o o f. Note that f_0 defined in (4.4) belongs to $L^\infty(\rho)$ iff $\operatorname{supp} M$ is bounded in $\mathbf{R}$. In this case $f_0 \in L^2(\rho)$ as well. Since $\operatorname{supp}(\rho \circ f_0^{-1}) = f_0(\operatorname{supp} \rho) = [a, b]$ where $a = \inf(\operatorname{supp} M)$, $b = \sup(\operatorname{supp} M)$, Theorem 2 gives the conclusion of Theorem 4.

Remark 3. If $Y(1) \stackrel{\mathrm{d}}{=} [0, \sigma^2, 0]$, i.e., $Y(t)$ is a Brownian Motion, then $Z(t) \stackrel{\mathrm{d}}{=} \frac{1}{\sqrt{2}} e^{-t} Y(1)$. In other words, $\sqrt{2e^{2t}} Z \stackrel{\mathrm{d}}{=} [0, \sigma^2, 0] = N(0, \sigma^2)$. By an easy calculation, $\mathbf{P}[\int_{(t,\infty)} e^{-s} dY(s) \in \cdot]$ satisfy *LDP* on $\mathbf{R}$, as $t \to \infty$, with the speed $\lambda(t) = 2e^{2t}$ and the rate $I(x) = x^2/2\sigma^2$, $x \in \mathbf{R}$ (well known!).

Corollary 4. *If* $Y(1) \stackrel{\mathrm{d}}{=} [0, \sigma^2, M]$ *and* $\operatorname{supp} M$ *is compact then*

$$\mathbf{P}[Z(t) = \int_{(t,\infty)} e^{-s} dY(s) \in \cdot]$$

satisfy LDP (on $\mathbf{R}$*) with speed* $\lambda(t) := te^t$ *and the rate function* $I_M(x)$ *(i.e., Gaussian part does not contribute to the rate function).*

P r o o f. Let us write $Y(t) = Y^1(t) + Y^2(t)$, where Y^1 and Y^2 are independent Lévy processes such that $Y^1(1) \stackrel{\mathrm{d}}{=} [0, \sigma^2, 0]$ and $Y^2(1) \stackrel{\mathrm{d}}{=} [0, 0, M]$. Defining

$$Z^i(t) := \int_{(t,\infty)} e^{-s} dY^i(s) , \quad i = 1, 2$$

one has $Z(t) = Z^1(t) + Z^2(t)$ (with two independent summands). By Remark 3, $\mathbf{P}(Z^1(t) \in \cdot]$ satisfy *LDP* with the speed $2e^{2t} >> te^t$ (as $t \to \infty$), it is then negligible for the large deviations with speed te^t. Consequently, $\mathbf{P}[Z(t) \in \cdot]$ satisfy the same *LDP* as $\mathbf{P}[Z^2(t) \in \cdot]$. Then the Corollary follows from Theorem 4.

Since in the theory of large deviations often one needs the existence of exponential moments we complete this section with the following facts about class L distributions (on Banach spaces).

Lemma. *Let* $Y(1) \stackrel{\mathrm{d}}{=} [a, R, M]$, Y *be a Banach space valued Lévy process. Then for any* $\lambda > 0$

$$\text{(4.6)} \qquad \mathbf{E}[\exp \lambda \| \int_0^\infty e^{-s} dY(s) \|] < \infty \quad \textit{iff} \quad \int_{[\|x\| > a]} \|u\|^{-1} e^{\lambda \|u\|} dM(u) < \infty$$

for all $a > 0$. *In particular, it is so whenever one has* $\mathbf{E}[\exp \lambda \|Y(1)\|] < \infty$.

P r o o f. For an infinitely divisible measure $\nu = [b, S, K]$ (on a Banach space B) and submultiplicative (or subadditive) functions $\Phi : E \to [0, \infty)$

$$\int_B \Phi(\|x\|) \nu(dx) < \infty \quad \text{iff} \quad \int_{\|x\| > a} \Phi(\|x\|) K(dx) < \infty , \quad \text{for all} \quad a > 0 .$$

(see: [JM], p. 36). If M is the Lévy spectral measure of $Y(1)$, the integral $\int_{(0,\infty)} e^{-s} dY(s)$ (class L distribution) has the Lévy spectral measure $\overline{M}$ given by

$$\overline{M}(A) := \int_0^\infty M(e^t A)dt \ , \quad A \text{ is a Borel subset (in } B) \ .$$

(cf. [JM] p.120). Hence

$$\begin{aligned}\int_{[\|x\|\geq a]} e^{\lambda\|x\|} d\overline{M}(x) &= \int_B \int_0^\infty 1_{[\|x\|\geq a]}(e^{-t}x) \exp(\lambda e^{-t}\|x\|) dt M(dx) \\ &= \int_{[\|x\|\geq a]} (\int_0^{ln(\|u\|/a)} \exp(\lambda e^{-t}\|u\|) dt M(du) \\ &= \int_{[\|x\|\geq a]} (\int_a^{\|u\|} (e^{\lambda s}/s) ds) M(du) \ .\end{aligned}$$

Since $\int_a^s e^{\lambda y}/y dy \sim e^{\lambda s}/s$, as $s \to \infty$, and M is finite on $[\|x\| > a]$ we conclude the proof of Lemma.

Corollary 5. *For* $Y(1) \stackrel{\mathrm{d}}{=} [0, \sigma^2, M]$ *on* $\mathbf{R}$ *let us assume that the limits*

$$\gamma^+ := -\overline{\lim}_{a\to+\infty} a^{-1} \log M(x > a) \ , \quad \gamma^- := -\overline{\lim}_{a\to+\infty} a^{-1} \log M(x < -a)$$

are finite and strictly positive. Then $\mathbf{P}[Z(t) = \int_{(t,\infty)} e^{-s} dY(s) \in \cdot]$ *satisfy LDP (on* $\mathbf{R}$*) with the speed* $\lambda(t) := e^t$ *and the rate function* $I(x) := \gamma^+ x$*, for* $x \geq 0$ *and* $I(x) := -\gamma^- x$*, for* $x < 0$*.*

P r o o f. As in the proof of Corollary 4 (or Theorem 4) we can assume that $\sigma^2 = 0$ and

$$Z(t) \stackrel{\mathrm{d}}{=} e^{-t} Z(0) = e^{-t} \int_0^\infty \int_{\mathbf{R}^*} e^{-s} x \widetilde{N}(ds, dx) \ , \quad t \geq 0$$

where $\rho(ds, dx) := ds \times M(dx)$ is the compensator. Observe for $f(s, x) := e^{-s}x$,

$$\begin{aligned}\overline{\lim}_{a\to+\infty} a^{-1} \log \rho((s,x) : \ e^{-s}x > a) &= \sup\left\{\lambda > 0 : \int_{[f\geq 1]} e^{\lambda f} d\rho < +\infty\right\} \\ &= \sup\left\{\lambda > 0 : \int_{[x>1]} e^{\lambda x}/x M(dx) < \infty\right\} \\ &= \sup\left\{\lambda > 0 : \int_{[x>1]} e^{\lambda x} M(dx) < \infty\right\} \\ &= \overline{\lim}_{a\to+\infty} a^{-1} \log M(x > a) = -\gamma^+ \ .\end{aligned}$$

Consequently we proved that

$$\overline{\lim}_{a\to\infty} a^{-1} \log \rho((s,x) : \ e^{-s}x > a) = -\gamma^+ \ , \tag{4.7}$$

and by similar arguments we also have

$$\overline{\lim}_{a\to+\infty} a^{-1} \log \rho((s,x):\ e^{-s}x < -a) = -\gamma^{-}\ . \tag{4.8}$$

Now applying Theorem 3 we conclude the *LDP* described in Corollary 5.

Remark 4. The two main results of this paper, Theorem 2 and 3, show that the behavior of the tail probability of $\widetilde{N}(f)$ (an element in the first chaos of the Poisson point process N), depends strongly on that of f. This is essentially different from the Brownian Motion case. A further interesting question is to investigate the large deviations of multiple random integrals (or element in the chaos of order ≥ 2), similarly to the work of Ledoux [L] on the Wiener space.

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