SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

HIDEATSU TSUKAHARA

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Séminaire de probabilités (Strasbourg), tome 33 (1999), p. 397-404

<http://www.numdam.org/item?id=SPS_1999__33__397_0>

A LIMIT THEOREM FOR THE PREDICTION PROCESS UNDER ABSOLUTE CONTINUITY

HIDEATSU TSUKAHARA

Abstract

Consider a stochastic process with two probability laws, one of which is absolutely continuous with respect to the other. Under each law, we look at a process consisting of the conditional distributions of the future given the past. Blackwell and Dubins showed in discrete case that those conditional distributions merge as we observe more and more; more precisely, the total variation distance between them converges to 0 a.s. In this paper we prove its extension to continuous time case using the prediction process of F. B. Knight.

1. Introduction. Let $(\mathbb{E}_n, \mathcal{E}_n)$, $n \in \mathbb{N}$ be measurable Lusin spaces and put $(\mathbb{E}, \mathcal{E}) = (\mathbb{E}_1 \times \mathbb{E}_2 \times \cdots, \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \cdots)$. Suppose that μ and ν are probability measures on $(\mathbb{E}, \mathcal{E})$ satisfying $\nu \ll \mu$. We then denote by $Z_n^{\mu}(x_1, \ldots, x_n)(\bullet)$ and $Z_n^{\nu}(x_1, \ldots, x_n)(\bullet)$ the regular conditional distributions for the future $(\mathbb{E}_{n+1} \times \cdots)$ given the past $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n$ under μ and ν respectively. Blackwell and Dubins (1962) showed that those conditional distributions merges as n becomes large; more precisely, the total variation distance between them converges to 0 ν-a.s. as $n \to \infty$. In what follows, we prove its extension to continuous time case using the prediction process of F. B. Knight (1975, 1992).

We start with introducing the prediction process, which consists of suitable versions of conditional distributions of the future given the past in continuous time setting. For our purpose, it is unnecessary to make any topological assumption on the state space. Thus we need only the prediction process in a measure-theoretic setting as developed in Chapter 1 of Knight (1992). Let $(\mathbb{E}, \mathcal{E})$ be a *measurable* Lusin space and $\mathbb{M}_{\mathbb{E}}$ the space of $\mathcal{B}_+/\mathcal{E}$ measurable functions as before. Let us define the *pseudo-path* filtration $(\mathcal{F}'_t)_{t \in \mathbb{R}_+}$ by

$$\mathcal{F}'_t \triangleq \sigma\left(\int_0^s f(w(u))\, du,\ s < t,\ f \in b\mathcal{E} \right),$$

and put $\mathcal{F}' = \mathcal{F}'_\infty = \bigvee_{t>0} \mathcal{F}'_t$. Note that each $\mathcal{F}'_t$ is countably generated and satisfies $\mathcal{F}'_{t-} = \mathcal{F}'_t$ for $t > 0$. We denote by Π the set of probability measures on $(\mathbb{M}_{\mathbb{E}}, \mathcal{F}')$, and set $\mathcal{G} \triangleq \sigma(z(A),\ A \in \mathcal{F}')$. The shift operator θ_t on $\mathbb{M}_{\mathbb{E}}$ is defined by $\theta_t w(s) = w(t+s)$ and is $\mathcal{F}'_{t+s}/\mathcal{F}'_s$ measurable for all $s, t \in \mathbb{R}_+$.

It is shown in Chapter 1 of Knight (1992) that the prediction process $(Z_t^z)_{t\in\mathbb{R}_+}$ on $(\mathbb{M}_{\mathbb{E}}, \mathcal{F}')$ is P^z-a.s. uniquely defined by the requirements:

(i) The mapping $(z,s,w) \mapsto Z_s^z(\bullet, w)$ on $\Pi \times [0,t] \times \mathbb{M}_{\mathbb{E}}$ is $\mathcal{G} \otimes \mathcal{B}[0,t] \otimes \mathcal{F}'_{t+\epsilon}/\mathcal{G}$ measurable for each $t \in \mathbb{R}_+$ and each $\epsilon > 0$.

(ii) For any $(\mathcal{F}'_{t+})$-optional T and $A \in \mathcal{F}'$, $Z_T^z(A) = P^z(\,\theta_T^{-1}A \mid \mathcal{F}'_{T+}\,)$ on $\{T < \infty\}$.

Analogously, the left-limit prediction process $(Z_{t-}^z)_{t>0}$ on $(\mathbb{M}_{\mathbb{E}}, \mathcal{F}')$ is P^z-a.s. uniquely defined by the requirements:

(i) The mapping $(z,s,w) \mapsto Z_{s-}^z(\bullet, w)$ on $\Pi \times [0,t] \times \mathbb{M}_{\mathbb{E}}$ is $\mathcal{G} \otimes \mathcal{B}[0,t] \otimes \mathcal{F}'_t/\mathcal{G}$ measurable for each $t > 0$.

(ii) For any $(\mathcal{F}'_t)$-predictable $T > 0$ and $A \in \mathcal{F}'$, $Z_{T-}^z(A) = P^z(\,\theta_T^{-1}A \mid \mathcal{F}'_T\,)$ on $\{T < \infty\}$.

We note that even when the space $\mathbb{E}$ has been given the prescribed Lusin topology, the processes (Z_t^z) and (Z_{t-}^z) are not related to each other through that topology of $\mathbb{E}$ (see Knight (1992)).

Furthermore, employing the notation of Meyer (1976), we define the processes K_t^z and K_{t-}^z by

$$K_t^z(f \circ \theta_t) = Z_t^z(f), \quad K_{t-}^z(f \circ \theta_t) = Z_{t-}^z(f)$$

for $f \in b\mathcal{F}'$. Hence Π is the state space of the K^z's and they satisfy, besides measurability conditions,

$$K_T^z(A) = P^z(\,A \mid \mathcal{F}'_{T+}\,) \quad \text{on } \{T < \infty\},$$

for any $(\mathcal{F}'_{t+})$-optional T and $A \in \mathcal{F}'$, and

$$K_{T-}^z(A) = P^z(\,A \mid \mathcal{F}'_T\,) \quad \text{on } \{T < \infty\},$$

for any $(\mathcal{F}'_t)$-predictable $T > 0$ and $A \in \mathcal{F}'$,

Following Meyer (1976), we define the optional and predictable σ-fields as follows. The optional σ-field $\mathcal{O}$ is generated by the càdlàg processes adapted to $(\mathcal{F}'_{t+})$, and the predictable σ-field $\mathcal{P}$ is generated by the left continuous processes adapted to $(\mathcal{F}'_{t-})$. The utility of K_t^z and K_{t-}^z lies in the following result due to Meyer (1976).

Proposition 1.1 *For every bounded measurable process X on $(\mathbb{M}_{\mathbb{E}}, \mathcal{F}')$ and for every $z \in \Pi$,*

$$(t,\omega) \mapsto \int K_t^z(dw,\omega)\, X_t(w)$$

defines an optional projection of X *for* z, *and*

$$(t,\omega) \mapsto \int K^z_{t-}(dw,\omega)\, X_t(w)$$

defines a predictable projection of X *for* z.

A simple monotone class argument proves the above theorem. We can actually improve on this result, using a similar monotone class argument. This is also due to Meyer (1976).

Proposition 1.2 *Let* $X(t,\omega,t',\omega')$ *be a bounded function that is* $\mathcal{O}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}'$ *measurable. Then an optional projection for* z *of the process* $X(t,\omega,t,\omega)$ *is given by*

$$(t,\omega) \mapsto \int K^z_t(dw,\omega)\, X(t,\omega,t,w).$$

Similarly, if $X(t,\omega,t',\omega')$ *is a bounded function that is* $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}'$ *measurable, then a predictable projection for* z *of the process* $X(t,\omega,t,\omega)$ *is given by*

$$(t,\omega) \mapsto \int K^z_{t-}(dw,\omega)\, X(t,\omega,t,w).$$

Remark 1.3 In Dellacherie and Meyer (1980), VI.43, optional and predictable projections are defined under the "usual conditions". Here we are not assuming them, but in view of Lemma 7 of Dellacherie and Meyer (1980), Appendix I, we can choose a version of the optional projection which is optional relative to $(\mathcal{F}'_{t+})$, and a version of the predictable projection which is predictable relative to $(\mathcal{F}'_t)$. Thus according to our definition of the optional and predictable σ-fields , no complications on those projections arise.

2. Main result. For $z,z'\in\Pi$, the *total variation distance* $\rho_{TV}(z,z')$ on Π is defined by

$$\rho_{TV}(z,z') \triangleq \sup_{A\in\mathcal{F}'} |z(A)-z'(A)|.$$

Our main result is the following theorems.

Theorem 2.1 *Let* z *and* z' *be two probabilities on* $(\mathbb{M}_{\mathbb{E}},\mathcal{F}')$ *satisfying* $z'\ll z$ *and let* (Z^z_t) *and* $(Z^{z'}_t)$ *be their prediction processes. Then the total variation distance between* (Z^z_t) *and* $(Z^{z'}_t)$ *converges to 0 as* $t\to\infty$, $P^{z'}$-a.s.

Theorem 2.2 *Let z and z' be two probabilities on $(\mathbb{M}_{\mathbb{E}}, \mathcal{F}')$ satisfying $z' \ll z$ and let (Z^z_{t-}) and $(Z^{z'}_{t-})$ be their left-limit prediction processes. Then the total variation distance between (Z^z_{t-}) and $(Z^{z'}_{t-})$ converges to 0 as $t \to \infty$. $P^{z'}$-a.s.*

To prove the theorems, we need some preliminary results. Let $L(w) = \frac{dz'}{dz}(w)$. We would like to show first that $K^{z'}_{t-}$ $[K^{z'}_t]$ is $P^{z'}$-a.s. absolutely continuous with respect to K^z_{t-} $[K^z_t]$ and find a version of Radon-Nikodym derivative. The following general lemma is well known in the filtering theory.

Lemma 2.3 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ a sub-σ-field of $\mathcal{F}$. Suppose that we have another probability measure Q which is absolutely continuous with respect to P. Set $L = \frac{dQ}{dP}$. Then for any Q integrable $\mathcal{F}$ measurable V, we have*

$$E^Q(\, V \mid \mathcal{G}\,) = \frac{E^P(\, LV \mid \mathcal{G}\,)}{E^P(\, L \mid \mathcal{G}\,)}, \quad Q\text{-a.s.}$$

It follows from the above lemma that for any $f \in b\mathcal{F}'_\infty$,

$$E^{z'}(\, f \mid \mathcal{F}'_t\,) = \frac{E^z(\, fL \mid \mathcal{F}'_t\,)}{E^z(\, L \mid \mathcal{F}'_t\,)}, \quad P^{z'}\text{-a.s..}$$

Let $L^z_t \triangleq K^z_t(L)$. Then L^z_t is a càdlàg version of the martingale $E^z(\, L \mid \mathcal{F}'_t\,)$ since it is an optional projection of the process L constant in t (L is not bounded, but it is positive). And we put $L^z_{t-} \triangleq \lim_{s \uparrow\uparrow t} L^z_s = K^z_{t-}(L)$, $t > 0$. By the same reasoning, L^z_{t-} is a predictable projection of L.

Proposition 2.4 *For $P^{z'}$-almost all ω, $K^{z'}_t(dw, \omega)$ is absolutely continuous with respect to $K^z_t(dw, \omega)$ with the Radon-Nikodym derivative $L(w)/L^z_t(\omega)$ for all t. Similarly, for $P^{z'}$-almost all ω, $K^{z'}_{t-}(dw, \omega)$ is absolutely continuous with respect to $K^z_{t-}(dw, \omega)$ with the Radon-Nikodym derivative $L(w)/L^z_t(\omega)$ for all $t > 0$.*

Proof. We give a proof for K^z_t case since K^z_{t-} case can be proved analogously. As in Chapter 1 of Knight (1992), we may assume that $\mathbb{E} = [0, 1]$. There exists a metric on $\mathbb{M}_{[0,1]}$ for which it is compact, and its Borel sets are $\mathcal{F}'$. Since $C(\mathbb{M}_{[0,1]})$ is separable, we can find a sequence (f_j) in $C(\mathbb{M}_{[0,1]})$ that is uniformly dense in $\{f \in C(\mathbb{M}_{[0,1]}) \colon 0 \le f \le 1\}$. By Lemma 2.3, for each $r \in \mathbb{Q}_+$ and $j \in \mathbb{N}$, there is a set $A_{r,j}$ with $P^z(A_{r,j}) = 1$ such that if $\omega \in A_{r,j}$,

$$K^{z'}_r(f_j, \omega) = \int f_j(w) \frac{L(w)}{L^z_r(\omega)} K^z_r(dw, \omega). \tag{2.1}$$

Set

$$A = \left(\bigcap_{r\in\mathbb{Q}_+} \bigcap_{j\in\mathbb{N}} A_{r,j} \right) \cap \left\{ K^{z'}(f_j), K^z(f_jL) \text{ and } K^z(L) \text{ are right continuous} \right\}$$

The processes $K^{z'}(f_j)$, $K^z(f_jL)$ and $K^z(L)$ are all right continuous $P^{z'}$-a.s. since they are optional projections of the processes which are constant in t. Thus we have $P^{z'}(A) = 1$. Since (f_j) is uniformly dense in $\{f \in C(\mathbb{M}_{[0,1]}) : 0 \le f \le 1\}$ and both sides of (2.1) are right continuous for $\omega \in A$, we conclude that if $\omega \in A$,

$$K_t^{z'}(f,\omega) = \int f(w) \frac{L(w)}{L_t^z(\omega)} K_t^z(dw,\omega),$$

for all $f \in C(\mathbb{M}_{[0,1]})$ and all $t \in \mathbb{R}_+$. In view of the fact that the continuous bounded functions separate the measures on a metric space, we see that the above is true for all $f \in b\mathcal{F}'$ and all $t \in \mathbb{R}_+$, which implies our assertion. ■

To get a similar result for the Z^z, we need the splicing operator on $\mathbb{M}_\mathbb{E} \times \mathbb{R}_+ \times \mathbb{M}_\mathbb{E}$ onto $\mathbb{M}_\mathbb{E}$ defined by

$$(\omega/t/w)_s \triangleq \begin{cases} \omega(s), & s < t, \\ w(s-t), & s \ge t. \end{cases}$$

The mapping $(\omega, t, w) \mapsto \omega/t/w$ is continuous, so it is $\mathcal{F}' \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}' / \mathcal{F}'$ measurable. It follows from Lemma 4 of Meyer (1976) that for P^z-almost every ω and all t, $K_t^z(dw,\omega)$ and $K_{t-}^z(dw,\omega)$ is concentrated on the atom of $\mathcal{F}_t'$ containing ω, which is the set of $w \in \mathbb{M}_\mathbb{E}$ such that $w(s) = \omega(s)$ for λ-a.e. $s \le t$. Moreover, denoting the mapping $w \mapsto \omega/t/w$ by $\gamma_{\omega,t}(w)$, we have

$$\begin{aligned} K_t^z(f,\omega) &= \int f(w/t/\theta_t w)\, K_t^z(dw,\omega) = \int f(\omega/t/\theta_t w)\, K_t^z(dw,\omega) \\ &= \int f(\omega/t/u)\, Z_t^z(du,\omega) = Z_t^z(f \circ \gamma_{\omega,t}, \omega), \end{aligned}$$

and similarly for K_{t-}^z and Z_{t-}^z. Thus $K_t^z(dw,\omega)$ and $K_{t-}^z(dw,\omega)$ are the image measures of $Z_t^z(dw,\omega)$ and $Z_{t-}^z(dw,\omega)$ respectively under $\gamma_{\omega,t}$. It then follows that $P^{z'}$-a.s.,

$$Z_t^{z'}(f,\omega) = \int f(w) \frac{L(\omega/t/w)}{L_t^z(\omega)} Z_t^z(dw,\omega), \quad f \in b\mathcal{F}',$$

for all $t \in \mathbb{R}_+$, and

$$Z_{t-}^{z'}(f,\omega) = \int f(w) \frac{L(\omega/t/w)}{L_t^z(\omega)} Z_{t-}^z(dw,\omega), \quad f \in b\mathcal{F}',$$

for all $t > 0$.

Proof of Theorem 2.1. It is easy to see that if $z' \ll z$, then, with $\ell = dz'/dz$,

$$\rho_{TV}(z,z') = \int_{\{\ell>1\}} (\ell - 1)\, dz.$$

Using this, we can evaluate the total variation distance between Z_t^z and $Z_t^{z'}$. We have, for any $\epsilon > 0$,

$$\begin{aligned}
\rho_{TV}\left(Z_t^z, Z_t^{z'}\right) &= \int_{\left\{w: \frac{L(\omega/t/w)}{L_t^z(\omega)} - 1 > 0\right\}} \left(\frac{L(\omega/t/w)}{L_t^z(\omega)} - 1\right) Z_t^z(dw,\omega) \\
&\leq \epsilon + \int_{\left\{w: \frac{L(\omega/t/w)}{L_t^z(\omega)} - 1 > \epsilon\right\}} \frac{L(\omega/t/w)}{L_t^z(\omega)} Z_t^z(dw,\omega) \\
&= \epsilon + Z_t^{z'}\left(\left\{w: \frac{L(\omega/t/w)}{L_t^z(\omega)} - 1 > \epsilon\right\}, \omega\right) \\
&= \epsilon + K_t^{z'}\left(\left\{w: \frac{L(\omega/t/\theta_t w)}{L_t^z(\omega)} - 1 > \epsilon\right\}, \omega\right).
\end{aligned}$$

Note that for $P^{z'}$-almost all ω and all t, we have $\omega/t/\theta_t w = w/t/\theta_t w = w$, $K_t^{z'}(\bullet,\omega)$-a.e. Thus we get

$$\rho_{TV}\left(Z_t^z, Z_t^{z'}\right) \leq \epsilon + K_t^{z'}\left(\left\{w: \frac{L(w)}{L_t^z(\omega)} - 1 > \epsilon\right\}, \omega\right). \tag{2.2}$$

Now let $f(x,y) = \mathbf{1}_{\{(x,y):y/x-1>\epsilon\}}(x,y)$ for $(x,y) \in \mathbb{R}^2$. Clearly this is $\mathcal{B}(\mathbb{R}^2)$ measurable. Put $X(t,\omega,w) = f(L_t^z(\omega), L(w))$, so that we have

$$X(t,\omega,w) = \mathbf{1}_{\left\{(t,\omega,w): \frac{L(w)}{L_t^z(\omega)} - 1 > \epsilon\right\}}(t,\omega,w).$$

$(t,\omega) \mapsto L_t^z(\omega)$ is $\mathcal{O}$ measurable since $L_t^z(\omega) = K_t^z(L,\omega)$ is the optional projection of L for P^z. Also L is $\mathcal{F}'$ measurable. Thus $(t,\omega,w) \mapsto (L_t^z(\omega), L(w))$ is $\mathcal{O} \otimes \mathcal{F}'/\mathcal{B}(\mathbb{R}^2)$ measurable, and hence $X(t,\omega,w)$ is $\mathcal{O} \otimes \mathcal{F}'$ measurable. It follows from Proposition 1.2 that

$$\int K_t^{z'}(\omega,dw)\, X(t,\omega,w) = K_t^{z'}\left(\left\{w: \frac{L(w)}{L_t^z(\omega)} - 1 > \epsilon\right\}, \omega\right) \tag{2.3}$$

is the optional projection of the process $X(t, \omega, \omega)$.

By the martingale convergence theorem, as $t \to \infty$, L_t^z converges to L, P^z-a.s. and hence $P^{z'}$-a.s. Since $L > 0$, $P^{z'}$-a.s., L/L_t^z converges to 1, $P^{z'}$-a.s. as $t \to \infty$. This implies that $X(t, \omega, \omega) \to 0$, $P^{z'}$-a.s. Finally, by Dellacherie and Meyer (1980), VI.50 c), the optional projection of $X(t, \omega, \omega)$, i.e. (2.3) converges to 0, $P^{z'}$-a.s. Therefore the right-hand side of (2.3) converges to 0 as $t \to \infty$, $P^{z'}$-a.s., and the theorem follows from (2.2). ■

By an entirely analogous argument using the predictable counterparts, one can prove Theorem 2.2. When $z' \ll z$ does not necessarily hold, we still have the following.

Corollary 2.5 *Let $T = \inf\{t \geq 0 : Z_t^{z'} \ll Z_t^z\}$, where z and z' are* **any** *two probabilities on $(\mathrm{M}_\mathrm{E}, \mathcal{F}')$. Then the conclusions of Theorem 2.1 and Theorem 2.2 holds on $\{T < \infty\}$.*

Proof. Since Π is the set of probability measures, the set $\Lambda = \{(z_1, z_2) \in \Pi^2 : z_1 \ll z_2\}$ belongs to $\mathcal{G}^2$. T is then the first entrance time into Λ by the process $(Z_t^{z'}, Z_t^z)$, so it is $\mathcal{F}'_{t+}$-optional.

Next we quote the following result due to Yor and Meyer (1976): For any $\mathcal{F}'_{t+}$-optional $S < \infty$ and any $z \in \Pi$, $Z_{S+t}^z(\bullet, w) = Z_t^{Z_S^z(w)}(\bullet, \theta_S w)$ for all $t \geq 0$,P^z-a.s. The corollary then easily follows from this, together with Theorems 2.1 and 2.2, and the section theorem, ■

Acknowledgements. The author would like to thank Frank Knight, Catherine Doléans-Dade and Ditlev Monrad for their helpful comments.

REFERENCES

Blackwell, D. and Dubins, L. (1962). Merging of opinions with increasing information, *Ann. Math. Statist.* **33**, 882-886.

Dellacherie, C and Meyer, P.-A. (1980). *Probabilités et Potentiel: Théorie des Martingales*, Chapitres V à VIII, Hermann, Paris.

Knight, F.B. (1975). A predictive view of continuous time processes, *Ann. Probab.* **3**, 573-596.

Knight, F.B. (1992). *Foundations of the Prediction Process*, Oxford University Press, Oxford.

Meyer, P.-A. (1976). La théorie de la prédiction de F. Knight, *Séminaire de Probabilités X, Lect. Notes in Math.* **511**, Springer-Verlag, Berlin Heidelberg New York, 86-103.

Yor, M. and Meyer, P.-A. (1976). Sur la théorie de la prédiction, et le problème de décomposition des tribus $\mathcal{F}^{\circ}_{t+}$, *Séminaire de Probabilités X, Lect. Notes in Math.* **511**, Springer-Verlag, Berlin Heidelberg New York, 104-117.