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A short proof of decomposition of strongly reduced martingales

Michał Morayne and Krzysztof Tabisz

Abstract

A short proof of the following theorem is given: If M is a martingale, $T > 0$ is a stopping time, $M = M^T$ and $E(|M_T||\mathcal{F}_t)1_{[t<T]}$ is bounded, then M is a sum of a BMO (and, thus, square-integrable) martingale and a martingale of integrable variation.

The purpose of this note is to give a short proof of P.A. Meyer's theorem ([Me]) stated in the abstract. Although the proof given here is very much in the spirit of that of [Me] it seems to be simpler and shorter (in particular, we do not use potentials and Riesz decomposition in the proof). The proof presented here reduces to a sequence of easy inequalities. A shortcut has been possible because of the fact that the dual predictable projection of a reduced process is reduced by the same stopping time.

Let $\mathbf{R}_+ = [0, \infty)$. Let $(\mathcal{F}_t, t \in \mathbf{R}_+)$ be a fixed right-hand side continuous complete filtration. We shall consider martingales with respect to this filtration, assuming always that they are CADLAG (i.e. that almost all their trajectories are right-hand side continuous and have left-hand side finite limits). For a process X by X_∞ we denote $\lim_{t\to\infty} X_t$, when such a limit exists a.s.

BMO denotes the class of those uniformly integrable martingales M for which $EM_\infty^2 < \infty$ and there exists a constant c such that for each stopping time S the following inequality is satisfied: $E((M_\infty - M_{S^-})^2|\mathcal{F}_S) < c$. $\mathcal{A}$ denotes the class of the processes of integrable variation i.e. $A \in \mathcal{A}$ if $E\,\mathrm{Var}|_0^\infty A_t < \infty$.

For a stopping time T we shall put $[[0,T]] = \{(\omega,t) : t \leq T(\omega)\}$, $[[T,\infty)) = \{(\omega,t) : t \geq T(\omega)\}$, $((T,\infty)) = \{(\omega,t) : t > T(\omega)\}$ and $[[T]] = \{(\omega,t) : T(\omega) = t\}$.

We say that a process X is *reduced* by a stopping time T if $X^T = X$, where $X_t^T = X_{\min(t,T)}$. If, in addition, $T > 0$ and for some constant C we have $\sup_{t<T} E(|X_T||\mathcal{F}_t) < C$ a.s. we say that X is *strongly reduced* by T.

We are going to prove the following theorem ([Me], Chap. IV, 8, p. 294 and Chap. V, 5c, p. 335).

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Theorem. If $T > 0$ is a stopping time and M is a martingale strongly reduced by T, then M can be expressed as a sum $M = N + A$, where $N \in BMO$ and $A \in \mathcal{A}$.

The following lemma is crucial for our proof.

Lemma. Let $T > 0$ be a stopping time, $\Phi \in L_1$, Φ be $\mathcal{F}_T$ - measurable, $E(|\Phi||\mathcal{F}_t)$ be strongly reduced by T, U be the dual predictable projection of the process $\Phi \mathbf{1}_{[[T,\infty))}$. Then there exists a constant c such that for each stopping time S

$$E((U_\infty - U_{S-})^2|\mathcal{F}_S) < c.$$

Proof. First we shall show that there exists such a constant c that for each stopping time S

$$E((U_\infty - U_S)^2|\mathcal{F}_S) < c. \tag{1}$$

Let W be the dual predictable projection of the process $|\Phi|\mathbf{1}_{[[T,\infty))}$. Let $P_t = E(|\Phi||\mathcal{F}_t)$. We have for a set $B \in \mathcal{F}_S$:

$$E((U_\infty - U_S)^2\mathbf{1}_B) \le E((W_\infty - W_S)^2\mathbf{1}_B) \le 2E\int_0^\infty \mathbf{1}_{((S,\infty))}\mathbf{1}_B(W_t - W_S)dW_t$$

$$= 2E\int_0^\infty \mathbf{1}_{((S,\infty))}\mathbf{1}_B(W_t - W_S)d(|\Phi|\mathbf{1}_{[[T,\infty))})_t = 2E(\mathbf{1}_B|\Phi|(W_T - W_S)) =$$

$$2E\int_0^\infty \mathbf{1}_{((S,\infty))}\mathbf{1}_B|\Phi|dW_t = 2E\int_0^\infty \mathbf{1}_{((S,\infty))}\mathbf{1}_B P_{t-}dW_t$$

$$(= 2E\int_0^\infty \mathbf{1}_{((S,\infty))}\mathbf{1}_B P_{t-}d(|\Phi|\mathbf{1}_{[[T,\infty))})_t \le 2E(P_{T-}|\Phi|) \le 2CE|\Phi| < \infty)$$

$$\le 2CE(\mathbf{1}_B\mathbf{1}_B(W_\infty - W_S)) \le 2C(P(B))^{1/2}(E(\mathbf{1}_B(W_\infty - W_S)^2))^{1/2}.$$

To get the second and the third equality and the next inequality we used the fact that the process W is reduced by T. The last step was possible by virtue of Schwarz inequality.

Comparing the second and the last term of the sequence of (in)equalities above and repeating the first inequality we get

$$E((U_\infty - U_S)^2\mathbf{1}_B) \le 4C^2P(B)$$

and this gives (1).

We still need to show that all the jumps of the process U are uniformly bounded. First let us notice that the process U, being predictable, possibly jumps only at countable number of graphs of predictable stopping times. So let us assume that S is a predictable stopping time which means that there exists a nondecreasing sequence of stopping times S_n such that $S_n < S$ on the set $[S > 0]$ and $\lim_n S_n = S$. Let $B \in \mathcal{F}_S$. Let $R_t = E(\mathbf{1}_B|\mathcal{F}_t)$. We have:

$$E(\mathbf{1}_B|U_S - U_{S-}|) \le E(\mathbf{1}_B(W_S - W_{S-})) = E\int_0^\infty \mathbf{1}_{[[S]]}\mathbf{1}_B dW_t =$$

$$E\int_0^\infty \mathbf{1}_{[[S]]} R_{t}\text{-}\, d(|\Phi|\mathbf{1}_{[[T,\infty]]})_t = E(\mathbf{1}_{[S=T]} R_{S}\text{-}\, |\Phi|) = EE(\mathbf{1}_{[S=T]} R_{S}\text{-}\, |\Phi||\mathcal{F}_{S}\text{-}) =$$

$$E(R_{S}\text{-}\, E(\mathbf{1}_{[S=T]}|\Phi||\mathcal{F}_{S}\text{-})) = E(R_{S}\text{-}\, \lim_n E(\mathbf{1}_{[S=T]}|\Phi||\mathcal{F}_{S_n})) \leq$$

$$E(R_{S}\text{-}\, \sup_n(\mathbf{1}_{[S_n<T]} P_{S_n})) \leq CER_{S}\text{-} = CP(B).$$

This finishes the proof.

Now we can give a proof of Theorem.

Proof of Theorem. Let U now denote the dual predictable projection of $M_T \mathbf{1}_{[[T,\infty))}$. The decomposition of M is defined as in [Me] as $M = N + A$, where $N = M - M_T\mathbf{1}_{[[T,\infty))} + U$ and $A = M_T\mathbf{1}_{[[T,\infty))} - U$ and we apply Lemma to get $N \in BMO$. It is obvious that $A \in \mathcal{A}$. This finishes the proof.

References

[Me] P.A. Meyer, *Un cours sur les integrales stochastiques*, Séminaire de Probabilités X, Lecture Notes in Mathematics 511, Berlin, Heidelberg, New York 1976.