Séminaire de probabilités (Strasbourg)

Shinzo Watanabe

The existence of a multiple spider martingale in the natural filtration of a certain diffusion in the plane

Séminaire de probabilités (Strasbourg), tome 33 (1999), p. 277-290

<http://www.numdam.org/item?id=SPS_1999__33__277_0>

The Existence of a Multiple Spider Martingale in the Natural Filtration of a Certain Diffusion in the Plane

Shinzo Watanabe, Kyoto University

Introduction

The notion of spider martingales (martingales-araignées) with n rays, $n = 2, 3, \ldots, \infty$, has been introduced by Yor ([Y]) by generalizing Walsh's Brownian motions. A spider martingale with 2 rays is essentially a continuous local martingale and, on the other hand, a non-trivial spider martingale with $n \geq 3$ rays is called a *multiple spider martingale.* By the recent works by Tsirelson ([T]) and Barlow, Emery, Knight, Song and Yor ([BEKSY]), it has been recognized that a multiple spider martingale plays an important role in distinguishing a filtration from a Brownian filtration or, more generally, from a filtration which is homomorphic to a Brownian filtration; that is, any filtration which is homomorphic to a Brownian filtration can not contain a multiple spider martingale. In other words, as a noise generating the randomness of probability models, multiple spider martingales could sometimes provide us with a more useful information than a usual martingale could do. So it seems important to study, for a given stochastic process, if there exists a multiple spider martingale or not in its natural filtration. For the convenience of readers, we give in Section 1 a brief survey on the isomorphism problem of filtrations in connection with spider martingales.

The filtration of a smooth diffusion, "smooth" in the sense that it can be obtained as a strong solution of an Itô's stochastic differential equation (SDE) driven by a Wiener process, can not contain a multiple spider martingale because, as a strong solution of SDE, the natural filtration of the diffusion is homomorphic to the Brownian filtration generated by the driving Wiener process. On the other hand, the natural filtration of a Walsh's Brownian motion on $n \geq 3$ rays is a typical (and, indeed, a trivial) example of a filtration containing a multiple spider martingale. In Section 2, as a main purpose of this note, we give a less trivial example of a diffusion process on the plane $\mathbf{R}^2$ whose natural filtration contains a multiple spider martingale. This diffusion process has been studied by Ikeda and Watanabe ([IW 1] or [IW 2]) as an example of diffusions whose infinitesimal generators are not differential operators in the classical sense.

1 The isomorphism problem of filtrations

As we said in Introduction, we give here a summary of recent important results on the isomorphism problem of filtrations by Tsirelson ([T]) and Barlow, Emery, Knight, Song and Yor ([BEKSY]). No proofs are given. The reader is recommended to refer to [T] and [BEKSY] for proofs and more details.

As usual, by a *filtration* $\mathbf{F} = (F(t))_{t \in [0,\infty)}$ on a complete probability space (Ω, F, P), we mean an increasing family of sub σ-fields of F satisfying the usual conditions, that is, it is right-continuous and $F(0)$ contains all P-null sets. We set $F(\infty) = \bigvee_{t\in[0,\infty)} F(t)$. Let $\mathbf{F}$ be a filtration on (Ω, F, P) and $\mathbf{F}'$ be another filtration on (Ω', F', P').

Definition 1.1. *By a morphism π from $\mathbf{F}$ to $\mathbf{F}'$, denoted by $\pi : \mathbf{F} \to \mathbf{F}'$, we mean a map*

$$\pi_* : L^0(\Omega'; F'(\infty)) \longrightarrow L^0(\Omega; F(\infty))$$

satisfying the following conditions (i)~(iii). ($L^0(\Omega; F(\infty))$, or $L^0(F(\infty))$, stands for the real vector space formed of all $F(\infty)$-measurable real random variables on Ω.)

(i) For any $X_1, \ldots, X_n \in L^0(\Omega'; F'(\infty))$,

$$([X_1, \ldots, X_n],\ P') \stackrel{d}{=} ([\pi_*(X_1), \ldots, \pi_*(X_n)],\ P).$$

(ii) For any $X_1, \ldots, X_n \in L^0(\Omega'; F'(\infty))$ and $f : \mathbf{R}^n \to \mathbf{R}$ which is Borel measurable,

$$\pi_*[f(X_1, \ldots, X_n)] = f(\pi_*(X_1), \ldots, \pi_*(X_n)).$$

(iii) For any $X \in L^1(\Omega'; F'(\infty))$, (then, obviously $\pi_*(X) \in L^1(\Omega; F(\infty))$,)

$$\pi_*[E(X|F'(t)] = E[\pi_*(X)|F(t)], \quad \text{for all} \quad t \geq 0.$$

A morphism is also called a homomorphism; *we say that* $\mathbf{F}'$ is homomorphic to $\mathbf{F}$ *if there exists a morphism π from $\mathbf{F}$ to $\mathbf{F}'$.*

Note that the map π_* is obviously one-to-one and *non-anticipative* in the sense that, for every $t \geq 0$, $\pi_*(X)$ is $F(t)$-measurable if X is $F'(t)$-measurable.

Definition 1.2. *A morphism π from $\mathbf{F}$ to $\mathbf{F}'$ is called an* isomorphism *from $\mathbf{F}$ to $\mathbf{F}'$ if $\pi_* : L^0(\Omega'; F'(\infty)) \longrightarrow L^0(\Omega; F(\infty))$ is onto. We say that* $\mathbf{F}'$ is isomorphic to $\mathbf{F}$ *or,* $\mathbf{F}$ and $\mathbf{F}'$ are isomorphic, *if there exists an isomorphism π from $\mathbf{F}$ to $\mathbf{F}'$.*

Remark 1.1. *Since the map π_* is one-to-one, we can say equivalently as follows: $\mathbf{F}$ and $\mathbf{F}'$ are isomorphic if and only if there exists a morphism π from $\mathbf{F}$ to $\mathbf{F}'$ and a morphism π' from $\mathbf{F}'$ to $\mathbf{F}$ such that $\pi' \circ \pi = \mathrm{id}$ and $\pi \circ \pi' = \mathrm{id}$, i.e., $\pi_* \circ \pi'_* = \mathrm{id}_*$ on $L^0(\Omega; F(\infty))$ and $\pi'_* \circ \pi_* = \mathrm{id}_*$ on $L^0(\Omega'; F'(\infty))$.*

Generally, for two filtrations $\mathbf{F} = (F(t))$ and $\mathbf{G} = (G(t))$ on the same probability space (Ω, F, P), we denote $\mathbf{G} \subset \mathbf{F}$ if $G(t) \subset F(t)$ for all $t \geq 0$.

A probability space (Ω, F, P) endowed with a filtration $\mathbf{F}$ is called a *filtered probability space* and is denoted by $\{(\Omega, F, P), \mathbf{F}\}$.

Definition 1.3. *For an $\mathbf{F}$-adapted stochastic process $X = (X(t))$ on $\{(\Omega, F, P), \mathbf{F}\}$ and an $\mathbf{F}'$-adapted stochastic process $Y = (Y(t))$ on $\{(\Omega', F', P'), \mathbf{F}'\}$, we say that Y has a* canonical representation *by X if the natural filtration $\mathbf{F}^Y (\subset \mathbf{F}')$ of Y is homomorphic to the natural filtration $\mathbf{F}^X (\subset \mathbf{F})$ of X.*

Definition 1.4. *If, in Def. 1.3, $\mathbf{F}^X$ and $\mathbf{F}^Y$ are isomorphic, then we say that Y has a* properly canonical representation *by X.*

We denote by $\mathcal{M}(\mathbf{F})$ the space of all locally square-integrable $\mathbf{F}$-martingales $M = (M(t))$ with $M(0) = 0$.

Proposition 1.1. *(1) Y has a canonical representation by X if and only if*

$$\exists\, Y' \stackrel{d}{=} Y \quad \text{such that} \quad \mathbf{F}^{Y'} \subset \mathbf{F}^X \quad \text{and} \quad \mathcal{M}(\mathbf{F}^{Y'}) \subset \mathcal{M}(\mathbf{F}^X).$$

(2) Y has a properly canonical representation by X if and only if

$$\exists\, Y' \stackrel{d}{=} Y \quad \text{such that} \quad \mathbf{F}^{Y'} = \mathbf{F}^X, \quad (\text{then, obviously, } \mathcal{M}(\mathbf{F}^{Y'}) = \mathcal{M}(\mathbf{F}^X).)$$

Remark 1.2. *By Proposition 1.1, we can see clearly that the notions of the canonical and properly canonical representations exactly correspond to Hida's ([H]) (in the case of linear representations of Gaussian processes by a Wiener process) and Nisio's ([N]) (in the case of nonlinear representations of general stochastic processes by a Wiener process).*

Remark 1.3. *We say that a map $\pi_* : L^0(F'(\infty)) \longrightarrow L^0(F(\infty))$ is a* morphism in the weak sense *if it satisfies the same conditions as in Def. 1.1 in which* (iii) *is replaced by:*
(iii)' If $X \in L^1(F'(\infty))$ is $F'(t)$-measurable, then $\pi_*(X)$ is $F(t)$-measurable for every $t \geq 0$.
The existence of a weak morphism corresponds, in the case of stochastic processes, to the condition: $\exists\, Y' \stackrel{d}{=} Y$ such that $\mathbf{F}^{Y'} \subset \mathbf{F}^X$. In such a case, we say that Y has a non-anticipative representation *by X. However, this notion is very weak compared to that of canonical or properly canonical representation. Indeed, denoting by $BM^0(m)$ an m-dimensional standard Brownian motion starting at 0, if $X = BM^0(m)$ and $Y = BM^0(n)$, then a non-anticipative representation of Y by X exists for any m and n. However, a canonical representation of Y by X exists if and only if $n \leq m$, and a properly canonical representation of Y by X exists if and only if $n = m$. (These facts follow immediately from Theorem 1.1 and its Corollary given below since the multiplicity of the natural filtration of $X = BM^0(m)$ is m.)*

In the problem of existence or non-existence of canonical and properly canonical representations for stochastic processes, or more generally, existence and non-existence of homomorphisms and isomorphisms for filtrations, a useful and well-known invariant is the *multiplicity* or the *rank* of filtrations (cf. Davis-Varaiya ([DV]), Skorohod [S], cf. also, Motoo-Watanabe ([MW]), Kunita-Watanabe ([KW])).

Let $\mathbf{F} = (F(t))$ be a filtration on $(\Omega, \mathcal{F}, P)$. We assume that the filtration is separable in the sense that the Hilbert space $\mathbf{L}_2(\Omega, F(\infty), P)$ is separable.

Theorem 1.1.

(1) There exist $M_1, M_2, \ldots \in \mathcal{M}(\mathbf{F})$ such that

$$\langle M_i, M_j \rangle = 0 \quad \text{if} \quad i \neq j, \quad \langle M_1 \rangle \gg \langle M_2 \rangle \gg \cdots$$

and every $M \in \mathcal{M}(\mathbf{F})$ can be represented as a sum of stochastic integrals for some $\mathbf{F}$-predictable processes Φ_i as

$$M = \sum_i \int \Phi_i dM_i.$$

If $N_1, N_2, \ldots$ is another such sequence, then

$$\langle M_1\rangle \approx \langle N_1\rangle,\ \langle M_2\rangle \approx \langle N_2\rangle,\ \cdots.$$

Such a system $M_1, M_2, \cdots$ is called a basis *of $\mathcal{M}(\mathbf{F})$. Here as usual, $\langle M, N\rangle$ for $M, N \in \mathcal{M}^2(\mathcal{F})$ is the quadratic covariational process, $\langle M\rangle = \langle M, M\rangle$ and, $\gg$ and $\approx$ denote the absolute continuity and the equivalence of increasing processes, respectively.*

In particular, the cardinal of a basis is an invariant of the filtration $\mathbf{F}$ which we call the multiplicity *of $\mathbf{F}$ and denote by* mult($\mathbf{F}$).

(2) *Let $\mathbf{F}' \subset \mathbf{F}$ be a sub-filtration of $\mathbf{F}$ and suppose $\mathcal{M}(\mathbf{F}') \subset \mathcal{M}(\mathbf{F})$. Let $\{M_i\}$ and $\{M_i'\}$ be the basis of $\mathcal{M}(\mathbf{F})$ and $\mathcal{M}(\mathbf{F}')$, respectively. Then*

$$\langle M_1'\rangle \ll \langle M_1\rangle,\ \langle M_2'\rangle \ll \langle M_2\rangle,\ \cdots.$$

In particular, $\text{mult}(\mathbf{F}') \leq \text{mult}(\mathbf{F})$.

Corollary 1.1. *If $\mathbf{F}'$ is homomorphic to $\mathbf{F}$, then* $\text{mult}(\mathbf{F}') \leq \text{mult}(\mathbf{F})$.

Let $\mathcal{M}^c(\mathbf{F})$ be the totality of continuous elements of $\mathcal{M}(\mathbf{F})$. Then the property that $\mathcal{M}^c(\mathbf{F}) = \mathcal{M}(\mathbf{F})$, is an invariant for the existence of homomorphisms: If $\mathcal{M}^c(\mathbf{F}) = \mathcal{M}(\mathbf{F})$ and $\mathbf{F}'$ is homomorphic to $\mathbf{F}$, then we must have $\mathcal{M}^c(\mathbf{F}') = \mathcal{M}(\mathbf{F}')$.

The notion of the multiplicity of filtrations is useful to distinguish various filtrations. However, it is by no means complete; in fact, we have several examples of filtrations $\mathbf{F}$ such that $\mathcal{M}^c(\mathbf{F}) = \mathcal{M}(\mathbf{F})$ with a single base M_1 such that $\langle M_1\rangle(t) = t$ and $\mathbf{F}$ is not a natural filtration of $BM^0(1)$. An example was given by Dubins, Feldman, Smorodinsky and Tsirelson ([DFST]) and recently, a conjecture of Barlow, Pitman and Yor ([BPY]) that the natural filtration of a Walsh's Brownian motion on $n \geq 3$ rays is not a Brownian filtration has been finally settled affirmatively by Tsirelson ([T]). For this, Tsirelson introduced another invariant notion for filtrations, the notion of *cosiness* of filtrations. We would formulate this notion as follows:

Definition 1.5. *A family $\mathbf{F}_\alpha = (F_\alpha(t)), \alpha \in [0,1]$, of filtrations on (Ω, F, P) is called a* T-system *(Tsirelson system) if it satisfies the following properties:*

(1) *There exists a filtration $\widehat{\mathbf{F}}$ such that, for every $\alpha \in [0,1]$, $\mathbf{F}_\alpha \subset \widehat{\mathbf{F}}$ and $\mathcal{M}(\mathbf{F}_\alpha) \subset \mathcal{M}(\widehat{\mathbf{F}})$ so that the injection $i_* : L^0(F_\alpha(\infty)) \to L^0(\widehat{F}(\infty))$ satisfies the conditions (i)~(iii) of Def. 1.1.*

(2) *For every $\alpha \in (0,1]$, there exists $0 < \rho(\alpha) < 1$ such that, for all $M \in \mathcal{M}(\mathbf{F}_0) \subset \mathcal{M}(\widehat{\mathbf{F}})$ and $N \in \mathcal{M}(\mathbf{F}_\alpha) \subset \mathcal{M}(\widehat{\mathbf{F}})$, the following holds:*

$$|\langle M, N\rangle(t)| \leq \rho(\alpha)\sqrt{\langle M\rangle(t)\cdot\langle N\rangle(t)}, \quad \forall\, t \geq 0,\ a.s..$$

(3) *$\forall\ \alpha \in (0,1]$, $\exists$ isomorphism $\pi_\alpha : \mathbf{F}_\alpha \to \mathbf{F}_0$, i.e., $(\pi_\alpha)_* : L^0(F_0(\infty)) \to L^0(F_\alpha(\infty))$ such that*

$$||X - (\pi_\alpha)_*(X)||_2 \to 0 \quad \text{as} \quad \alpha \to 0$$

for all $X \in L^2(F_0(\infty))$. Note that, by (1), $X \in L^2(F_0(\infty)) \subset L^2(\hat{F}(\infty))$ and $(\pi_\alpha)_*(X) \in L^2(F_\alpha(\infty)) \subset L^2(\hat{F}(\infty))$.

A typical example is that induced from a family of Brownian filtrations as follows:

Example 1.1. *On a suitable probability space (Ω, F, P), we set up a $BM^0(2d)$ as $B = (B(t)) = (B_1(t), B_2(t))$ where B_1, B_2 are two mutually independent $BM^0(d)$'s. Set*

$$W_\alpha(t) = \sqrt{1-\alpha^2}B_1(t) + \alpha B_2(t), \quad \alpha \in [0,1].$$

Then W_α is a $BM^0(d)$ for all α. Set $\widehat{\mathbf{F}} = \mathbf{F}^B$ and $\mathbf{F}_\alpha = \mathbf{F}^{W_\alpha}$. Then, $\{(\Omega, F, P), \{\mathbf{F}_\alpha\}, \widehat{\mathbf{F}}\}$ is a T-system.

Indeed, if $(\mathbf{W}^d, F(\mathbf{W}^d), \mu^d)$ is the d-dimensional Wiener space, i.e.,

$$\mathbf{W}^d = \{w \in C([0,\infty) \to \mathbf{R}^d) \mid w(0) = 0\}$$

and μ^d is the d-dimensional Wiener measure defined on the σ-field $F(\mathbf{W}^d)$ of μ^d-measurable sets, then for every $X \in L^0(F_0(\infty))$, there exists a unique $\widetilde{X} \in L^0(F(\mathbf{W}^d))$ such that $X(\omega) = \widetilde{X}(W_0(\omega))$. Define $(\pi_\alpha)_* : L^0(F_0(\infty)) \to L^0(F_\alpha(\infty))$ by $(\pi_\alpha)_*(X) = \widetilde{X}(W_\alpha(\omega))$. (2) can be deduced, by taking $\rho(\alpha) = \sqrt{1-\alpha^2}$, from the martingale representation theorem for $\mathcal{M}(\mathbf{F}_\alpha)$ and the relation $\langle W_0, W_\alpha\rangle(t) = \sqrt{1-\alpha^2}\cdot t \cdot I$, I being $d\times d$-identity matrix. Finally, (3) can be deduced from the relation

$$||X - (\pi_\alpha)_*(X)||_2 = ||\widetilde{X} - T_t\widetilde{X}||_{L^2(\mu^d)}, \quad X \in L^2(F_0(\infty)).$$

Here, $1-\alpha^2 = e^{-2t}$ and T_t is the Ornstein-Uhlenbeck semigroup on the Wiener space.

Definition 1.6. *A filtration $\mathbf{F}$ on (Ω, F, P) is said to be* cosy *if there exist a T-system $\{(\Omega', F', P'), \{\mathbf{F}'_\alpha\}, \widehat{\mathbf{F}}'\}$ and a morphism $\pi : \mathbf{F}'_0 \to \mathbf{F}$; that is, $\mathbf{F}$ is homomorphic to $\mathbf{F}'_0$.*

From this definition and Example 1.1, we can easily deduce the following proposition:

Proposition 1.2. *(1) The Brownian filtration, i.e., the natural filtration of a $BM^0(d)$, for any dimension d, is cosy.*

(2) If $\mathbf{F}$ is cosy and $\mathbf{F}'$ is homomorphic to $\mathbf{F}$, then $\mathbf{F}'$ is cosy.

The notion of spider martingales (martingales-araignées) has been introduced by Yor ([Y], p.110). We follow the definition given in [BEKSY]: Before proceeding, we give some notions and notations. Let $n \geq 2$ and $\mathbf{E}$ be a real vector space of $n-1$ dimension. Let $\mathbf{U} = \{u_1, \ldots, u_n\}$ be a set of n nonzero vectors in $\mathbf{E}$ such that $\mathbf{U}$ spans the whole space and $\sum_{k=1}^n u_k = 0$. Let

$$\mathbf{T}(=\mathbf{T}(\mathbf{U})) := \bigcup_{k=1}^n \{\lambda u_k \mid \lambda \in [0,\infty)\} \subset \mathbf{E}.$$

$\mathbf{T}$ is called a *web* (une toile d'araignée) of n-rays. When $n = 2$, then $\mathbf{U} = \{u_1, u_2 = -u_1\}$ and

$$\mathbf{T} = \{\lambda u_1 \mid \lambda \in [0,\infty)\} \cup \{\lambda u_1 \mid \lambda \in (-\infty, 0]\} = \{\lambda u_1 \mid \lambda \in \mathbf{R}\} \cong \mathbf{R},$$

so that a web of 2 rays is essentially a real line.

Definition 1.7. *A spider martingale is a* $\mathbf{T}(\subset \mathbf{E})$*-valued continuous local martingale* $M = (M(t))$ *with* $M(0) = 0$ *for some web* $\mathbf{T}$ *in* $\mathbf{E}$*. If* $\mathbf{T}$ *is a web of* n *rays, then* M *is said a spider martingale with* n *rays.*

Thus a spider martingale with 2 rays is essentially a continuous local martingale M with $M(0) = 0$.

Definition 1.8. *A nontrivial spider martingale with* $n \geq 3$ *rays is called a* multiple *spider martingale.*

When martingales are referred to a filtration $\mathbf{F}$, we say an $\mathbf{F}$-spider martingale.

We denote by $WBM^x(n;p_1,\ldots,p_n)$ a Walsh's Brownian motion (cf. [BPY]) on n rays from the origin with the rate of excursions at the origin given by $p_1,\ldots,p_n,(p_i > 0, \sum_{i=1}^n p_i = 1)$ and starting at x. A Walsh's Brownian motion $WBM^0(n;1/n,\ldots,1/n)$ is a typical example of a spider martingale with n rays and, indeed, a spider martingale with n rays is essentially obtained from a $WBM^0(n;1/n,\ldots,1/n)$ by a time change.

Theorem 1.2. *(Tsirelson 1996, cf. [T], [BEKSY]). If there exists a multiple* $\mathbf{F}$*-spider martingale in a filtration* $\mathbf{F}$*, then* $\mathbf{F}$ *is not cosy. In particular, such a filtration can not be homomorphic to a Brownian filtration in any dimension. In other words, when a filtration* $\mathbf{F}$ *is the natural filtration of a stochastic process* $X = (X(t))$ *and a multiple* $\mathbf{F}$*-spider martingale exists, then* X *can not have a canonical representation by a Wiener process in any dimension.*

It is not difficult to see that a Walsh's Brownian motion $WBM^x(n;p_1,\ldots,p_n)$ contains a multiple spider martingale with n rays in its natural filtration. Also, the following result was obtained by Barlow et al. (cf. [BEKSY]).

Theorem 1.3. *If* $\mathbf{F}$ *is a filtration which is homomorphic to the natural filtration of a* $WBM^x(n;p_1,\ldots,p_n)$ *and if* $m > n$*, there does not exist any multiple* $\mathbf{F}$*-spider martingale with* m *rays.*

We recall the operation of *time change* on filtrations (cf. [IW 3], p.102). Given a filtration $\mathbf{F}$ on (Ω, F, P), we mean, by a *process of time change* with respect to $\mathbf{F}$, an $\mathbf{F}$-adapted increasing process $A = (A(t))$ such that, $A(0) = 0$, $t \to A(t)$ is continuous, strictly increasing and $\lim_{t\uparrow\infty} A(t) = \infty$, almost surely. If A is a process of time change with respect to $\mathbf{F}$, then, for each $t \geq 0$, $A^{-1}(t) = \inf\{u|A(u) = t\}$ is an $\mathbf{F}$-stopping time and the σ-field $F(A^{-1}(t)) := F^{(A)}(t)$ is defined as usual. Then, we have a filtration $\mathbf{F}^{(A)} = (F^{(A)}(t))$ and, with respect to which, the increasing process $A^{-1} = (A^{-1}(t))$ is a process of time change. We can easily see that $\mathbf{F} = \{\mathbf{F}^{(A)}\}^{(A^{-1})}$: More generally, if A is a process of time change with respect to $\mathbf{F}$ and B is a process of time change with respect to $\mathbf{F}^{(A)}$, then $C = B \circ A = \{C(t) := B(A(t))\}$ is a process of time change with respect to $\mathbf{F}$ and $\mathbf{F}^{(C)} = \{\mathbf{F}^{(A)}\}^{(B)}$. Also, the following proposition can be easily deduced:

Proposition 1.3. *If* $\pi : \mathbf{F} \to \mathbf{F}'$ *is a morphism (i.e.,* $\mathbf{F}'$ *is homomorphic to* $\mathbf{F}$*) and* $A' = (A'(t))$ *is a process of time change with respect to* $\mathbf{F}'$*, then the process* $A = (A(t))$ *defined by* $A(t) = \pi_*(A'(t))$ *is a process of time change with respect to* $\mathbf{F}$ *and the same map* $\pi_* : L^0(F'(\infty)) \to L^0(F(\infty))$ *induces the homomorphism* $\pi : \mathbf{F}^{(A)} \to \mathbf{F}'^{(A')}$.

Remark 1.4. *The property of cosiness of filtrations is not invariant under the time change; indeed, as we shall see, the filtration of a sticky Brownian motion, which is a time change of a Brownian filtration, is not cosy.*

The following strengthens a little Theorem 1.2.

Theorem 1.4. *If there exists an* $\mathbf{F}$*-multiple spider martingale in a filtration* $\mathbf{F}$*, then* $\mathbf{F}$ *can not be homomorphic to any time change of a cosy filtration, in particular, to any time change of a Brownian filtration in any dimension.*

Indeed, if $\mathbf{F}$ is homomorphic to a time change $\mathbf{G}^{(A)}$ of a cosy filtration $\mathbf{G}$, then $\mathbf{G}^{(A)}$ contains a $\mathbf{G}^{(A)}$-multiple spider martingale and hence, $\mathbf{G}$ contains a $\mathbf{G}$-multiple spider martingale. However, this contradicts with Theorem 1.2.

Finally, we introduce the notion of the *direct product* of filtrations. Given filtrations $\mathbf{F}^{(i)}$ on $(\Omega^{(i)}, F^{(i)}, P^{(i)})$, $i = 1, \dots, m$, define their direct product $\otimes_{i=1}^m \mathbf{F}^{(i)}$, as a filtration on the product probability space $(\Pi_{i=1}^m \Omega^{(i)}, \otimes_{i=1}^m F^{(i)}, \otimes_{i=1}^m P^{(i)})$, by

$$\otimes_{i=1}^m \mathbf{F}^{(i)} = (\otimes_{i=1}^m F^{(i)}(t))_{t \geq 0},$$

where $\otimes_{i=1}^m F^{(i)}(t)$ is the usual product σ-field. Then the following propositions are easily deduced:

Proposition 1.4. *If* $\pi_i : \mathbf{F}^{(i)} \to \mathbf{F}'^{(i)}$, $i = 1, \dots, m$ *are morphisms, then there exists a unique morphism*

$$\otimes_{i=1}^m \pi_i : \otimes_{i=1}^m \mathbf{F}^{(i)} \to \otimes_{i=1}^m \mathbf{F}'^{(i)}$$

such that $(\otimes_{i=1}^m \pi_i)_*(\otimes_{i=1}^m X_i) = \otimes_{i=1}^m [(\pi_i)_*(X_i)]$ *for* $X_i \in L^0(\Omega'^{(i)}; F'^{(i)}(\infty))$.

Here, as usual, $[\otimes_{i=1}^m X_i](\omega'_1, \dots, \omega'_m) = \Pi_{i=1}^m X_i(\omega'_i)$, $(\omega'_1, \dots, \omega'_m) \in \Pi_{i=1}^m \Omega'^{(i)}$.

Proposition 1.5. *If* $\{(\Omega^{(i)}, F^{(i)}, P^{(i)}), \{\mathbf{F}_\alpha^{(i)}\}, \widehat{\mathbf{F}}^{(i)}\}$, $i = 1, \dots, m$, *are T-systems, then* $\{(\Pi_{i=1}^m \Omega^{(i)}, \otimes_{i=1}^m F^{(i)}, \otimes_{i=1}^m P^{(i)}), \{\otimes_{i=1}^m \mathbf{F}_\alpha^{(i)}\}, \otimes_{i=1}^m \widehat{\mathbf{F}}^{(i)}\}$ *is a T-system.*

Corollary 1.2. *If* $\mathbf{F}^{(i)}$ *are cosy for all* i*, then their direct product* $\otimes_{i=1}^m \mathbf{F}^{(i)}$ *is cosy.*

2 The existence of a multiple spider martingale in a diffusion on the plane

In the plane $\mathbf{R}^2$, let $L_1, \dots, L_n$ be n different straight half-lines (rays) starting at the origin 0. Let $\mathbf{e}^{(k)} = (e_1^k, e_2^k)$, $k = 1, \dots, n$, be the unit direction vector of L_k, respectively. Let $\mathcal{D}_0$ be the space of all C^∞-functions on the plane $\mathbf{R}^2$ with a compact support and vanishing also in a neighborhood of the origin 0. Define the following bilinear form for $f, g \in \mathcal{D}_0$;

$$\mathcal{E}(f,g) = \frac{1}{2} \int_{\mathbf{R}^2} \sum_{i=1}^2 \frac{\partial f(x)}{\partial x^i} \frac{\partial g(x)}{\partial x^i} dx + \sum_{k=1}^n \int_{L_k} \sum_{i,j=1}^2 e_i^k e_j^k \frac{\partial f(x)}{\partial x^i} \frac{\partial g(x)}{\partial x^j} d\mu_k(x) \tag{1}$$

where $d\mu_k$ is the (one-dimensional) Lebesgue measure on the half-line L_k. Then, setting $D = \mathbf{R}^2 \setminus \{0\}$, $\mathcal{E}(f,g)$ with domain $\mathcal{D}_0$ is a closable Markovian form on $L^2(D; dx)$ and its closure is a regular and local Dirichlet form. Hence, by the general theory of

Dirichlet forms ([FOT]), there corresponds a unique diffusion process $\mathbf{X} = \{X(t), P_x\}$ on D. Locally, the sample paths of this diffusion can be constructed by a skew product of two mutually independent BM·(1)'s so that the diffusion is precisely defined for every starting point in D. As is proved in [IW 1] or [IW 2], we have that $P_x(\zeta < \infty) = 1$ and $P_x(\lim_{t\uparrow\zeta} X(t) = 0) = 1$ for every $x \in D$, where ζ is the lifetime of $\mathbf{X}$ and the terminal point (cemetery) can be identified with the origin 0. $\mathbf{X}$ is a symmetric diffusion with respect to the Lebesgue measure dx on D and it possesses the continuous transition density $p(t,x,y)$, $(t,x,y) \in (0,\infty) \times D \times D$, so that $p(t,x,y) = p(t,y,x)$.

We say that a continuous function $u(x)$ on D is **X**-*harmonic* if $u(x) = E_x[u(X(\sigma_{U^c}))]$ for every $x \in D$ and every bounded neighborhood U of x such that $\overline{U} \subset D$, where $\sigma_{U^c} = \inf\{t | X(t) \notin U\}$. $u(x)$ is **X**-harmonic if and only if, writing $L_k^o = L_k \cap D$,

(i) $u(x)$ is continuous in D,

(ii) $u(x)$ is harmonic in the usual sense in the open set $D \setminus \{\bigcup_{k=1}^n L_k\}$,

(iii) for each $k = 1, \ldots, n$

$$u|_{L_k^o} \in C^2(L_k^o) \quad \text{and} \quad 2\frac{\partial^2 u}{\partial \xi^2}(\xi, 0) = \frac{\partial u}{\partial \eta}(\xi, 0+) - \frac{\partial u}{\partial \eta}(\xi, 0-)$$

where we introduce a local coordinate (ξ, η) of $y \in U$, in a sufficiently small neighborhood U of $x \in L_k^o$, by $y - x = \xi \cdot \mathbf{e}^{(k)} + \eta \cdot \mathbf{e}^{(k)\perp}$; $(\mathbf{e}^{(k)\perp} = (-e_2^k, e_1^k)$: the unit vector perpendicular to $\mathbf{e}^{(k)}$.)

It was shown in [IW 1] or [IW 2] that, for each $k = 1, \ldots, n$, there exists a unique bounded **X**-harmonic function $u_k(x)$ such that

$$\lim_{x \to 0, x \in L_j^o} u_k(x) = \delta_{jk}, \quad j = 1, \ldots, n.$$

It satisfies $0 < u_k(x) < 1$ and, furthermore, every bounded **X**-harmonic function $u(x)$ can be expressed as

$$u(x) = \sum_{k=1}^n c_k u_k(x), \quad c_k \in \mathbf{R},$$

the expression being unique because $c_k = \lim_{x \to 0, x \in L_k^o} u(x)$. In particular,

$$\sum_{k=1}^n u_k(x) \equiv 1.$$

If we set

$$\Xi^{(k)} = \{X(t) \to 0 \text{ as } t \uparrow \zeta \text{ tangentially along } L_k\}, \ k = 1, \ldots, n,$$

(for the precise meaning of "tangentially along," cf. [IW 1] or [IW 2]), then

$$u_k(x) = P_x(\Xi^{(k)}), \quad x \in D, \quad k = 1, \ldots, n.$$

For each $k = 1, \ldots, n$, $u_k(x)$ is an $\mathbf{X}$-excessive function and we can define the u_k-subprocess $\mathbf{X}^{(k)} = (X(t), P_x^{(k)})$, i.e., the diffusion on D obtained from $\mathbf{X}$ by the transformation by the multiplicative functional (cf. [FOT], Chap. 6.3),

$$M(t) = l_{\{\zeta > t\}} \cdot \frac{u_k(X(t))}{u_k(X(0))}.$$

This process satisfies

$$P_x^{(k)}(\Xi^{(k)}) = 1 \tag{2}$$

for all $x \in D$. For $j = 1, \ldots, n$, $u_j(x)u_k(x)^{-1}$ is an $\mathbf{X}^{(k)}$-excessive function and, by the symmetry of $\mathbf{X}$, the measure $u_j(x)u_k(x)dx$ is $\mathbf{X}^{(k)}$-excessive measure. Then, we can construct the $\mathbf{X}^{(k)}$-*Markovian measure* $\mathbf{N}_{jk}$, called also the *approximate process* or *quasi-process*, associated to $\mathbf{X}^{(k)}$-excessive measure $u_j(x)u_k(x)dx$, cf. e.g. Weil ([We]): $\mathbf{N}_{jk}$ is a σ-finite measure on the path space

$$\begin{aligned} \mathcal{W} \;=\; \{\, & w \in \mathcal{C}([0,\infty) \to \mathbf{R}^2) \mid w(0) = 0,\ \exists\, \sigma(w) \in (0,\infty) \ \text{ such that} \\ & w(t) \in \mathbf{R}^2 \setminus \{0\} \text{ for } t \in (0, \sigma(w)) \ \text{ and } w(t) = 0 \text{ for } t \geq \sigma(w) \,\} \end{aligned}$$

endowed with σ-field $\mathcal{B}(\mathcal{W})$ generated by Borel cylinder sets, uniquely determined by the following properties:

(i)

$$\int_0^\infty dt \int_{\mathcal{W}} f(w(t)) 1_{\{\sigma(w) > t\}} \mathbf{N}_{jk}(dw) = \int_D f(x) u_j(x) u_k(x) dx, \quad f \in \mathcal{C}_0(D),$$

(ii) for $t > 0, E \in \mathcal{B}(D)$ and $U \in \mathcal{B}(\mathcal{W})$,

$$\mathbf{N}_{jk}(\{\, w \mid w(t) \in E, \theta_t(w) \in U \}) = \int_{\mathcal{W}} P_{w(t)}^{(k)}(X \in U) 1_{\{w(t) \in E\}} \mathbf{N}_{jk}(dw), \tag{3}$$

where $\theta_t(w)$ is the shifted path: $\theta_t(w)(s) = w(t+s)$.

Since $\mathbf{X}$ is symmetric, we can deduce the following property under the time reversal:

$$\mathbf{N}_{jk}\{T^{-1}(U)\} = \mathbf{N}_{kj}(U), \quad U \in \mathcal{B}(\mathcal{W}), \quad j,k = 1, \ldots, n \tag{4}$$

where $T : \mathcal{W} \to \mathcal{W}$ is the time reversal operator:

$$(Tw)(t) = \begin{cases} w(\sigma(w) - t), & 0 \leq t \leq \sigma(w), \\ 0, & t \geq \sigma(w). \end{cases}$$

If we set, for $k = 1, \ldots, n$,

$$\tilde{\Xi}^{(k)} = \{w \in \mathcal{W} \mid w(t) \to 0 \text{ as } t \uparrow \sigma(w) \text{ tangentially along } L_k\}$$

and

$$\tilde{\Pi}^{(k)} = T^{-1}(\tilde{\Xi}^{(k)}) = \{w \in \mathcal{W} \mid w(t) \text{ starts at } 0 \text{ tangentially along } L_k\},$$

then, obviously, $\tilde{\Xi}^{(1)}, \cdots, \tilde{\Xi}^{(n)}$ are mutually disjoint and so are also $\tilde{\Pi}^{(1)}, \cdots, \tilde{\Pi}^{(n)}$. From (2), (3) and (4), we can deduce the following:

Proposition 2.1.

$$\mathbf{N}_{jk}(\mathcal{W} \setminus \{\tilde{\Pi}^{(j)} \cup \tilde{\Xi}^{(k)}\}) = 0, \quad j, k = 1, \ldots, n.$$

If we set

$$\mathbf{N}_j = \sum_{k=1}^{n} \mathbf{N}_{jk}, \quad j = 1, \ldots, n, \tag{5}$$

then $\mathbf{N}_j$ is the $\mathbf{X}$-Markovian measure associated to $\mathbf{X}$-excessive measure $u_j(x)dx$.

Now, the possible extension of $\mathbf{X}$ to a diffusion on the whole plane can be obtained by applying Itô's theory of excursion point processes (cf. [I]).

Theorem 2.1. *([IW 1] or [IW 2].) An extension $\widehat{\mathbf{X}} = (X(t), \widehat{P}_x)$, for which the origin 0 is not a trap, is determined by the nonnegative parameters $p_1, \ldots, p_n$ and m such that $\sum_{k=1}^{n} p_k = 1$. $m = 0$ if and only if $\int_0^\infty 1_{\{0\}}(X(t))dt = 0$ a.s. with respect to $\widehat{P}_x$ for every $x \in \mathbf{R}^2$. $\widehat{\mathbf{X}}$ is symmetric with respect to some measure on $\mathbf{R}^2$ if and only if $p_1 = \cdots = p_n = 1/n$ and, then, a symmetrizing measure is given by $m(dx) = dx + m \cdot \delta_{\{0\}}(dx)$. In this case, the corresponding Dirichlet form is the closure on $L^2(m(dx))$ of the $\mathcal{E}(f, g)$ given by (1) with the domain $C_0^\infty(\mathbf{R}^2)$.*

The sample paths of $\widehat{\mathbf{X}}$ starting at the origin 0 can be constructed as follows. Let $\mathbf{N} = \sum_{k=1}^{m} p_k \mathbf{N}_k$ which is a σ-finite measure on $(\mathcal{W}, \mathcal{B}(\mathcal{W}))$ with infinite total mass. We set up a Poisson point process p on the state space $\mathcal{W}$ with the characteristic measure $\mathbf{N}$ (cf. [I] or [IW 3], p.43 and p.123-130). Note that each sample of p is a point function $p : \mathbf{D}_p \in t \mapsto p_t \in \mathcal{W}$, where the domain $\mathbf{D}_p$ of p is a countable subset of $(0, \infty)$. Set

$$A(t) = mt + \sum_{s \in \mathbf{D}_p, s \le t} \sigma(p_s).$$

Then, it is a càdlàg increasing process with stationary independent increments and $A(0) = 0$. Since $\mathbf{N}(\mathcal{W}) = \infty$, $t \mapsto A(t)$ is strictly increasing and $\lim_{t \uparrow \infty} A(t) = \infty$, a.s. Hence, for each $t \ge 0$, there exists unique $s \ge 0$ such that $A(s-) \le t \le A(s)$. $s \in \mathbf{D}_p$ if and only if $A(s) > A(s-)$. Set, for each $t \in [0, \infty)$,

$$X(t) = \begin{cases} p_s(t - A(s-)), & s \in \mathbf{D}_p, A(s-) \le t \le A(s), \\ 0, & t = A(s-) = A(s). \end{cases}$$

Then, $\widehat{\mathbf{X}} = (X(t))$ is the sample path starting at 0 of the diffusion which is the extension of $\mathbf{X}$ corresponding to the parameters $p_1, \ldots, p_n$ and m.

Let $\mathbf{A} = \{k \in \{1, \ldots, n\} |\ p_k > 0\}$. Let $\mathbf{F} = \mathbf{F}^{\widehat{\mathbf{X}}}$ be the natural filtration of the diffusion $\widehat{\mathbf{X}}$ constructed above.

Theorem 2.2. *Assume that the set $\mathbf{A}$ contains $l \ge 2$ elements. Then there exists a non-trivial $\mathbf{F}$-spider martingale with l rays. Hence, if $\mathbf{A}$ contains $l \ge 3$ elements, there exists a multiple $\mathbf{F}$-spider martingale so that the filtration $\mathbf{F}$ is not cosy.*

Proof. Let $\widehat{\mathbf{X}} = (X(t))$ be the diffusion constructed above and $\mathbf{F} = \{F(t)\}$ be the natural filtration of $\widehat{\mathbf{X}}$. For $t > 0$, set

$$g(t) = \sup\{s \in [0, t] \mid X(s) = 0\}.$$

Then, $g(t)$ is an $\mathbf{F}$-honest time and, by Proposition 2.1 and an excursion theory, we deduce

$$F(g(t)+) = F(g(t)) \bigvee \{\Theta_k, k \in \mathbf{A}\},$$

where

$$\Theta_k = [\theta_{g(t)}X \in \tilde{\Pi}^{(k)}], \quad k \in \mathbf{A}.$$

The existence of a non-trivial $\mathbf{F}$-spider martingale with l rays follows from a general result in [BEKSY].

Or, we can give a more direct construction of a non-trivial $\mathbf{F}$-spider martingale by piecing out some part of each excursion by the method given in [Wat], (the collection of excursions is the point process p from which we have constructed the process $\widehat{\mathbf{X}}$).

Multidimensional extensions of Theorem 2.2 are of course possible. We give a typical example in the case of a three dimensional diffusion process. We define a diffusion $\mathbf{X}$ on $D = \mathbf{R}^3 \setminus \{0\}$ similarly as above by the following Dirichlet form: Let $\Pi_j, j = 1, \ldots, m$, be m different planes in $\mathbf{R}^3$, each passing through the origin 0 , and let $L_k, k = 1, \ldots, n$, be n different half lines, each, starting at the origin and lying on some plane Π_j. Let $\mathcal{D}_0$ be, as above, the space of all C^∞-functions with a compact support and vanishing also in a neighborhood of origin. Let $D(u, v)$, $u, v \in \mathcal{D}_0$, be the usual Dirichlet integral on $\mathbf{R}^3$, $D_{\Pi_j}(u, v)$ be the two-dimensional Dirichlet integral for $u|_{\Pi_j}, v|_{\Pi_j}$ on Π_j (by regarding Π_j as two-dimensional Euclidean space by the imbedding) and $D_{L_k}(u, v)$ be the one-dimensional Dirichlet integral for $u|_{L_k}, v|_{L_k}$ on L_k. For positive constants $\mu_j, j = 1, \ldots, m$ and $\nu_k, k = 1, \ldots, n$, define a bilinear form on $\mathcal{D}_0$ by

$$\mathcal{E}(u, v) = \frac{1}{2}D(u, v) + \sum_{j=1}^{m} \mu_j D_{\Pi_j}(u, v) + \sum_{k=1}^{n} \nu_k D_{L_k}(u, v), \quad u, v \in \mathcal{D}_0.$$

Then, it is a closable Markovian form on $L^2(D; dx)$ and its closure is a regular Dirichlet form. Therefore, there corresponds a unique diffusion $\mathbf{X}$ on D with a finite life time. We can obtain similar results as above: the space of bounded $\mathbf{X}$-harmonic functions are n-dimensional and the possible extensions of $\mathbf{X}$ as diffusions on $\mathbf{R}^3$ are determined in exactly the same way as Theorem 2.1. Also, Theorem 2.2 is valid in the same way: Namely, if an extension $\widehat{\mathbf{X}}$ which corresponds to nonnegative parameters $p_1, \ldots, p_n$ and m, is such that $\sharp\{ k \mid p_k > 0\} = l$, then the natural filtration of $\widehat{\mathbf{X}}$ contains a multiple spider martingale with l rays.

3 An application to sticky Brownian motions

Here we apply Theorem 2.2 to show the non-cosiness of the filtration of one-dimensional Brownian motion which is sticky at the origin 0.

For given $c \geq 0$, $\rho \geq 0$ with $c + \rho > 0$, consider the following stochastic differential equation for a continuous $\mathbf{F}$-semimartingale $X = (X(t))$ on $\mathbf{R}$ on a filtered probability space $\{(\Omega, F, P), \mathbf{F}\}$:

$$d[X(t) \vee 0] = 1_{\{X(t)>0\}} \cdot dB(t) + c \cdot d\phi(t), \quad d[X(t) \wedge 0] = 1_{\{X(t)<0\}} \cdot dB(t) - c \cdot d\phi(t) \quad (6)$$

where $B(t)$ is an $\mathbf{F}$-Wiener process with $B(0) = 0$, $\phi(t)$ is an $\mathbf{F}$-adapted, continuous increasing process with $\phi(0) = 0$ such that

$$\phi(t) = \int_0^t 1_{\{X(s)=0\}} \cdot d\phi(s), \quad \int_0^t 1_{\{X(s)=0\}} \cdot ds = \rho\phi(t). \tag{7}$$

Given $x \in \mathbf{R}$, a solution $X = (X(t))$ with $X(0) = x$ exists on a suitable filtered probability space and it is unique in the law sense ([IW 3]). When $c = 0$, then the solution $X(t)$ is the Brownian motion $x + B(t)$ stopped at $\sigma_0 = \min\{u \mid x + B(u) = 0\}$. When $\rho = 0$, the solution is irrelevant to c and coincides with the Brownian motion $x + B(t)$. In these two extreme cases, the natural filtration $\mathbf{F}^X$ of the solution X is either homomorphic or isomorphic to the Brownian filtration $\mathbf{F}^B$ so that it is a cosy filtration.

So we assume $c > 0, \rho > 0$ and, replacing $c\phi$ by ϕ and ρ by ρ/c, we can always assume $c = 1$ in the equation (6). In the following, we assume that $X(0) = 0$, for simplicity, and denote the solution by X_ρ.

Theorem 3.1. *The natural filtration $\mathbf{F}^{X_\rho}$ of X_ρ is not cosy.*

Proof. Without loss of generality, we may assume $\rho = 1$. Let $X^{(1)}$ and $X^{(2)}$ be independent copies of X_1 and define a diffusion $\mathbf{X} = (X(t))$ in the plane $\mathbf{R}^2$ by $\mathbf{X} = (X^{(1)}, X^{(2)})$. Then, $\mathbf{F}^{\mathbf{X}} = \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1}$. Let L_1, L_2, L_3, L_4 be the positive part of x-axis, the positive part of y-axis, the negative part of x-axis and the negative part of y-axis, respectively, so that $\mathbf{L} = L_1 \cup L_2 \cup L_3 \cup L_4$ coincides with the union of x- and y-axes. Let

$$A(t) = \int_0^t 1_{\{\mathbf{R}^2 \setminus \mathbf{L}\}}(X(s))ds.$$

Then, we easily deduce that $t \mapsto A(t)$ is strictly increasing and $\lim_{t\uparrow\infty} A(t) = \infty$, a.s. so that $A = (A(t))$ is a process of time change. Define $\widehat{X}(t) = X(A^{-1}(t))$ and $\widehat{\mathbf{X}} = ((\widehat{X}(t))$. Then, the natural filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ is just the time change $(\mathbf{F}^{\mathbf{X}})^{(A)}$. A key observation is that the diffusion process $\widehat{\mathbf{X}}$ is a particular case of diffusions given in Theorem 2.1: It is the case of $n = 4$ with L_1, L_2, L_3, L_4 given above and, $m = 0$, $p_1 = p_2 = p_3 = p_4 = 1/4$, cf. [IW 1], p.118. Hence, by Theorem 2.2, the filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ contains an $\mathbf{F}^{\widehat{\mathbf{X}}}$-multiple spider martingale with 4 rays. Since the filtration $\mathbf{F}^{\mathbf{X}}$ is obtained from the filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ by a time change as $\mathbf{F}^{\mathbf{X}} = \{\mathbf{F}^{\widehat{\mathbf{X}}}\}^{(A^{-1})}$, it also contains an $\mathbf{F}^{\mathbf{X}}$-multiple spider martingale with 4 rays. By Theorem 1.2, we can conclude that the filtration $\mathbf{F}^{\mathbf{X}} = \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1}$ is not cosy. Now, the non-cosiness of the filtration $\mathbf{F}^{X_1}$ follows from Corollary 1.2.

Remark 3.1. *Recently, J. Warren ([War]) proved directly that the natural filtration of a reflecting sticky Brownian motion is not cosy. His result is stronger than ours because there is a homomorphism from the filtration of a bilateral sticky Brownian motion to a reflecting sticky Brownian motion. However, we can give the following argument from which we can also deduce that the natural filtration of a reflecting sticky Brownian motion is not cosy.*

Let $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ be independent copies of X_1 and define a diffusion process $\mathbf{X} = (X(t))$ in $\mathbf{R}^3$ by $\mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)})$. Let Π_1, Π_2, Π_3 be coordinate planes in $\mathbf{R}^3$ and let $\mathbf{\Pi}$ be their union. Let

$$A(t) = \int_0^t 1_{\{\mathbf{R}^3 \setminus \mathbf{\Pi}\}}(X(s))ds \quad \text{and} \quad \widehat{X}(t) = X(A^{-1}(t)).$$

Then, the process $\widehat{\mathbf{X}} = ((\widehat{X}(t))$ is exactly a kind of diffusions discussed in Section 2 as multi-dimensional extensions of Theorem 2.2: It is the case that $m = 3$, Π_1, Π_2, Π_3 are coordinate planes as above and, $n = 6$, $L_1, \ldots, L_6$ are six half coordinate axes each starting at the origin. Furthermore, $\mu_j = \nu_k \equiv 1$, and $\widehat{\mathbf{X}}$ corresponds to parameters $p_1 = \ldots = p_6 = 1/6$ and $m = 0$. Hence, we can conclude that the natural filtration $\mathbf{F}^{\widehat{\mathbf{X}}}$ of $\widehat{\mathbf{X}}$ contains a multiple $\mathbf{F}^{\widehat{\mathbf{X}}}$-spider martingale with 6 rays. Then, the natural filtration $\mathbf{F}^{\mathbf{X}} = \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1} \otimes \mathbf{F}^{X_1}$ of $\mathbf{X}$ contains also a multiple $\mathbf{F}^{\mathbf{X}}$-spider martingale with 6 rays because $\mathbf{F}^{\mathbf{X}}$ is obtained from $\mathbf{F}^{\widehat{\mathbf{X}}}$ by a time change. From this, we can deduce that the filtration $\mathbf{F} = \mathbf{F}' \otimes \mathbf{F}' \otimes \mathbf{F}'$, where $\mathbf{F}'$ is the natural filtration of $|X_1| = (|X_1(t)|)$, has a multiple $\mathbf{F}$-spider martingale with 3 rays. Therefore, $\mathbf{F}$ is not cosy by Theorem 1.2 and hence, by Corollary 1.2, the filtration $\mathbf{F}'$, which is the natural filtration of the reflecting sticky Brownian motion $|X_1|$, is not cosy.

References

[BEKSY] M. T. Barlow, M. Émery, F. B. Knight, S. Song et M. Yor, Autour d'un théorème de Tsirelson sur des filtrations browniennes et non browniennes, *Séminaire de Probabilités XXXII*, **LNM 1686**, Springer, Berlin (1998), 264-305

[BPY] M. T. Barlow, J. W. Pitman and M. Yor, On Walsh's Brownian motions, *Séminaire de Probabilités XXIII*, **LNM 1372**, Springer, Berlin (1989), 275-293

[DV] M. H. Davis and P. Varaiya, The multiplicity of an increasing family of σ-fields, Annals of Probab., **2**(1974), 958-963

[DFST] L. Dubins, J. Feldman, M. Smorodinsky and B. Tsirelson, Decreasing sequences of σ-fields and a measure change for Brownian motion, Annals of Probab., **24**(1996), 882-904

[EY] M. Émery et M. Yor, Sur un théorème de Tsirelson relatif à des mouvements browniens corrélés et à la nullité de certains temps locaux, *Séminaire de Probabilités XXXII*, **LNM 1686**, Springer, Berlin (1998), 306-312

[FOT] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin-New York, 1994

[H] T. Hida, Canonical representation of Gaussian processes and their applications, Mem. Colleg. Sci. Univ. Kyoto, A, **33**(1960), 109-155

[IW 1] N. Ikeda and S. Watanabe, *The Local Structure of Diffusion Processes, (Kakusan-Katei no Kyokusho Kôzô)*, Seminar on Probab. **Vol. 35**, Kakuritsuron Seminar, 1971 (in Japanese)

[IW 2] N. Ikeda and S. Watanabe, The local structure of a class of diffusions and related problems, *Proc. 2nd. Japan-USSR Symp*, **LNM 330**, Springer, Berlin (1973), 124-169

[IW 3] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Amsterdam/Tokyo, 1988

[I] K. Itô, Poisson point processes attached to Markov processes, *in Kiyosi Itô, Selected Papers*, Springer, New York (1987), 543-557, Originally published in Proc. Sixth Berkeley Symp. Math. Statist. Prob. **III**(1970), 225-239

[KW] H. Kunita and S. Watanabe, On square integrable martingales, *Nagoya Math Jour.* **30**(1967), 209-245

[MW] M. Motoo and S. Watanabe, On a class of additive functionals of Markov processes, J. Math. Kyoto Univ. **4**(1965), 429-469

[N] M. Nisio, Remark on the canonical representation of strictly stationary processes, *J. Math. Kyoto Univ.* **1**(1961), 129-146

[RY] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, Berlin, 1991

[S] A. B. Skorokhod, Random processes in infinite dimensional spaces, *Proc. ICM. 1986, Berkeley*, Amer. Math. Soc., 163-171 (in Russian).

[T] B. Tsirelson, Triple points: From non-Brownian filtrations to harmonic measures, Geom. Func. Anal. **7**(1997), 1096-1142

[War] J. Warren, On the joining of sticky Brownian motion, *in this volume.*

[Wat] S. Watanabe, Construction of semimartingales from pieces by the method of excursion point processes, Ann. Inst. Henri Poincaré **23**(1987), 293-320

[We] M. Weil, Quasi-processus, *Séminaire de Probabilités IV*, **LNM 124**, Springer, Berlin (1970), 216-239

[Y] M. Yor, *Some Aspects of Brownian Motion,* Part **II**, Lectures in Math. ETH Zürich, Birkhäuser, 1997