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JONATHAN WARREN

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On the joining of sticky Brownian motion

J. WARREN[1]

Abstract

We present an example of a one-dimensional diffusion that cannot be innovated by Brownian motion. We do this by studying the ways in which two copies of sticky Brownian motion may be joined together and applying Tsirel'son's criteria of cosiness.

There has been much recent interest in Tsirel'son's idea [9] of studying the filtration of Walsh Brownian motion through the behaviour of pairs of such processes. A general technique has been developed by Tsirel'son [10] and others which involves taking two copies of a filtration and jointly immersing them in a larger set-up. See also Émery and Yor [5], Beghdadi-Sakrani and Émery [4] and Barlow et al. [2]. This note is motivated by applying these ideas to a particular process - sticky Brownian motion.

Let θ be a real constant satisfying $0 < \theta < \infty$. Suppose that $\big(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ is a filtered probability space satisfying the usual conditions, and that $\big(X_t; t \geq 0\big)$ is a continuous, adapted process taking values in $[0,\infty)$ which satisfies the stochastic differential equation

$$X_t = x + \int_0^t 1_{(X_s>0)} dW_s + \theta \int_0^t 1_{(X_s=0)} ds, \tag{0.1}$$

where $\big(W_t; t \geq 0\big)$ is a real-valued $\mathcal{F}_t$-Brownian motion and $x \geq 0$ is some constant. We say that X is sticky Brownian motion with parameter θ started from x, and refer to W as its driving Brownian motion. Unless stated otherwise we will assume $x = 0$. Sticky Brownian motion arose in the work of Feller [6] on strong Markov processes taking values in $[0,\infty)$ that behave like Brownian motion away from 0. In fact it can be constructed quite simply as a time change of reflected Brownian motion so that the resulting process is slowed down at zero, and so spends a real amount of time there. However here our interest will be focused on it arising as a solution of the above SDE. This equation does not admit a strong solution, it is not possible to construct X directly from W, and the filtration $\mathcal{F}$ is not generated by W alone. Warren [12] obtained a description of the extra randomness (hereafter referred to as the singular contribution) in terms of a mutation process on trees. Here we will suppose that our set-up carries two $\mathcal{F}_t$-Brownian motions $W^{(1)}$ and $W^{(2)}$ and two adapted processes $X^{(1)}$ and $X^{(2)}$ such that each pair $\big(X^{(i)}, W^{(i)}\big)$ satisfies an equation of the same form as (0.1), the value of θ being the same in both. We refer to this as a joining of sticky Brownian motion.

In the first section of this note we consider the case $W^{(1)} \equiv W^{(2)}$, and show that there is a family of different joinings such that this is so, which may be parameterised by $p \in [0,1]$. This parameter may be thought of as the correlation between the singular contributions. If $p = 1$ then the singular contributions are identical and hence so are $X^{(1)}$ and $X^{(2)}$, whereas for any $p < 1$ the process $\big(X^{(1)}, X^{(2)}\big)$ can and does spend time away from the 'diagonal'.

[1]University of Warwick, United Kingdom

In the second section, following Tsirel'son's method, we consider joinings with the instantaneous correlation between $W^{(1)}$ and $W^{(2)}$ bounded in modulus away from 1, and investigate what happens as this correlation is allowed to approach 1. We will find that the limiting law is that of the joining constructed in the previous section with $p = 0$.

Tsirel'son's concept of cosiness is a necessary condition for Brownian innovation. By this we mean the existence of a probabilistic setup $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ carrying a $\mathcal{F}_t$-adapted sticky Brownian motion X, a $\mathcal{F}_t$-Brownian motion W, with the pair X and W satisfying equation (0.1), and such that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of a Brownian motion (necessarily different from W). For a discussion of innovation see [11]. The limiting behaviour of the joinings we observe in the second section is exactly the failure of the cosiness criteria, and so we deduce that Brownian innovation of sticky Brownian motion is impossible. In particular the filtration generated by X and W is not Brownian.

1 Correlated mutations and some singular planar diffusions

Theorem 1. *Let $p \in [0,1]$. There exists a joining of sticky Brownian motion with $W^{(1)} \equiv W^{(2)}$ such that*

$$L_t^0(X^{(1)} \vee X^{(2)}) = 2(2-p)\theta A_t^{00}, \tag{1.1}$$

where $A_t^{00} = \int_0^t 1_{(X_s^{(1)}=X_s^{(2)}=0)} ds$, and L^0 is the semimartingale local time of $X^{(1)} \vee X^{(2)}$ at 0. The joint law of the processes $X^{(1)}$, $X^{(2)}$, and the common driving Brownian motion is uniquely determined.

Proof. We begin by considering an arbitrary joining with $W^{(1)}$ and $W^{(2)}$ equal to some common process W. We can write $X^{(1)} \vee X^{(2)}$ as the sum of three contributions, of which, at any time, at most one is non-zero.

$$X_t^{(1)} \vee X_t^{(2)} = Z_t^{(=)} + Z_t^{(1)} + Z_t^{(2)},$$

where

$$Z_t^{(=)} = X_t^{(1)} 1_{\left(X_t^{(1)}=X_t^{(2)}\right)} = X_t^{(2)} 1_{\left(X_t^{(1)}=X_t^{(2)}\right)},$$

and for $i = 1, 2$, with j denoting $3-i$,

$$Z_t^{(i)} = X_t^{(i)} 1_{\left(X_t^{(i)}>X_t^{(j)}\right)}.$$

It follows from the formula for balayage of semimartingales, see [7] and the appendix of this note, that the processes $Z^{(=)}, Z^{(1)}$, and $Z^{(2)}$ are themselves continuous semimartingales, and that,

$$Z_t^{(=)} = \int_0^t 1_{\left(Z_s^{(=)}>0\right)} dW_s + \tfrac{1}{2} L_t^0(Z^{(=)}),$$
$$Z_t^{(i)} = \int_0^t 1_{\left(Z_s^{(i)}>0\right)} dW_s + \tfrac{1}{2} L_t^0(Z^{(i)}),$$

where, as always, $L^a(Z)$ denotes the local time at level a of the semimartingale Z. For each of the three processes $Z^{(=)}, Z^{(1)}$ and $Z^{(2)}$, the measure $dL^0(Z)$ is supported on the set of times $\{t : X_t^{(1)} = X_t^{(2)} = 0\}$. We must also observe that

$$Y_t^{(i)} \stackrel{\text{def}}{=} X_t^{(j)} 1_{\left(X_t^{(i)} > X_t^{(j)}\right)} = \int_0^t 1_{\left(Y_s^{(i)} > 0\right)} dW_s + \theta \int_0^t 1_{\left(Z_s^{(i)} > 0,\, Y_s^{(i)} = 0\right)} ds.$$

Notice that this time the balayage formula does not introduce an additional term which grows when $X_t^{(1)} = X_t^{(2)} = 0$. That this is so may be deduced from an appropriate application of théorème 2 of [7] (see the appendix again!).

Next we have that

$$L_t^0\left(X^{(1)} \vee X^{(2)}\right) = L_t^0\left(Z^{(=)}\right) + L_t^0\left(Z^{(1)}\right) + L_t^0\left(Z^{(2)}\right),$$

and since

$$X_t^{(i)} = Z_t^{(=)} + Z_t^{(i)} + Y_t^{(j)},$$

we also find that

$$2\theta A_t^{00} = L_t^0\left(Z^{(=)}\right) + L_t^0\left(Z^{(i)}\right).$$

Thus if $L_t^0\left(X^{(1)} \vee X^{(2)}\right) = 2(2-p)\theta A_t^{00}$ for some fixed $p \in [0,1]$ then we infer that

$$L_t^0\left(Z^{(=)}\right) = 2p\theta A_t^{00}, \qquad \text{and} \qquad L_t^0\left(Z^{(i)}\right) = 2(1-p)\theta A_t^{00}.$$

Let $|Z_t| = X_t^{(1)} \vee X_t^{(2)}$ then the process $\left(|Z_t|; t \geq 0\right)$ is itself a sticky Brownian motion[2] with parameter $(2-p)\theta$. For $i = 1, 2$ let

$$A_t^{(i)} = \int_0^t 1_{(Z_s^{(i)} > 0)} ds = \int_0^t 1_{(X_s^{(i)} > X_s^{(j)})} ds,$$

and $\alpha^{(i)}$ be the right continuous inverse of $A^{(i)}$. Then define $\tilde{Y}_t^{(i)} = Y_{\alpha_t}^{(i)}$ and construct the Brownian motions

$$\tilde{W}_t^{(i)} = \int_0^{\alpha_t^{(i)}} 1_{(X_s^{(i)} > X_s^{(j)})} dW_s.$$

Each pair $\left(\tilde{Y}^{(i)}, \tilde{W}^{(i)}\right)$ satisfies an equation analogous to (0.1).

A (rather laborious) construction of $\left(X^{(1)}, X^{(2)}\right)$ now suggests itself. We will describe it informally- there are no real difficulties here. Start with a Brownian motion W, and choose $|Z|$ according to the conditional law of sticky Brownian motion with parameter $(2-p)\theta$ given W as its driving Brownian motion. Independently assign each excursion of $|Z|$ to be an excursion of $Z^{(=)}$, $Z^{(1)}$ or $Z^{(2)}$ with probability $(1-p)/(1+p)$, $p/(1+p)$ and $p/(1+p)$ respectively. Now for $i = 1, 2$ construct the Brownian motions $\tilde{W}^{(i)}$ as above, and then choose $\tilde{Y}^{(i)}$ according to the conditional law of sticky Brownian motion with parameter θ given $\tilde{W}^{(i)}$ as its driving Brownian

[2] $|Z|$ to recall the modulus of the Walsh Brownian motion on three rays.

motion, and independently of anything else. Finally put $Y_t^{(i)} = \tilde{Y}^{(i)}_{A_t^{(i)}}$ and then let $X_t^{(i)} = Z_t^{(=)} + Z_t^{(i)} + Y_t^{(j)}$.

If we consider any joining with the same value of p the joint law of the processes W, $Z^{(=)}$, $Z^{(1)}$, $Z^{(2)}$, $Y^{(1)}$ and $Y^{(2)}$ has the same structure as we have just constructed, and the uniqueness assertion follows from this. Note that we are using here that the joint law of W and X solving (0.1) is unique, and also that there is uniqueness for the martingale problem formulation of the Walsh process on 3 rays, see [3]. □

Observe that, had we not known that sticky Brownian motion was not generated by its driving Brownian motion, we would be now able to deduce this from the existence of the non-diagonal joinings displayed in the preceding theorem. This is precisely the technique used by Barlow in [1], although he, dealing with a general class of SDEs which have no strong solutions, has to do much work to see non-diagonal joinings exist. Here things are much easier because we understand the nature of the singular contribution very well.

Recall the description of the law of X_t conditional on W, given in [12].

Theorem 2. *Suppose that (X, W) satisfy the SDE equation (0.1). Let $L_t = \sup_{s\le t}(-W_s)$. For fixed t, the conditional law of X_t given W is determined by*

$$(X_t, W) \overset{law}{=} \Big((W_t + L_t - T)^+, W \Big),$$

where T is an independent exponential random variable with mean $1/2\theta$.

We may make repeated application of Theorem 2 to the construction of Theorem 1, and hence obtain the following description of the conditional law of $(X_t^{(1)}, X_t^{(2)})$ given the common driving Brownian motion. Those familiar with interpreting Theorem 2 in terms of mutations on trees will easily extend the idea to cover the present case.

Corollary 3. *Let $p \in [0,1]$ and consider the corresponding joining of sticky Brownian motion constructed in Theorem 1. The conditional law of $(X_t^{(1)}, X_t^{(2)})$ given the common driving Brownian motion W is determined by*

$$(X_t^{(1)}, X_t^{(2)}, W) \overset{law}{=} \Big((W_t + L_t - T^{(1)})^+, (W_t + L_t - T^{(2)})^+, W \Big),$$

where, $L_t = \sup_{s\le t}(-W_s)$, and the law of $(T^{(1)}, T^{(2)})$ is described as follows.

Let $(M_y)_{y\ge 0} = (M_y^{(1)}, M_y^{(2)})_{y\ge 0}$ be a Markov chain with state space $\{0,1\}^2$, and let its transition rates be given by the following diagram.

$$\begin{tikzcd}
(0,1) \arrow[r, "2\theta"] & (1,1) \\
(0,0) \arrow[u, "2(1-p)\theta"] \arrow[r, "2(1-p)\theta"'] \arrow[ur, "2p\theta"'] & (1,0) \arrow[u, "2\theta"']
\end{tikzcd}$$

Then we take:

$$\left(T^{(1)}, T^{(2)}\right) \overset{law}{=} \left(\inf\{y : M_y^{(1)} = 1\}, \inf\{y : M_y^{(2)} = 1\}\right).$$

As particular cases, if we take $p = 1$, then $X^{(1)} \equiv X^{(2)}$, while at the other extreme, $p = 0$, and $X^{(1)}$ and $X^{(2)}$ are conditionally independent given the common driving Brownian motion W.

2 Non-cosiness of sticky Brownian motion

We have just seen that when a joining possesses common driving Brownian motions there is a 'hidden' parameter p which may be thought of as describing the correlation of the singular contributions. We want to know whether this possibility exists even if the driving Brownian motions are not identical. The answer to this does not seem, *a priori*, obvious. With any joining the pair $\left(X^{(1)}, X^{(2)}\right)$ spends plenty of time at the origin- which is where they need to be to do something mischievous. However the argument of the next paragraph shows that, at least in a special case, nothing untoward happens.

We consider the case $\langle W^{(1)}, W^{(2)}\rangle \equiv 0$. In this case the four martingales

$$\int_0^t 1_{(X_s^{(1)}=0)} dW_s^{(1)}, \quad \int_0^t 1_{(X_s^{(1)}>0)} dW_s^{(1)}, \quad \int_0^t 1_{(X_s^{(2)}=0)} dW_s^{(2)}, \text{ and } \int_0^t 1_{(X_s^{(2)}>0)} dW_s^{(2)}$$

are mutually orthogonal. Consequently Knight's theorem tells us that if we time change each martingale to obtain a Brownian motion, then these resulting Brownian motions are mutually independent. But, for $i = 1, 2$, the pair $\left(X^{(i)}, W^{(i)}\right)$ is measurable with respect to the two Brownian motions arising from the two stochastic integrals with respect to $W^{(i)}$. Thus $\left(X^{(1)}, W^{(1)}\right)$ and $\left(X^{(2)}, W^{(2)}\right)$ are independent. Hence we see that there is a unique (in law) joining such that the driving processes $W^{(1)}$ and $W^{(2)}$ are orthogonal, and in this case the singular contributions are necessarily independent.

Throughout this section we will consider joinings such that there exists a $\rho_{max} < 1$ such that $|\langle W^{(1)}, W^{(2)}\rangle_t - \langle W^{(1)}, W^{(2)}\rangle_s| \leq \rho_{max}|t - s|$ for all $t, s \geq 0$, we say the maximal correlation of the joining is less than 1.

Lemma 4. *Any random variable belonging to $\mathcal{L}^2\left(X, W\right)$ can be expressed as a stochastic integral with respect to W.*

By virtue of this representation property (which is proved in the appendix), the maximal correlation of the joining being less than 1 makes available to us the important *hypercontractivity* inequality, see Tsirelson [10] for an outline of the proof.

Lemma 5. *Suppose that a joining satisfies, for some $\rho_{max} < 1$*

$$|\langle W^{(1)}, W^{(2)}\rangle_t - \langle W^{(1)}, W^{(2)}\rangle_s| \leq \rho_{max}|t - s|, \qquad \textit{for all } t, s \geq 0.$$

Then if Φ is a bounded path functional

$$\mathbb{E}\left[\Phi(X^{(1)})\Phi(X^{(2)})\right] \leq \mathbb{E}\left[\Phi(X)^{(1+\rho_{max})}\right]^{2/(1+\rho_{max})},$$

where X possesses the common law of $X^{(1)}$ and $X^{(2)}$.

Theorem 6. *If the maximal correlation of the joining is less than* 1, *then*

$$L^0_t(X^{(1)} \vee X^{(2)}) = 4\theta A^{00}_t,$$

where $A^{00}_t = \int_0^t 1_{(X^{(1)}_s = X^{(2)}_s = 0)} ds$, *and* L^0 *is the semimartingale local time of* $X^{(1)} \vee X^{(2)}$ *at* 0.

Proof. Observe that

$$L^0_t(X^{(1)} + X^{(2)}) = 4\theta A^{00}_t.$$

Now as a consequence of the occupation time formula (see Revuz and Yor [8]) we find that

$$|L^0_t(X^{(1)} + X^{(2)}) - L^0_t(X^{(1)} \vee X^{(2)})| \leq \limsup_{\epsilon \downarrow 0} \frac{4}{\epsilon} \int_0^t 1_{(0<X^{(1)}_s<\epsilon)} 1_{(0<X^{(2)}_s<\epsilon)} ds.$$

We use the preceding Lemma to show the expectation of the righthand-side is zero. A simple computation confirms that, if X is a sticky Brownian motion, then,

$$\mathbb{E}\left[1_{(0<X_s<\epsilon)}\right] = O(\epsilon),$$

uniformly for all s. Hence, by virtue of hypercontractivity, for some $\rho_{max} < 1$,

$$\mathbb{E}\left[1_{(0<X^{(1)}_s<\epsilon)} 1_{(0<X^{(2)}_s<\epsilon)}\right] = O(\epsilon^{2/(1+\rho_{max})}),$$

uniformly for all s, and this suffices. □

This proof displays very clearly how the process $(X^{(1)}, X^{(2)})$, living in the positive quadrant $\mathbb{R}^2_+$, uses motion along axes to visit the origin. But be careful in interpreting this.

Now recall Tsirelson's definition, [10], of cosy. In order for the filtration generated by sticky Brownian motion to be cosy there must exist a sequence of joinings each with maximal correlation less than 1, such that if Φ is any bounded path functional then

$$\Phi(X^{(1)}) - \Phi(X^{(2)}) \xrightarrow{\mathcal{L}^2} 0,$$

as we tend along the sequence. We are actually considering (to use Tsirelson's terminology more properly) self-joinings of the filtration generated by the sticky Brownian motion and the driving motion together. But this distinction, is in fact, unimportant, since any self-joining of the filtration generated by sticky Brownian motion alone is easily enriched to become a joining of the type we are considering.

Corollary 7. *The filtration generated by sticky Brownian motion is non-cosy.*

Proof. Fix $\lambda > 0$ and let $\gamma = \sqrt{2\lambda}$. If X is sticky Brownian motion started from 0 then

$$e^{-\gamma X_t - \lambda t} + (\gamma\theta + \lambda) \int_0^t e^{-\lambda s} 1_{(X_s=0)} ds$$

is a martingale, and

$$\mathbb{E}\left[\int_0^\infty e^{-\lambda s} 1_{(X_s=0)} ds\right] = \frac{1}{\gamma\theta+\lambda}.$$

For any joining of sticky Brownian motion $X_t^{(1)} \vee X_t^{(2)}$ is a submartingale with quadratic variation process $\int_0^t 1_{(X_s^{(1)} \vee X_s^{(2)} > 0)} ds$. If the joining has maximal correlation less than 1, then, by virtue of the preceding Theorem, $L_t^0(X^{(1)} \vee X^{(2)}) = 4\theta A_t^{00}$, and an application of Itô's formula shows that

$$e^{-\gamma X_t^{(1)} \vee X_t^{(2)} - \lambda t} + (2\gamma\theta + \lambda) \int_0^t e^{-\lambda s} 1_{(X_s^{(1)} = X_s^{(2)} = 0)} ds$$

is a supermartingale. In this case we may deduce that

$$\mathbb{E}\left[\int_0^\infty e^{-\lambda s} 1_{(X_s^{(1)} = X^{(2)} = 0)} ds\right] \leq \frac{1}{2\gamma\theta + \lambda}.$$

But if the filtration generated by sticky Brownian motion was cosy then as we tend along some sequence of joinings, for each $t \geq 0$,

$$1_{(X_t^{(1)}=0)} - 1_{(X_t^{(2)}=0)} \xrightarrow{\mathcal{L}^2} 0,$$

and hence

$$\mathbb{E}\left[\int_0^\infty e^{-\lambda s} 1_{(X_s^{(1)} = X_s^{(2)} = 0)} ds\right] \to \mathbb{E}\left[\int_0^\infty e^{-\lambda s} 1_{(X_s=0)} ds\right].$$

In view of the above computations this is impossible. □

A little more effort tidies things up. By the law of a joining we mean the joint law of $(X^{(1)}, X^{(2)}, W^{(1)}, W^{(2)})$.

Corollary 8. *Let $\mathbb{P}_n$ for $n \geq 1$, be the laws of a sequence of joinings of sticky Brownian motion, each with maximal correlation less than 1. Suppose that the law of $(W^{(1)}, W^{(2)})$, as we tend along the sequence, converges to the law of the diagonal process (W, W), where W is a Brownian motion. Then, as n tends to infinity, $\mathbb{P}_n$ converges weakly to the law of the joining constructed in Theorem 1 with $p = 0$, that is with independent singular contributions.*

Proof. The sequence $\mathbb{P}_n$ is tight because the marginal laws of $(X^{(i)}, W^{(i)})$ are constant. Suppose $\mathbb{Q}$ is the limit of a convergent subsequence. $\mathbb{Q}$ is evidently the law of a joining of sticky Brownian motion, with identical driving Brownian motions. Because of the uniqueness assertion of Theorem 1 it suffices to identify that, almost surely under $\mathbb{Q}$,

$$L_t^0(X^{(1)} \vee X^{(2)}) = 4\theta A_t^{00}.$$

Consider $f \in C_b(\mathbb{R}_+)$ with $f' \geq 0$, and $f''(0+) = 4\theta f'(0+)$. Note, for such f,

$$M_t^f = f(X_t^{(1)} \vee X_t^{(2)}) - \tfrac{1}{2} \int_0^t f''(X_s^{(1)} \vee X_s^{(2)}) ds$$

is a submartingale under any $\mathbb{P}_n$, whence it is also a submartingale under $\mathbb{Q}$. We may deduce from this that

$$L_t^0(X^{(1)} \vee X^{(2)}) \geq 4\theta A_t^{00},$$

and re-examining the proof of Theorem 1 we see that this can only happen with equality. □

Acknowledgments. This work was done while visiting MSRI, where I was very lucky to be able to talk with both Boris Tsirel'son and Ruth Williams.

Appendix. The first part of this appendix contains an explanation as to the use of balayage to deduce that the processes $Z^{(=)}, Z^{(1)}$, and $Z^{(2)}$ are continuous semimartingales.

First we denote by H the random closed set

$$\{t : X_t^{(1)} = X_t^{(2)} = 0\},$$

and observe that each of the processes

$$\left(1_{\left(X_t^{(1)}=X_t^{(2)}\right)}\right)_{t\geq 0}, \left(1_{\left(X_t^{(1)}>X_t^{(2)}\right)}\right)_{t\geq 0}, \text{ and } \left(1_{\left(X_t^{(1)}<X_t^{(2)}\right)}\right)_{t\geq 0}$$

is constant on each component of H^c. For each $t > 0$ we define random times D_t and τ_t by,

$$D_t = \inf\{u > t : u \in H\},$$
$$\tau_t = \sup\{s < t : s \in H\}.$$

Then we consider a process K defined by

$$K_t = \liminf_{u \downarrow\downarrow t} 1_{\left(X_u^{(1)}=X_u^{(2)}\right)}.$$

Because K is bounded and progressive, and $X_{D_t}^{(1)} = 0$ for all t, the balayage formula, see [7], tells us that

$$K_{\tau_t} X_t^{(1)} = \int_0^t \kappa_s dX_s^{(1)} + \mathcal{R}_t,$$

where κ is the previsible projection of K and $\mathcal{R}$ is an adapted, finite-variation process, constant on each component of H^c. In particular $K_{\tau_t} X_t^{(1)}$ is a continuous semimartingale. Now K_t is equal to 1 if t is the left-hand end of an excursion into $\{x^{(1)} = x^{(2)} > 0\}$, but equal to zero at the left-hand end of other components of H^c. Thus we see that

$$K_{\tau_t} X_t^{(1)} = Z_t^{(=)}.$$

Finally we note that $\kappa_t = 1_{\left(X_t^{(1)}=X_t^{(2)}\right)}$ on H^c, whence we see that the semimartingale decomposition of $Z^{(=)}$ must be as claimed in the proof of Theorem 1. By making appropriate changes to the definition of K we may consider $Z^{(1)}$ and $Z^{(2)}$ in the same way.

When we turn to considering the processes $Y^{(1)}$ and $Y^{(2)}$ we need to alter our choice of the closed random set H and the process K. Let us now take H to be set of times at which $X^{(2)}$ is zero, and define

$$K_t = 1_{\left(X_t^{(1)} > X_t^{(2)}\right)}.$$

This choice of K is previsible, and on applying the balayage formula we find that

$$K_{\tau_t} X_t^{(2)} = \int_0^t 1_{\left(X_s^{(1)} > X_s^{(2)}\right)} dX_s^{(2)} = \int_0^t 1_{\left(X_s^{(1)} > X_s^{(2)} > 0\right)} dW_s + \theta \int_0^t 1_{\left(X_s^{(1)} > X_s^{(2)} = 0\right)} ds.$$

with no additional finite-variation term. The argument is completed by observing that $K_{\tau_t} X_t^{(2)} = Y_t^{(1)}$. The process $Y^{(2)}$ may be obtained by making obvious changes to the indices in these formulae.

The final part of this appendix contains a proof of Lemma 4. Introduce the two Brownian motions W^+ and W^0 defined by

$$W_t^+ = \int_0^\infty 1_{\left(X_s > 0, A_s^+ \le t\right)} dW_s,$$
$$W_t^0 = \int_0^\infty 1_{\left(X_s = 0, A_s^0 \le t\right)} dW_s,$$

where $A_t^+ = \int_0^t 1_{\left(X_s > 0\right)} ds$ and $A_t^0 = \int_0^t 1_{\left(X_s = 0\right)} ds$. Notice that the two stochastic integrals above are orthogonal. We find that we are able to write exponential random variables of the form

$$\exp\left\{\sum \lambda_i \left(W_{t_{i+1}}^+ - W_{t_i}^+\right) + \mu_i \left(W_{t_{i+1}}^0 - W_{t_i}^0\right)\right\}$$

as stochastic integrals against W. But these exponential variables are total in $\mathcal{L}^2(W^+, W^0)$, and moreover

$$\mathcal{L}^2(W, X) = \mathcal{L}^2(W^+, W^0),$$

whence the martingale representation property extends to all of $\mathcal{L}^2(W, X)$.

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