Paul Mc Gill

# Brownian motion, excursions, and matrix factors

*Séminaire de probabilités (Strasbourg)*, tome 32 (1998), p. 412-425
<http://www.numdam.org/item?id=SPS_1998__32__412_0>

# Brownian motion, excursions, and matrix factors

*Paul McGill*[1]

*Laboratoire de Probabilités, Université de Lyon I, 69622 Villeurbanne, France*[2]

The Wiener-Hopf problem in analysis asks how one can factor a matrix function on the line as $\mathfrak{A} = \mathfrak{A}^+ * \mathfrak{A}^-$ where $*$ denotes convolution and $\mathfrak{A}^\pm$ are supported on $\mathbf{R}^\pm$ respectively. Existence is known [7], but a general algorithm seems to be out of reach: even the $2 \times 2$ case [5] presents an unexpected degree of complication.

The probabilistic counterpart (in dimension one) involves characterising a Lévy process $Y$ in the form $(Y^+, Y^-)$ where the laws $Y^\pm$ are supported on $\mathbf{R}^\pm$. These factors can be constructed by decomposing the sample path into its excursions from the maximum: $Y^+$ is then a Lévy process while $Y^-$ takes its values in the space of paths. The probabilistic factorisation implies the (unique) factorisation of the generator $\mathcal{G} = \mathcal{G}^+ * \mathcal{G}^-$ — $Y^\pm$ are not unique — and allows us to describe the connection $Y^\pm \leftrightarrow \mathcal{G}^\pm$ in more detail. It is natural therefore to ask for an extension to higher dimensions: is there a probabilistic setting in which one can perform a sample path factorisation of certain matrices?

The difficulty is to know where to start. This note presents an example based on the idea in [10] and we recall the general problem: given a Markov process $X_t \in \mathcal{E}$ and a fluctuating additive functional $V$, there is a decomposition of the state space $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ determined by $V$ increasing/decreasing on $\mathcal{E}^+/\mathcal{E}^-$. This leads to a probabilistic decomposition $X \to (X^+, X^-)$ and one studies the relationship $(X, V) \leftrightarrow (X^+, X^-)$; in terms of generators

$$(\mathcal{G}, V) \leftrightarrow \begin{pmatrix} \mathcal{G}_{++} & \mathcal{G}_{+-} \\ \mathcal{G}_{-+} & \mathcal{G}_{--} \end{pmatrix}$$

but see [10] for more explanation. The analogy here is with two-point Markov chains and excursions with 'interaction': $\mathcal{G}_{++}$ is the generator of $X^+$ in the interior of $\mathcal{E}^+$ while $\mathcal{G}_{-+}$ describes how $X^+$ returns to $\mathcal{E}^+$ after interacting with $\mathcal{E}^-$.

We treat only the simplest case where $X = B$ a real Brownian motion; the boundary set $\partial = \bar{\mathcal{E}}^+ \cap \bar{\mathcal{E}}^-$ then has a local time. As a first step towards calculating $\mathcal{G}_{-+}$ we derive a (vector) convolution equation for the entrance law of $(B, V)$ into the right half plane. The equation formally resembles the first passage relation for a real Lévy process and is solved by Wiener-Hopf factorisation of a certain matrix $\mathfrak{S}$ where, and quite remarkably, the factors can be seen from a sample path decomposition. The proof echoes [4] — it uses excursions from maxima of $V$ observed in

[1] Supported by the National Science Foundation and the Air Force Office of Scientific Research Grant No: F49620-92-J-0154 and the Army Research Office Grant No: DAAL03-92-G-0008.

[2] While visiting: Center for Stochastic Processes, University of North Carolina, Chapel Hill, NC 27599, USA.

the boundary local time — but the need to simultaneously track $B$ means that the theory of homogeneous regenerative sets does not apply directly. Instead we employ Maisonneuve's theory [6] in multiple timescales, which we then patch together by viewing the 'boundary chain at the maximum' in its equilibrium distribution.

The matrix $\mathfrak{S}$ is quite special here: it is the generator of a two-dimensional Markov process and tri-diagonal to boot. Nevertheless, and even if the other manageable case, that of finite Markov chains [10], lies beyond our compass since there is nothing to play the crucial role of the boundary $\partial$, one suspects that the method should apply more generally.

## 1. Problem

We work with real Brownian motion $B_t$ and a fixed $V_t = \int L_t^a m(da)$; $L_t^a$ is a bicontinuous version of the $B$ local time and the Hahn decomposition of the Radon measure $m = m^+ - m^-$ splits $\mathbf{R} = \mathcal{E}^+ \oplus \mathcal{E}^-$ with $\mathcal{E}^+$ defined as the closed support of $m^+$. We will assume throughout that the boundary $\partial = \mathcal{E}^+ \cap \bar{\mathcal{E}}^-$ is discrete — no limit points. Our results are not proved in full generality however; further restrictions, pertaining mainly to the set of maxima and used to simplify an already complicated proof, are stated at the beginning of sections 5 and 6.

To define the splitting of $B$ induced by $V$ we consider the increasing process $\bar{V}_t = \sup\{V_s : 0 < s \leq t\}$ and take $\tau_t^+ = \tau_t$ as its right continuous inverse. Then $X_t^+ = B_{\tau_t} \in \mathcal{E}^+$, but note that the process can jump since in general the intervals of constancy of $\bar{V}$ strictly contain $\{t : B_t \in \mathcal{E}^-\}$. A similar description of $X^-$ provides the desired factorisation $(B, V) \leftrightarrow (X^+, X^-)$.

This external approach to defining $(X^+, X^-)$ is quite useless. One should look instead for an intrinsic (internal) characterisation of each factor[1] which would amount to decomposing the generator $\mathcal{G}$ as above. As a first step in this direction we obtain an expression containing $\Pi^+(x, dy) = \mathbf{P}_x[B_\tau \in dy; \tau < \infty]$ with $\tau = \tau_0^+ = \inf\{t > 0 : V_t > 0\}$ the equalisation time. The method is to solve a vector equation of Wiener-Hopf type which we derive in the next section.

## 2. Equation for $\Pi^+(x, dy)$

We will derive an equation containing $\Pi^+(x, dy) = \mathbf{P}_x[B_\tau \in dy; \tau < \infty]$. Take $T$ as the $B$ hitting time of the boundary $\partial$ and let $L^\partial = \sum_{j \in \partial} L^j$ denote the boundary local time. Then, for each $j \in \partial$ we derive one scalar equation; our notation is that $j_r$ is the closest boundary point to the right of $j$, $j_\ell$ its closest point on the left.

Define functions $u(x, j) = \mathbf{E}_x\left[e^{-zV_T}; B_T = j\right]$ noting that they solve (eg.)

$$du_x = 2zudm \quad ; \quad u(j, j) = 1,\ u(j_r, j) = 0$$

[1] Think of $B$ timechanged to stay above zero and its characterisation via Skorokhod's equation.

if $x \in (j, j_r)$. So by Itô one can write the martingale $\int_0^t e^{-\lambda L_s^\theta} e^{-zV_s} u_x(B_s, j) dB_s$ in the form

$$e^{-\lambda L_t^\theta} e^{-zV_t} u(B_t, j) + \lambda \int_0^t e^{-\lambda L_s^\theta} e^{-zV_s} dL_s^j - \tfrac{1}{2}\Delta u_x(j,j) \int_0^t e^{-\lambda L_s^\theta} e^{-zV_s} dL_s^j$$

$$- \tfrac{1}{2}\Delta u_x(j_r, j) \int_0^t e^{-\lambda L_s^\theta} e^{-zV_s} dL_s^{j_r} - \tfrac{1}{2}\Delta u_x(j_\ell, j) \int_0^t e^{-\lambda L_s^\theta} e^{-zV_s} dL_s^{j_\ell}$$

the process being uniformly bounded for $z$ purely imaginary and $\lambda > 0$. If we stop it at first return to the boundary after time $\tau$ and take the expectation then

$$\mathbf{E}_x\left[e^{-\lambda L_\tau^\theta} e^{-zV_T \circ \theta_\tau}; B_T \circ \theta_\tau = j\right] = u(x, j) - \kappa_{jj}(z)\Psi_\lambda(z, j, x)$$

$$- \kappa_{jj_r}(z)\Psi_\lambda(z, j_r, x) - \kappa_{jj_\ell}(z)\Psi_\lambda(z, j_\ell, x) - \lambda\Psi_\lambda(z, j, x)$$

with $\Psi_\lambda(z, j, x) = \mathbf{E}_x\left[\int_0^\tau e^{-\lambda L_s^\theta} e^{-zV_s} dL_s^j\right]$. In vector form this reads

$$\mathbf{\Phi}_\lambda^+(z, x) = \mathbf{\Upsilon}^-(z, x) - \mathbf{K}(z)\mathbf{\Psi}_\lambda^-(z, x) - \lambda\mathbf{\Psi}_\lambda^-(z, x) \tag{2.1}$$

where $\mathbf{K}$, called the symbol, is a tri-diagonal matrix indexed by $\partial \times \partial$ and the superscripts $\pm$ are meant to suggest analyticity properties — the function $z \to \mathbf{\Phi}_\lambda^+(z, x)$ is bounded and analytic on the right half plane.

The solution of 2.1 is best explained analytically so we invert the Laplace transform, this being straightforward for all terms bar one, the symbol $\mathbf{K}(z)$ which we must examine in more detail. For the diagonal terms $\mathbf{K}_{jj}$ we denote by $\eta_t = \eta(j, t)$ the right continuous inverse of the local time $L^j$, whereupon $\kappa_{jj}(z) = -\frac{1}{2}\Delta u_x(j, j) = \int (1 - e^{-zx})\, \nu_{jj}(dx)$ with $\nu_{jj}$ the Lévy measure of the excursion functional $t \to \int_{j_\ell}^{j_r} L(a, \eta_t) m(da)$ under the law of $B_t$ conditioned to return to $j$ before hitting the points $j_r$ and $j_\ell$ — right and left excursions are therefore calculated by using the respective Bessel laws $\mathfrak{bes}_{j_r}(3)$ and $\mathfrak{bes}_{j_\ell}(3)$. This implies the existence of a Schwartz distribution defined on test functions by $\mathfrak{S}_{jj}(f) = \int [f(0) - f(x)]\, \nu_{jj}(dx)$, hence $\mathfrak{S}_{jj}(e^{-z\cdot}) = \mathbf{K}_{jj}(z)$. For the off-diagonal terms we have a similar description: $\kappa_{jj_r}(z) = \frac{1}{2}\Delta u_x(j_r, j) = \int (1 - e^{-zx})\, \nu_{j_r j}(dx)$ where $\nu_{j_r j}$ is the Lévy measure of the excursion functional $\int_j^{j_r} L(a, T) m(da)$ computed under the law of $B$ exiting $j_r$ and conditioned to re-enter $\partial$ at $j_r$, in other words the excursion law of $\mathfrak{bes}_j(3)$ from $j_r$.

This means that we can interpret 2.1 as the Laplace transform of a vector convolution equation

$$\mathbf{\Pi}_\lambda^+(x, dt) \quad = \quad \Sigma(x, dt) \quad - \quad \mathfrak{S} * \mathbf{r}_\lambda(x, dt) \quad - \quad \lambda\mathbf{r}_\lambda(x, dt) \tag{2.2}$$

where $\mathfrak{S}$ is a distributional matrix such that $\mathfrak{S}(e^{-z\cdot}) = \mathbf{K}(z)$ (since each $\mathfrak{S}_{jj}$ is the sum of a finite measure and a distribution with compact support $\mathfrak{S} * \mathbf{r}_\lambda$ is well-defined [9]). Note how the left side of 2.2 is supported on $[0, \infty)$ whereas the terms on the right — bar the distribution $\mathfrak{S}$ which is 'mixed' — are all supported on $(-\infty, 0]$. We therefore have an equation of Wiener-Hopf type for $\mathbf{\Pi}_\lambda^+(x, dt) = (\mathbf{E}_x[e^{-\lambda L_\tau^\theta} \mathbf{P}_{B_\tau}[V_T \in dt; B_T = j]])$.

## 3. Solution

The method for solving 2.2 is well-known [2] and depends on factoring the distributional matrix $\mathfrak{S} = \mathfrak{S}^+ * \mathfrak{S}^-$. Probabilistic proofs of the following results are postponed to sections 5 and 6 where the interpretation will make it plain as to why the various convolutions can be defined.

**Lemma 3.1** The distribution $\mathfrak{S}$ has Wiener-Hopf factors $\mathfrak{S}^\pm$ where $(\mathfrak{S}^+)^{-1}$ is a matrix-valued positive $\sigma$-finite Radon measure on $(0,\infty)$ and $\mathfrak{S}^-$ is supported on $(-\infty, 0]$. Moreover:
(1) $(\mathfrak{S}^+)^{-1}_{ji} \le (\mathfrak{S}^+)^{-1}_{ii}$.
(2) For $K$ a fixed compact set, $\lim_{\lambda\downarrow 0} \lambda (\mathfrak{S}^+)^{-1} * \mathbf{r}_\lambda(K) = 0$.
(3) $\lim_{z\downarrow -\infty} z^{-1}\mathfrak{S}^-(e^{-z\cdot}) = 0$.

*Remarks:* (1) By definition the $\mathfrak{S}^\pm$ are unique modulo a distribution supported at the origin; 3.1 (3) restricts the choice even further.
(2) Although $\mathfrak{S}$ is tri-diagonal its factors are not and this poses a difficulty when $\partial$ is infinite. Strictly speaking, one should specify the solution space for 2.2 as perhaps an inductive limit space of vector measures on $\mathbf{R}$. But this would take us too far afield, particularly since our main concern here is with the factorisation method and not the solution of the equation. So convergence in 3.1, and elsewhere, will always be interpreted coordinatewise.

Recall the method for solving 2.2. Convolution on the left with $(\mathfrak{S}^+)^{-1}$ gives

$$(\mathfrak{S}^+)^{-1} * \Pi^+_\lambda \quad = \quad (\mathfrak{S}^+)^{-1} * \Sigma \quad - \quad \mathfrak{S}^- * \mathbf{r}_\lambda \quad - \quad \lambda (\mathfrak{S}^+)^{-1} * \mathbf{r}_\lambda$$

and the idea is to eliminate the middle term on the right by projection onto $\mathbf{R}^+$, denoted $\mathbf{P}^+$. We must therefore verify that the distribution does not charge the origin. In the case of $\mathbf{r}_\lambda$ this is straightforward, since by the definition of $\partial$ in the closed support of $m^+$ we have $\mathbf{P}_\partial[\tau > 0] = 0$. Moreover, since the singular support at zero of $\mathfrak{S}^-$ is necessarily a linear combination of a Dirac mass and its derivatives, and since 3.1 (3) excludes the latter, the distribution $\mathfrak{S}^- * \mathbf{r}_\lambda$ cannot charge zero. We therefore have $\mathbf{P}^+[\mathfrak{S}^- * \mathbf{r}_\lambda] = 0$, and taking $\lambda \downarrow 0$ and applying 3.1 (2) we deduce

$$\Pi^+_0(x, dt) = \mathfrak{S}^+ * \mathbf{P}^+ \left[ (\mathfrak{S}^+)^{-1} * \Sigma(x, .) \right] (dt)$$

*Remarks:* (1) If $\partial$ is a singleton then the relation with the analytic problem is transparent: $\mathfrak{S}^+$ can be interpreted as the generator of $V_t$ observed in the local time scale at its maximum. The complex variable approach, in the case of symmetric stable processes, is described in Ray's paper [8].
(2) We omit the details of how one recovers $\Pi^+(x, dy)$ from $\Pi^+_0(x, dt)$ since this involves inverting an integral transform and can lead to delicate uniqueness questions.
(3) In some simple cases 2.1 can be solved explicitly for $\Pi^+(x, dy)$ — but apparently not by using the factorisation described here. This is a mystery even for real processes.

**4. Excursions**

We prove 3.1 by decomposing the path at the maxima of $V_t$ observed when $L_t^\partial$ increases, this process being finite almost surely since $\partial$ is a discrete set. Start by writing $\sigma^\partial$ for its right continuous inverse and define $Y_t = V(\sigma_t^\partial)$. Then the maximum $\bar{Y}_t = \sup\{0 < s \le t : Y_s\}$ defines a random set $\mathcal{M} = \{t : Y_t = \bar{Y}_t \text{ or } Y_{t-} = \bar{Y}_t\}$. We will factor our matrix by using excursions of $Y$ from $\mathcal{M}$, but since the set is not homogeneous the method of [4] does not apply directly. Instead, we modify their argument by using excursions from the maximum as observed from a point on the boundary.

To make this precise we introduce $\mathcal{M}^j = \{t \in \mathcal{M} : B(\sigma_t^\partial) = j\}$. By the strong Markov property of $B$ this is a closed regenerative set in the sense of Maisonneuve [6] and, being optional in the $\mathcal{B}(\sigma_t^\partial)$ filtration, it has an adapted local time $L^{\mathcal{M}^j}$ whose right continuous inverse we denote $\sigma^j$. Each gap $(\sigma_{t-}^j, \sigma_t^j)$ then defines an excursion with corresponding excursion measure denoted $Q^j$. The basic formula of [6] says that if $A$ is an additive functional of $Y$ then

$$\sum_{0<s\le t} \mathcal{A}_s \circ \theta_{\sigma_{s-}^j} - \int_0^t Q^j [\mathcal{A}_s] \, ds$$

is a martingale[1] — here $\mathcal{A}_t = \Delta A_{\sigma_t}$ is the change in the value of $A$ over the excursion interval $(\sigma_{t-}^j, \sigma_t^j)$. By stochastic integration of the above martingale against a bounded predictable process $h$ it follows that

$$\sum_{0<s\le t} h_s \mathcal{A}_s \circ \theta_{\sigma_{s-}^j} - \int_0^t h_s Q^j [\mathcal{A}_s] \, ds$$

is also a martingale. We remark for later (and frequent) use that if $h$ is càglàd then in particular $h_{t+}$ will be defined and can be substituted for $h_t$ in the integral on the right.

The excursion measures $Q^j$ can be decomposed still further, according to whether the excursion from $\mathcal{M}^j$ straddles a point in $\mathcal{M}^k$ or not. If not, then we have an excursion interval common to $\mathcal{M}$ and $\mathcal{M}^j$ with excursion measure denoted $Q^{jj}$. On the other hand, excursions from $\mathcal{M}^j$ which do straddle points in $\mathcal{M}$ have their excursion measure labelled $Q^{jk}$ where $k$ is the first point of maximum after leaving $\mathcal{M}^j$: for $k \ne j$ an excursion governed by $Q^{jk}$ first re-enters $\mathcal{M}$ at a point of $\mathcal{M}^k$, but it does not die there, instead continuing on until it reaches a point of $\mathcal{M}^j$.

The advantage of working with $\mathcal{M}^j$, and not $\mathcal{M}$, is that the $\mathcal{M}^j$ are homogeneous regenerative sets [6]; the downside is that this involves the manipulation of multiple timescales $L_t^{\mathcal{M}^j}$, unique only up to constants, and it would be better if, somehow, we could patch all these together to obtain $L^{\mathcal{M}} = \sum_{j\in\partial} L^{\mathcal{M}^j}$. To do this canonically let us introduce $N_t = B_{\sigma_t^\partial} \in \partial$ and denote by $\sigma^{\mathcal{M}}$ the right-continuous inverse of $L^{\mathcal{M}}$. The process $N_{\sigma^{\mathcal{M}}}$ is a Markov chain which we call 'the boundary chain at the

[1] We emphasise that the original filtration has been timechanged twice — first do $t \to \sigma_t^\partial$ and then $t \to \sigma_t^j$.

maximum'. If $\mathcal{M}$ is recurrent, $N_{\sigma^{\mathcal{M}}}$ has an invariant measure $\{\pi_j : j \in \partial\}$ and we can normalise its exponential holding times by $\lambda_j = \pi_j$.

*Notation:* When convenient, we shall simplify by writing $\sigma$ in place of $\sigma^{\mathcal{M}}$ or $\sigma^j$.

## 5. Existence

This is the main part of the paper. We prove existence of the factorisation under the assumption that the boundary is discrete, $Y$ spends zero time in $\mathcal{M}$, and the boundary chain at the maximum is recurrent. From Rogozin's trichotomy [1] for Lévy processes, any individual $\mathcal{M}^j$ is either recurrent or transient almost surely; by the strong Markov property it follows that the $\mathcal{M}^j$ are either all unbounded or all transient. We prove existence in the former case only — if the maximum is transient, and $-Y$ satisfies the conditions indicated, then one can use the set of minima instead.

First we outline the method, starting with the problem of finding factors $\mathbf{K}^{\pm}$ such that $\mathbf{K}\mathbf{K}^{-}\mathbf{K}^{+} = \mathbf{I}$ where the tri-diagonal matrix $\mathbf{K}$ is defined by the martingale of section two

$$e^{-zV_t}e^{-\lambda L_t^\partial}u(B_t,j) + \kappa_{jj}(z)\int_0^t e^{-\lambda L_s^\partial}e^{-zV_s}\,dL_s^j + \kappa_{jj_r}(z)\int_0^t e^{-\lambda L_s^\partial}e^{-zV_s}\,dL_s^{j_r}$$

$$+\kappa_{jj_l}(z)\int_0^t e^{-\lambda L_s^\partial}e^{-zV_s}\,dL_s^{j_l} + \lambda\int_0^t e^{-\lambda L_s^\partial}e^{-zV_s}\,dL_s^j$$

For $z$ purely imaginary and $\lambda > 0$ this is uniformly bounded. Starting at $B_0 = i \in \partial$ and applying martingale stopping as $t \uparrow$ then, since $L_t^\partial \uparrow$, boundedness of $u(x,j)$ gives

$$\kappa_{jj_l}(z)\mathbf{E}_i\left[\int_0^\infty e^{-\lambda L_s^\partial - zV_s}\,dL_s^{j_l}\right] + \kappa_{jj}(z)\mathbf{E}_i\left[\int_0^\infty e^{-\lambda L_s^\partial - zV_s}\,dL_s^{j}\right]$$

$$+\kappa_{jj_r}(z)\mathbf{E}_i\left[\int_0^\infty e^{-\lambda L_s^\partial - zV_s}\,dL_s^{j_r}\right] + \lambda\mathbf{E}_i\left[\int_0^\infty e^{-\lambda L_s^\partial - zV_s}\,dL_s^{j}\right] = \delta_{ji}$$

We do the factorisation from this. But first let us simplify our notation. Recall that when factorising real Lévy processes one does the calculations with $\lambda > 0$, taking $\lambda \downarrow 0$ at the end. In our case such reasoning leads to

$$\kappa_{jj_l}(z)\mathbf{E}_i\left[\int_0^\infty e^{-zV_s}\,dL_s^{j_l}\right] + \kappa_{jj}(z)\mathbf{E}_i\left[\int_0^\infty e^{-zV_s}\,dL_s^{j}\right]$$

$$+\kappa_{jj_r}(z)\mathbf{E}_i\left[\int_0^\infty e^{-zV_s}\,dL_s^{j_r}\right] = \delta_{ji}$$

where the expectations are interpreted as weak limits in $\lambda$. So the *claim* is that this doubly infinite family of equations indexed by $\partial \times \partial$ is precisely $\mathbf{K}\mathbf{K}^{-}\mathbf{K}^{+} = \mathbf{I}$ and that, moreover, one can determine the entries of $\mathbf{K}^{\pm}$ by decomposing the above integrals appropriately. This would give our desired factorisation $\mathfrak{S} = \mathfrak{S}^{+} * \mathfrak{S}^{-}$ in the form

$$\kappa^{-}(z) = \left(\mathfrak{S}^{-}\right)^{-1} e^{-z\cdot} \quad ; \quad \kappa^{+}(z) = \left(\mathfrak{S}^{+}\right)^{-1} e^{-z\cdot}$$

using invertibility in the sense of distributions.

Of course to make this rigorous one should do the argument with $\lambda > 0$. The difficulty there is that keeping track of all the different timescales would present a notational nightmare and consequently, since $\lambda$ does not figure in the final answer, we suppress all mention of it in our calculations with the *caveat* that $\lambda > 0$ is essential for justifying the various manipulations.

With this in mind we set out to identify the factor matrices $\kappa^{\pm}$. Recalling the notation $Y = V_{\sigma^\theta}$ and $N = B_{\sigma^\theta}$, we start from our conjecture in the form

$$\sum_k \kappa_{ik}^{-}\kappa_{kj}^{+} = \left(\kappa^{-}\kappa^{+}\right)_{ij} = \mathbf{E}_j\left[\int_0^{\infty} e^{-zV_s} dL_s^i\right] = \mathbf{E}_j\left[\int_0^{\infty} e^{-zY_s} 1_{(N_s=i)} ds\right] \quad (5.1)$$

and the idea of calculating with excursions from $\mathcal{M}$, the set of maxima of $Y$. Since $Y$ spends no time in $\mathcal{M}$ we see that $\sigma = \sigma^j$, the right continuous inverse of $L^{\mathcal{M}^j}$, is a pure jump process. The integral therefore decomposes into its excursions from $\mathcal{M}^j$

$$\sum_{t>0} e^{-zY_{\sigma_t -}} \int_{\sigma_{t-}}^{\sigma_t} e^{-zY_u + zY_{\sigma_t -}} 1_{(N_u=i)} du = \sum_{t>0} e^{-zY_{\sigma_t -}} \left[\int_0^{\zeta} e^{-zY_u} 1_{(N_u=i)} du\right] \circ \theta_{\sigma_t -}$$

where $\zeta^j = \zeta$ is the excursion lifetime. Now apply the excursion theorem to see that

$$\sum_{0<s\leq t} e^{-zY(\sigma_s -)} \int_0^{\zeta} e^{-zY_u} 1_{(N_u=i)} du \circ \theta_{\sigma_s -} - \int_0^t e^{-zY(\sigma_s)} ds Q^j \left[\int_0^{\zeta} e^{-zY_u} 1_{(N_u=i)} du\right]$$

is a martingale[1]. Taking the expectation therefore gives[2]

$$\mathbf{E}_j\left[\int_0^{\infty} e^{-zY_s} 1_{(N_s=i)} ds\right] = \mathbf{E}_j\left[\int_0^{\infty} e^{-zY(\sigma_s^j)} ds\right] Q^j \left[\int_0^{\zeta} e^{-zY_u} 1_{(N_u=i)} du\right]$$

which we propose to write as $\sum_k \kappa_{ik}^{-}/\kappa_{kj}^{+}$ by exploiting a suitable decomposition of $Q^j$ — for convenience we take $\kappa_{ij}^{-} = \kappa_{ij}^{-}$ but $\kappa_{ij}^{+} = 1/\kappa_{ij}^{+}$.

The diagonal entries of $\kappa^{+}$ have the most transparent definition: since $Y_{\sigma^i}$ is a subordinator, with Laplace exponent $\kappa_{ii}^{+}$ (say), we can take $\kappa_{ii}^{+} = 1/\kappa_{ii}^{+}$. For the other entries, recall that in the previous section we decomposed the excursion measures as $Q^i = \sum_{j\in\partial} Q^{ij}$; the measure $Q^{ii}$ is supported on excursions from $\mathcal{M}^i$ which do not straddle any points in $\mathcal{M}$ and, since $Y_u \leq 0$ throughout, it follows that $Q^{ii}[\int_0^{\zeta} e^{-zY_u} 1_{(N_u=i)} du]$ is the Laplace transform of a measure supported on $(-\infty, 0]$. The temptation then is to place these terms along the diagonal of $\kappa^{-}$. But this is wrong. We show later that $\kappa_{ii}^{-}$ is more complicated and contains additional terms

---

[1] Thanks to our phantom $\lambda$.

[2] The price we pay for homogeneity is that $Q^j$ is not necessarily supported on $(-\infty,0]$.

coming from a path decomposition of the $Q^{ij}$. This observation — that 5.1 entails complicated cross-cancellations amongst the various excursion terms — prompted us to devise a notation for keeping track of the relevant components of the path. Our argument is best understood in the

2 × 2 case

Fix $i \neq j$ throughout. We start by decomposing $\mathbf{E}_i[\int_0^\infty e^{-zY_s} 1_{(N_s=i)} ds]$ using excursions from $\mathcal{M}^i$ in the timescale $\sigma_t^i = \sigma_t$. These split naturally into two kinds: a $Q^i$ excursion either straddles a point of $\mathcal{M}^j$ or it doesn't. By the definition of $\kappa_{ii}^+$, the excursion theorem gives

$$(\kappa^-\kappa^+)_{ii} = Q^{ii}\left[\int_0^\zeta e^{-zY_s} 1_{(N_s=i)} ds\right]/\kappa_{ii}^+ + Q^{ij}\left[\int_0^\zeta e^{-zY_s} 1_{(N_s=i)} ds\right]/\kappa_{ii}^+$$

which we want in the form $\kappa_{ii}^-/\kappa_{ii}^+ + \kappa_{ij}^-/\kappa_{ji}^+$. The idea is to use path decomposition inside $Q^{ij}[\int_0^\zeta e^{-zY_u} 1_{(N_u=k)} du]$ by noting that an excursion from $\mathcal{M}^i$ which straddles a point in $\mathcal{M}^j$ has three distinct components, each non-trivial:
$\mathfrak{a}$) The initial excursion until we arrive in $\mathcal{M}^j$.
$\mathfrak{b}$) Excursions from $\mathcal{M}^j$ such that, even if $N$ visits $i$, $Y$ cannot achieve a maximum there.
$\mathfrak{c}$) The final excursion from $\mathcal{M}^j$ back to $\mathcal{M}^i$.
We write all this as $Q_k^{ij}[\mathfrak{a}+\mathfrak{b}+\mathfrak{c}]$ and remark immediately that $Q_k^{ij}[\mathfrak{a}]$ is the LT of a measure supported on $(-\infty, 0]$, the same being true for

$$Q_k^{i\to\mathcal{M}} = Q^{ii}\left[\int_0^\zeta e^{-zY_u} 1_{(N_u=k)} du\right] + Q_k^{ij}[\mathfrak{a}]$$

It is therefore legitimate to define the entries of $\kappa^-$ by

$$\kappa_{ij}^- = \kappa_{ij}^- = Q_i^{j\to\mathcal{M}} \quad ; \quad \kappa_{ii}^- = \kappa_{ii}^- = Q_i^{i\to\mathcal{M}}$$

thereby reducing the problem to defining the off-diagonal entries of $\kappa^+$ in a manner consistent with

$$(\kappa^-\kappa^+)_{ii} = Q_i^{i\to\mathcal{M}}/\kappa_{ii}^+ + Q_i^{ij}[\mathfrak{b}+\mathfrak{c}]/\kappa_{ii}^+$$

We start by writing $Q_k^{ij}[\mathfrak{b}+\mathfrak{c}]$ in a more convenient form. First modify our notation so that $T = T_{\mathcal{M}}$ represents the first time $Y$ enters $\mathcal{M}$ and recall that $N_{\sigma\mathcal{M}}$ is the 'boundary chain at the maximum' whose exponential holding times have parameters $\{\lambda_i : i \in \partial\}$. The passages $\mathcal{M}^i \to \mathcal{M}\setminus\mathcal{M}^i = \mathcal{M}^j$ are then Poisson processes of rate $\lambda_i$ so that in obvious notation $Q^{ij}[e^{-zY_T}] = \lambda_i \mathbf{E}^{ij}[e^{-zY_T}]$. Applying the excursion theorem from $\mathcal{M}^j$ (*sic*) now gives

$$\begin{aligned} Q_k^{ij}[\mathfrak{b}] &= \lambda_i \mathbf{E}^{ij}\left[e^{-zY_T}\right]\mathbf{E}^{jj}\left[\int_0^\xi e^{-zY(\sigma_t)}dt\right] Q^{jj}\left[\int_0^\zeta e^{-zY_u} 1_{(N_u=k)} du\right] \\ &= \lambda_i\lambda_j^{-1}\mathbf{E}^{ij}\left[e^{-zY_T}\right]\mathbf{E}^{jj}\left[e^{-zY_{\sigma_\xi}}\right] Q^{jj}\left[\int_0^\zeta e^{-zY_u} 1_{(N_u=k)} du\right] \end{aligned}$$

where $\xi = \xi_j = L_\zeta^{\mathcal{M}^j} \sim \exp(\lambda_j)$ is the holding time at $j$. The final part of the $Q^{ij}$ excursion is dealt with similarly: we use the strong Markov property to expand

$$\begin{aligned} Q_k^{ij}[\mathfrak{c}] &= \lambda_i \mathbf{E}^{ij}\left[e^{-zY_T}\right]\mathbf{E}^{jj}\left[e^{-zY_{\sigma_\xi}}\right]\mathbf{E}^{ji}\left[\int_0^T e^{-zY_u}1_{(N_u=k)}du\right] \\ &= \lambda_i\lambda_j^{-1}\mathbf{E}^{ij}\left[e^{-zY_T}\right]\mathbf{E}^{jj}\left[e^{-zY_{\sigma_\xi}}\right]Q_k^{ji}[\mathfrak{a}] \end{aligned}$$

The shorthand[1]

$$e^{ij} = \lambda_i\lambda_j^{-1}\mathbf{E}^{ij}\left[e^{-zY_T}\right]\mathbf{E}^{jj}\left[e^{-zY_{\sigma_\xi}}\right] \quad ; \quad E_k^{ij} = \mathbf{E}^{ij}\left[\int_0^T e^{-zY_u}1_{(N_u=k)}du\right]$$

now lets us write $Q_k^{ij}[\mathfrak{b}+\mathfrak{c}] = e^{ij}Q_k^{j\to\mathcal{M}}$ which, in light of the desired decomposition of $(\mathrm{K}^-\mathrm{K}^+)_{ii}$, forces

$$\mathrm{K}_{ji}^+ = 1/\kappa_{ji}^+ = e^{ij}/\kappa_{ii}^+$$

We now see that we have a proof of 5.1 and so the proof of factorisation for the $2\times 2$ case is complete.

The next step is to look at the

## 3 × 3 case

Here the main difference is that a $Q^{ij}$ excursion may visit $\mathcal{M}^k$ for $k \neq i,j$. Taking our cue from the above, we define $\mathrm{K}_{ii}^+ = 1/\kappa_{ii}^+ = \mathbf{E}[\int_0^\infty e^{-zY(\sigma_t^i)}dt]$, and denoting $Q_k^{ij} = Q^{ij}[\int_0^\zeta e^{-zY_t}1_{(N_t=k)}dt]$, we propose

$$\mathrm{K}_{ij}^- = \kappa_{ij}^- = Q_i^{j\to\mathcal{M}} = Q_i^{jj} + \sum_{k\neq j} Q_i^{jk}[\mathfrak{a}] = Q_i^{jj} + \lambda_j \sum_{k\neq j} p_{jk}E_i^{jk}$$

The problem is now to determine the off-diagonal entries in $\mathrm{K}^+$. For this we will decompose $Q_k^{ij}[\mathfrak{b}+\mathfrak{c}] = Q_k^{ij} - Q_k^{ij}[\mathfrak{a}]$ by tracking the boundary chain $N_{\sigma\mathcal{M}}$ whose transition matrix and holding times we denote respectively by $p_{ij}$ and $\xi_j \sim \exp(\lambda_j)$. Also, taking $i,j,j'$ all distinct, we introduce the notation $\mathfrak{f}_{ij} = 1 - p_{jj'}p_{j'j}e^{jj'}e^{j'j}$ with $e^{ij}$ defined as before. Now to calculate. By definition a $Q^{ij}$ excursion first visits $\mathcal{M}^j$ and the journey gives rise to a factor $\lambda_i p_{ij}\mathbf{E}^{ij}\left[e^{-zY_T}\right]$. The contribution from its initial sojourn in $\mathcal{M}^j$, which lasts for time $\xi_j$, is

$$\lambda_i p_{ij}\mathbf{E}^{ij}\left[e^{-zY_T}\right]\lambda_j^{-1}\mathbf{E}^{jj}\left[e^{-zY_{\sigma_\xi}}\right]Q_k^{jj} = p_{ij}e^{ij}Q_k^{jj}$$

with a factor $p_{ij}\lambda_j e^{ij}$ prefixing what remains — either termination at $\mathcal{M}^i$ which gives $p_{ij}\lambda_j e^{ij}p_{ji}E_k^{ji}$, or else a passage to $\mathcal{M}^{j'}$ which adds on

$$p_{ij}\lambda_j e^{ij}\left[p_{jj'}E_k^{jj'} + \frac{p_{jj'}Q_k^{ij'}[\mathfrak{b}+\mathfrak{c}]}{\lambda_i p_{ij'}\mathbf{E}^{ij'}\left[e^{-zY_T}\right]}\right]$$

[1] Rem: caps for negative, positive in lower case.

The result is

$$\mathcal{Q}_k^{ij}[\mathfrak{b}+\mathfrak{c}] = p_{ij}e^{ij}\left[Q_k^{j\to\mathcal{M}} + \frac{p_{jj'}}{p_{ij'}}\frac{e^{jj'}}{e^{ij'}}\mathcal{Q}_k^{ij'}[\mathfrak{b}+\mathfrak{c}]\right]$$

Noting that $i$ is fixed here, switching $j \leftrightarrow j'$ yields a $2\times 2$ system of equations of the form

$$\rho^j = Q_k^{j\to\mathcal{M}} + p_{jj'}e^{jj'}\rho^{j'}$$

which we can solve for $\rho^j = \mathcal{Q}_k^{ij}[\mathfrak{b}+\mathfrak{c}]/(p_{ij}e^{ij})$ to obtain

$$\mathcal{Q}_k^{ij}[\mathfrak{b}+\mathfrak{c}] = \frac{p_{ij}e^{ij}}{\mathfrak{f}_{ij}}\left[Q_k^{j\to\mathcal{M}} + p_{jj'}e^{jj'}Q_k^{j'\to\mathcal{M}}\right]$$

The diagonal term $\mathbf{E}_i[\int_0^\infty e^{-zY_t}1_{(N_t=i)}dt]$ therefore decomposes like

$$\sum_j Q^{ij}\left[\int_0^\zeta e^{-zY_t}1_{(N_t=i)}dt\right]/\kappa_{ii}^+ = Q_i^{i\to\mathcal{M}}/\kappa_{ii}^+ + \sum_{j\neq i}\mathcal{Q}_i^{ij}[\mathfrak{b}+\mathfrak{c}]/\kappa_{ii}^+$$

$$= Q_i^{i\to\mathcal{M}}/\kappa_{ii}^+ + \sum_{j\neq i}p_{ij}e^{ij}\left[Q_i^{j\to\mathcal{M}} + p_{jj'}e^{jj'}Q_i^{j'\to\mathcal{M}}\right]/\mathfrak{f}_{ij}\kappa_{ii}^+ = \sum_j Q_i^{j\to\mathcal{M}}/\kappa_{ji}^+$$

which means we ought to take

$$\mathbf{K}_{ji}^+ = 1/\kappa_{ji}^+ = \frac{p_{ij}e^{ij} + p_{ij'}p_{j'j}e^{ij'}e^{j'j}}{\mathfrak{f}_{ij}\kappa_{ii}^+} \qquad (i\neq j)$$

This completes the proof of the $3\times 3$ case.

## Discrete Boundary

The boundary chain at the maximum is recurrent and so it has an invariant measure $\{\pi_i : i\in\partial\}$; as explained in §4 we normalise the holding times by $\lambda_i = \pi_i$. The proof here follows the plan for the $3\times 3$ case: the diagonal terms of $\mathbf{K}^+$ are defined as above, likewise the entries of $\mathbf{K}^-$, and we look at the decomposition of $\mathcal{Q}_k^{ij}[\mathfrak{b}+\mathfrak{c}]$ to help discover the rest of $\mathbf{K}^+$. If we define $\rho^j = \mathcal{Q}_k^{ij}[\mathfrak{b}+\mathfrak{c}]/(p_{ij}e^{ij})$ then arguing as before gives a system of equations

$$\rho^j = Q_k^{j\to\mathcal{M}} + \sum_{j'\neq i,j}p_{jj'}e^{jj'}\rho^{j'}$$

which we write in vector form as $\rho = \mathbf{Q} + {}_i\mathfrak{P}\rho$ ($i$ is a taboo point). The solution is therefore $\rho = \sum_{n=0}^\infty {}_i\mathfrak{P}^n\mathbf{Q}$: for $z\geq 0$ the series converges since (with notation from [3]) the entries of the sum matrix are dominated by $\sum_{n=0}^\infty p_{jk}^{(n)}\lambda_j\lambda_k^{-1} = \lambda_i\lambda_j^{-1}{}_i\pi_{ij}$ where ${}_i\pi_{ij}$ is a multiple of the invariant measure. This gives us

$$\mathcal{Q}_k^{ij}[\mathfrak{b}+\mathfrak{c}] = p_{ij}e^{ij}\left[\sum_{n=0}^\infty {}_i\mathfrak{P}^n\mathbf{Q}\right]_j$$

which must square with

$$\mathbf{E}_i\left[\int_0^\infty e^{-zY_t}1_{(N_t=i)}dt\right] = Q_i^{i\to\mathcal{M}}/\kappa_{ii}^+ + \sum_k Q_i^{ik}[\mathfrak{b}+\mathfrak{c}]/\kappa_{ii}^+$$

$$= Q_i^{i\to\mathcal{M}}/\kappa_{ii}^+ + \sum_k p_{ik}e^{ik}\left[\sum_{n=0}^\infty {}_i\mathfrak{P}^n\mathbf{Q}\right]_k/\kappa_{ii}^+ = \sum_j \mathbf{K}_{ij}^-\mathbf{K}_{ji}^+ = \sum_j Q_i^{j\to\mathcal{M}}\mathbf{K}_{ji}^+$$

Comparing coefficients of $Q_i^{j\to\mathcal{M}}$ we find

$$\mathbf{K}_{ji}^+ = \frac{1}{\kappa_{ii}^+}\sum_k p_{ik}e^{ik}\sum_{n=0}^\infty {}_i\mathfrak{P}_{kj}^n \qquad (j\neq i)$$

Since our decomposition satisfies 5.1, the proof of factorisation in the general discrete case is now complete.

The proof of 3.1 (1) follows, since for $z\geq 0$ we have $\mathbf{K}_{ji}^+/\mathbf{K}_{ii}^+ \leq \sum_k p_{ik}p_{kj}\lambda_k^{-1}\lambda_{ji}\pi_{ik} = \sum_k p_{ik}p_{kj} < 1$ using the normalisation $\lambda_j = \pi_j$ and the formula ${}_i\pi_{ik} = \pi_k/\pi_i$ of [3] 11.24. An immediate consequence is that the $\mathfrak{G}_{ij}^{-1}$ are positive $\sigma$-finite measures — they are dominated by $1/\kappa_{ii}$ which is the LT of a $\sigma$-finite Radon measure on the line (the potential of the Lévy process $Y_{\sigma^i}$). We have therefore proved the existence part of 3.1.

*Remarks:* (1) If $\partial$ is a singleton then the above probabilistic argument can be deduced from [4] but note that they deal with the process killed at an independent exponential time; this corresponds to factoring $\lambda + \mathbf{K}$, a task more difficult than factoring the symbol $\mathbf{K}$ alone.
(2) We obtain an analytic interpretation for our factorisation by noting that $\mathbf{K}$ is the LT of the generator of the Markov process $(V_{\sigma^\theta}, B_{\sigma^\theta})$.
(3) The set $\mathcal{M}$ is not homogeneous since the excursions of $Y = V_{\sigma^\theta}$ are only conditionally independent given the boundary process $N_{\sigma^\mathcal{M}}$. Nevertheless, one can use the exit system of $\mathcal{M}$ [6] to see (cf. after 5.1) that

$$\sum_{0<s\leq t} e^{-\lambda\sigma_{s-}^\mathcal{M} - zY(\sigma_{s-}^\mathcal{M})}\int_0^\zeta e^{-\lambda u - zY_u}1_{(N_u=i)}du\circ\theta_{\sigma_{s-}^\mathcal{M}}$$

$$-\int_0^t e^{-\lambda\sigma_s^\mathcal{M} - zY(\sigma_s^\mathcal{M})}Q^\mathcal{M}\left[\int_0^\zeta e^{-\lambda u - zY_u}1_{(N_u=i)}du; N_{\sigma_{s-}^\mathcal{M}}\right]ds$$

is a uniformly integrable martingale; $\mathbf{E}[\int_0^\infty e^{-\lambda L_t^\theta - zV_t}1_{(B_t=i)}dt]$ then takes the form

$$\mathbf{E}\left[\int_0^\infty e^{-\lambda\sigma_t^\mathcal{M} - zY(\sigma_t^\mathcal{M})}Q^\mathcal{M}\left[\int_0^{\zeta^\mathcal{M}} e^{-\lambda u - zY_u}1_{(N_u=i)}du; N_{\sigma_{t-}^\mathcal{M}}\right]dt\right]$$

and our factorisation is obtained by conditionally decomposing the expectation according to the values of $N_{\sigma_{t-}^\mathcal{M}}$.

## 6. Estimates

We turn now to the last step in justifying the solution of 2.1, which is the proof of 3.1 (2)-(3) — we dealt with 3.1 (1) at the end of §5. Our running assumption is that the maximum is recurrent and so $N_{\sigma\mathcal{M}}$ has an invariant measure $\{\pi_i : i \in \partial\}$. For the proof of the last part of 3.1 (3) we will assume that $\partial$ is finite.

In the previous section we obtained factors of the symbol matrix in the form $\kappa\kappa^-\kappa^+ = 1$ where the off-diagonal elements of $\kappa^{\pm}(z) = (\mathfrak{S}^{\pm})^{-1}(e^{-z\cdot})$ are

$$(\mathfrak{S}^-)^{-1}_{ij}(e^{-z\cdot}) = Q_i^{j\to\mathcal{M}} \quad ; \quad (\mathfrak{S}^+)^{-1}_{ji}(e^{-z\cdot}) = \frac{1}{\kappa^+_{ii}}\sum_k p_{ik}e^{ik}\sum_{n=0}^{\infty} {}_i\mathfrak{P}^n_{kj}$$

From [9] we know that convolution is well-defined for

a) bounded measures,

b) distributions supported on the same half-line,

c) two distributions if one of them has compact support.

But to justify all the steps in the solution of 2.2 we also need the following: for $\mu = \{\mu_i\}$ a vector measure supported on $(-\infty, 0]$, such that $\sum_i \mu_i(\mathbf{R}^-) < \infty$, the formula $\sum_i (\mathfrak{S}^+)^{-1}_{ji} * \mu_i$ defines a Radon measure (we used $\mu = \Sigma, \mathbf{r}_\lambda$). To show this it suffices, by 3.1 (1), to prove that for any compact $K$ containing zero

$$\sum_i (\mathfrak{S}^+)^{-1}_{ii} * \mu_i(K) = \sum_i \int_{-\infty}^0 \mu_i(dy)\mathbf{E}_i\left[\int_0^\infty 1_{(Y_t\in K-y)}dL_t^{\mathcal{M}^i}\right] < \infty$$

However, by the strong Markov property at first entry into $K-y$ and translation invariance, we can replace $K-y$ by $K$. Using $L^{\mathcal{M}^i} \le L^{\mathcal{M}}$ now gives the bound

$$\sum_i \mu_i(\mathbf{R}^-)\mathbf{E}_i\left[\int_0^\infty 1_{(Y_t\in K)}dL_t^{\mathcal{M}}\right] = \sum_i \mu_i(\mathbf{R}^-)\mathbf{E}_{\mu^\partial}\left[\int_0^\infty 1_{(Y_t\in K)}dL_t^{\mathcal{M}}\right]$$

with $\mu^\partial\{i\} = \mu_i(\mathbf{R}^-)/\sum_i \mu_i(\mathbf{R}^-)$ as initial probability. The expectation is finite since the Markov process $(Y_{\sigma\mathcal{M}}, N_{\sigma\mathcal{M}})$ is transient, and the result is proved.

The same reasoning gives a proof of 3.1 (2), where for fixed compact set $K$ we need $\lim_{\downarrow 0}\lambda\sum_i(\mathfrak{S}^+)^{-1}_{ji} * r^i_\lambda(K) = 0$. Taking $\lambda \le 1$ we bound by

$$\sum_i r^i_\lambda(\mathbf{R}^-)\mathbf{E}_i\left[\int_0^\infty 1_{(Y_t\in K)}dL_t^{\mathcal{M}}\right] \le \sum_i r^i_\lambda(\mathbf{R}^-)\mathbf{E}_{\mu_1}\left[\int_0^\infty 1_{(Y_t\in K)}dL_t^{\mathcal{M}}\right]$$

with $\mu_1\{i\} = r^i_1(\mathbf{R}^-)/\sum_i r^i_\lambda(\mathbf{R}^-)$ and, since the expectation is bounded, it suffices to show $\lim_{\lambda\downarrow 0}\lambda\sum_i r^i_\lambda(\mathbf{R}^-)$. Recalling $r^i_\lambda(dx) = \mathbf{E}[\int_0^\tau e^{-\lambda L_t^\partial}1_{(V_t\in dx)}dL_t^i]$ we get

$$\lambda\sum_i r^i_\lambda(\mathbf{R}^-) = \sum_i \lambda\mathbf{E}\left[\int_0^\tau e^{-\lambda L_t^\partial}dL_t^i\right] = \lambda\mathbf{E}\left[\int_0^\tau e^{-\lambda L_t^\partial}dL_t^\partial\right] = \mathbf{E}\left[1 - e^{-\lambda L_\tau^\partial}\right]$$

This converges to zero by dominated convergence and $\tau < \infty$ a.s.

Turning now to the proof of 3.1 (3), we will write $(\mathfrak{S}^-)^{-1} = \mathfrak{D} - \mathfrak{O}$ where $\mathfrak{D}_{ii} = (\mathfrak{S}^-)^{-1}_{ii}$ is diagonal. The proof exploits the expansion $\mathfrak{S}^- = \mathfrak{D}^{-1} \sum_{n\geq 0} \mathfrak{D}^{-n}\mathfrak{O}^n$; we show:

d) $\lim_{z\downarrow -\infty} z^{-1}\mathfrak{D}^{-1}(e^{-z\cdot}) = 0$ pointwise.

e) $\sum_{n\geq 0} \mathfrak{D}^{-n}\mathfrak{O}^n(e^{z\cdot})$ converges in the supremum norm when $\Re z < 0$.

Let us start with d) where it suffices to see that $\lim_{z\downarrow -\infty} zQ_i^{ii} = \infty$ for each $i \in \partial$. Consider the Lévy process $\bar{Y}$ obtained by deleting from $Y = V_{\sigma^\theta}$ all excursions from $\mathcal{M}^i$ which enter $\mathcal{M} \setminus \mathcal{M}^i$. By a duality argument (meaning here decomposition at the minimum) $1/Q_i^{ii}$ appears as the Laplace exponent of the positive factor $\bar{Y}^+$, and the result follows.

For the proof of e) it suffices to see that the entries of $\mathfrak{D}^{-1}\mathfrak{O}$ are uniformly strictly less than one. But these have the form $Q_i^{j\to\mathcal{M}}/Q_i^{i\to\mathcal{M}}$ where the only contribution comes when $N$ visits $i$; we write $T_i$ for the time of first passage. By the strong Markov property $Q_i^{j\to\mathcal{M}}$ decomposes as

$$Q_i^{j\to\mathcal{M}}\left[e^{-zY_{T_i}}\mathbf{E}_i\left[\int_0^\infty e^{-z\tilde{Y}_u}1_{(\tilde{N}_u=i,\tilde{Y}_u\leq -Y_{T_i})}du\right];T_i<\zeta\right]$$

with $(Y,N) \sim (\tilde{Y},\tilde{N})$ independent. Evaluating the expectation by using the excursion theorem for semi-regenerative sets (cf. remark at the end of §5) we have

$$\mathbf{E}_i\left[\int_0^\infty e^{-z\tilde{Y}(\sigma_u^i)}1_{(\tilde{Y}(\sigma_u^i)\leq -Y_{T_i})}du\right]Q_i^{i\to\mathcal{M}}$$

which gives the ratio $Q_i^{j\to\mathcal{M}}/Q_i^{i\to\mathcal{M}}$ in the form

$$Q_i^{j\to\mathcal{M}}\left[e^{-zY_{T_i}}\mathbf{E}_i\left[\int_0^\infty e^{-z\tilde{Y}(\sigma_u^i)}1_{(\tilde{Y}(\sigma_u^i)\leq -Y_{T_i})}du\right];T_i<\zeta\right]$$

As $z \downarrow -\infty$ this converges to zero and so we obtain a proof of 3.1(3) when $\partial$ is finite.

## REFERENCES


1. **Bingham N.** *Fluctuation theory in continuous time. Adv. Appl. Prob.* 7 (1975) 705-766.

2. **Dieudonne J.** *Eléments d'analyse, Tome 7.* Gauthier-Villars, Paris 1978.

3. **Feller W.** *An Introduction to Probability Theory and its Applications, Vol. 1. 3rd Ed.*, Wiley, New York 1968.

4. **Greenwood P. and Pitman J.** *Fluctuation identities for Lévy processes and splitting at the maximum. Adv. Appl. Prob.*, 12 (1980) 893-902.

5. **Jones D.S.** *Wiener-Hopf splitting of a* 2 × 2 *matrix. Proc. Roy. Soc. Lond. A.* 434 (1991) 419-433.

6. **Maisonneuve B.** *Systèmes régénératifs. Astérisque No. 15*, Société Mathématique de France, Paris 1974.

7. **Muskelishvili N.I.** *Singular integral equations.* Noordhoff, Groningen 1953.

8. **Ray D.B.** *Stable processes with absorbing barrier.* Trans. *Amer. Math. Soc.* 89 (1958) 16-24.

9. **Schwartz L.** *Théorie des distributions.* Hermann, Paris 1966.

10. **Williams D.** *Some aspects of Wiener-Hopf factorization. Phil.* Trans. *R. Soc. Lond. A.* 335 (1991) 593-608.