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Perturbed Bessel Processes

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There has been some interest in the literature in Brownian motion perturbed at its maximum; that is a process $(X_t; t \geq 0)$ satisfying

$$X_t = B_t + \alpha M_t^X, \tag{0.1}$$

where $M_t^X = \sup_{0 \leq s \leq t} X_s$ and $(B_t; t \geq 0)$ is Brownian motion issuing from zero. The parameter α must satisfy $\alpha < 1$. For example arc-sine laws and Ray-Knight theorems have been obtained for this process; see Carmona, Petit and Yor [3], Werner [16], and Doney [7]. Our initial aim was to identify a process which could be considered as the process X conditioned to stay positive. This new process behaves like the Bessel process of dimension three except when at its maximum and we call it a perturbed three-dimensional Bessel process. We establish Ray-Knight theorems for the local times of this process, up to a first passage time and up to infinity (see Theorem 2.3), and observe that these descriptions coincide with those of the local times of two processes that have been considered in Yor [18]. We give an explanation for this coincidence by showing, in Theorem 2.2, that these processes are linked to the perturbed three dimensional Bessel process by space-time transformations and time-reversal.

A process which could be termed a perturbed one-dimensional Bessel process (or perturbed reflected Brownian motion) has already been studied, originally by Le Gall and Yor [11] in connection with windings of Brownian motion, and more recently by Chaumont and Doney [5] as a time change of the positive part of doubly perturbed Brownian motion. We are therefore motivated to introduce perturbed Bessel processes of dimension d, for any $d \geq 1$. Our fundamental result about these processes is Theorem 1.1, which shows how a perturbed Bessel process of dimension d is related to an ordinary Bessel process of dimension d via a space-time transformation. From this we deduce several extensions of results known for ordinary Bessel processes. Thus these processes have the Brownian scaling property, a power of a perturbed Bessel process is a time-change of another perturbed Bessel process (see Theorem 4.2), and there are descriptions of the local times of these processes which show that the Ciesielski-Taylor identity extends to this situation (see Theorem 5.2). On the other hand, some familiar properties of Bessel processes do not extend to perturbed Bessel processes. Thus they are not Markov processes, squares of perturbed Bessel processes do not have the additivity property, and the law of a perturbed 3-dimensional Bessel process up to a first hitting time is not invariant under time reversal (see Theorem 2.2).

We also show that some of these results extend to the case $0 < d < 1$ (see section 3) and to the case where the perturbation factor is replaced by a function of M_t^X (see section 6). Finally, in section 7 we discuss briefly a class of processes which can be thought of as Bessel processes of dimension $d > 2$ perturbed at their future minimum.

1. An h-transform of perturbed Brownian motion

We begin by observing that if X satisfies (0.1) then

$$M_t^X = \frac{1}{1-\alpha} M_t^B, \tag{1.1}$$

where $M_t^B = \sup_{0\le s\le t} B_s$, and consequently we can construct X from B thus,

$$X_t = B_t + \frac{\alpha}{1-\alpha} M_t^B. \tag{1.2}$$

From this we can see that the bivariate process $(X_t, M_t^X; t \ge 0)$ is Markov, and the classical theory of h-transforms of Markov processes tells us how to proceed in order to condition on X_t being positive for all time. We must look for a function h, strictly positive on $\{(x,m) : x > 0\}$ and zero on the set $\{(x,m) : x = 0\}$, such that $h(X_t, M_t^X)$ is a martingale for the bivariate process killed when X is first zero. Applying Itô's formula we find that h is given by

$$h(x,m) = cxm^{-\alpha}, \tag{1.3}$$

for some constant c. Consequently one introduces, for each $a > 0$,

$$\mathbb{P}_a^{3,\alpha}|_{\mathcal{F}_t} = \frac{1}{a^{1-\alpha}} \frac{X_{t\wedge T_0}}{(M_{t\wedge T_0}^X)^\alpha} \cdot \mathbb{P}_a^{(\alpha)}|_{\mathcal{F}_t}$$

where $T_0 = \inf\{u : X_u = 0\}$ and $\mathbb{P}_a^{(\alpha)}$ is the law of X started from a. We have

$$B_t = \tilde{B}_t + \int_0^t ds\, \frac{h'_x}{h}(X_s, M_s^X), \tag{1.4}$$

where $\tilde{B}$ is a $\mathbb{P}_a^{3,\alpha}$-Brownian motion, and so we find that under this latter law X has the following semimartingale decomposition,

$$X_t = a + \tilde{B}_t + \int_0^t \frac{ds}{X_s} + \alpha(M_t^X - a). \tag{1.5}$$

Of course, when $\alpha = 0$, this reduces to the equation which defines the ordinary Bessel process of dimension three, and it is well known that this has an extension to dimension $d \ge 1$. This motivates the following definition of the *perturbed Bessel processes of dimension* $d \ge 1$. We say that a continuous, $\mathbb{R}^+$-valued process $(R_{d,\alpha}(t); t \ge 0)$ is an α-perturbed Bessel process of dimension $d > 1$ starting from $a \ge 0$ if it satisfies

$$R_{d,\alpha}(t) = a + B_t + \frac{d-1}{2} \int_0^t \frac{ds}{R_{d,\alpha}(s)} + \alpha(M_t^R - a), \tag{1.6}$$

and an α-perturbed Bessel process of dimension 1 if it satisfies

$$R_{1,\alpha}(t) = a + B_t + \frac{1}{2} l^R(t) + \alpha(M_t^R - a), \tag{1.7}$$

where $M_t^R = \sup_{0\le s\le t} R_{d,\alpha}(s)$, B is a Brownian motion, and l_t^R is the semimartingale local time of $R_{1,\alpha}$ at zero, it being clear from (1.6) and (1.7) that $R_{d,\alpha}$ is a semimartingale for $d \ge 1$.

Theorem 1.1. *Let $d \ge 1$ and $\alpha < 1$. Suppose that $R_{d,\alpha}$ is defined from a given Bessel process $\tilde{R}$ of dimension d starting at $\tilde{a} \ge 0$ via the time-change*

$$\tilde{M}_u^{\alpha^*} \tilde{R}_u = R_{d,\alpha}\left(\int_0^u dv\, \tilde{M}_v^{2\alpha^*} \right), \tag{1.8}$$

where $\tilde{M}_t = \sup_{0\le s\le t} \tilde{R}_s$ and $\alpha^* = \alpha/1-\alpha$. Then $R_{d,\alpha}$ satisfies (1.6) when $d > 1$ and (1.7) when $d = 1$ with $a = R_{d,\alpha}(0) = \{\tilde{a}\}^{1/1-\alpha}$. Conversely, given a perturbed Bessel *process $R_{d,\alpha}$ starting from a the process $\tilde{R}$ defined via the time-change*

$$\frac{R_{d,\alpha}(t)}{M_t^\alpha} = \tilde{R}\left(\int_0^t \frac{ds}{M_s^{2\alpha}}\right), \tag{1.9}$$

where $M_t = \sup_{0\le s\le t} R_{d,\alpha}(s)$, is a Bessel process of dimension d starting from $\tilde{a} = a^{1-\alpha}$.

Proof. Suppose $d > 1$ and (1.6) holds. Then, from an application of Itô's formula we see that

$$\frac{R_{d,\alpha}(t)}{M_t^\alpha} = a^{1-\alpha} + \int_0^t \frac{dB_s}{M_s^\alpha} + \frac{d-1}{2}\int_0^t \frac{ds}{M_s^{2\alpha}}\frac{M_s^\alpha}{R_{d,\alpha}(s)}.$$

Now replacing t by $a(t)$, the inverse of $A(t) = \int_0^t ds/M_s^{2\alpha}$, we see that $\tilde{R}$ is a Bessel process starting from $a^{1-\alpha}$. Moreover, we have

$$\tilde{M}(A_t) = M_t^{1-\alpha}, \tag{1.10}$$

so inverting the time-change we see that (1.9) and (1.8) are equivalent. If $d > 1$ and $\alpha > 0$ we start with the equation which $\tilde{R}$ satisfies (i.e. (1.6) with $\alpha = 0$), and the same argument shows that $R_{d,\alpha}$ defined by (1.8) satisfies (1.6). For $d = 1$ the argument is virtually the same. ∎

- It is a consequence of the representation (1.8) that equation (1.6) enjoys the uniqueness in law property.
- It is now known that in the case $d = 1$ equation (1.7) enjoys the pathwise uniqueness property: see [5]. However the corresponding question for (1.6) has not yet been resolved.
- Note also that if R is a Bessel process of dimension d starting from 0, and we use Theorem 1.1 with $\tilde{R}(.) = R_{T_a+.}$ to construct a family of $R_{d,\alpha}$ processes starting at $a \ge 0$ then these processes vary (in the uniform topology) continuously with a, and hence so do their laws. In particular, for $d = 3$ we see that

  $$\mathbb{P}_a^{3,\alpha} \Rightarrow \mathbb{P}_0^{3,\alpha} \text{ as } a \downarrow 0,$$

  so that one can also think of $\mathbb{P}_0^{3,\alpha}$as the law of perturbed Brownian motion starting from zero conditioned to stay positive.
- An important deduction from (1.6) and (1.7) is that $R_{d,\alpha}$ *has the Brownian scaling property.*
- A further deduction is that, just as in the case $\alpha = 0$, the point 0 is instantaneously reflecting for $d < 2$ and polar for $d \ge 2$.
- We mention that although $R_{d,\alpha}$ does not have the Markov property when $\alpha \neq 0$, the pair $\{R_{d,\alpha}, M^{R_{d,\alpha}}\}$ is strong Markov.
- Henceforth we will write PBES(d,α) for a perturbed Bessel process of dimension d, and $\mathbb{P}_a^{d,\alpha}$for its law if it starts from $a \ge 0$. For $\alpha = 0$ these will be abbreviated to BES(d) and $\mathbb{P}_a^d$.

2. Some Ray-Knight theorems on local time

We consider the perturbed Bessel processes of dimension $d = 3$, and write $\delta = 2(1-\alpha)$. Let $l_t^a(X)$ denote the (jointly continuous version of the) semimartingale local time attained before time t by a process X at the level a. As is now standard, $\mathbb{Q}_a^\delta$ denotes the law of the squared Bessel process of dimension δ starting from a, and

$\mathbb{Q}^{\delta}_{a\to b}$ the bridge of this process to a level b at time 1. Yor describes in [18], following Le Gall-Yor [10], the construction of two processes for which Ray-Knight theorems involving these squared Bessel processes are known. Specifically, given a Brownian motion B we define

$$\Sigma_t^{\delta} = |B_t| + \frac{2}{\delta} l_t^0(B), \tag{2.1}$$

and then define $(D_t^{\delta}; t < T_1)$ via the space-time change

$$\frac{\Sigma_t^{\delta}}{1+\Sigma_t^{\delta}} = D^{\delta}\left(\int_0^t \frac{ds}{(1+\Sigma_s^{\delta})^4}\right). \tag{2.2}$$

Note that if (2.1) holds and $J_t^{\Sigma} = \inf_{s\geq t} \Sigma_s^{\delta}$, then $J_t^{\Sigma} = \frac{2}{\delta} l_t^0(B)$. It is then easy to see that (2.1) is equivalent to the existence of a Brownian motion $\hat{B}$ such that

$$\Sigma_t^{\delta} = \hat{B}_t + (1+\delta/2) J_t^{\Sigma}. \tag{2.3}$$

The result is

Theorem 2.1. *(Le Gall-Yor) The following descriptions of the local times of Σ^{δ} and D^{δ} hold.*

$$(l_{\infty}^a(\Sigma^{\delta}); a \geq 0) \textit{ has law } \mathbb{Q}_0^{\delta},$$

and

$$(l_{T_1}^a(D^{\delta}); 0 \leq a \leq 1) \textit{ has law } \mathbb{Q}_{0\to 0}^{\delta}.$$

We are going to establish a similar result for the PBES$(3,\alpha)$ processes but first we need the following.

Theorem 2.2. *Suppose that $R_{3,\alpha}$ is a PBES$(3,\alpha)$ process starting from zero. Then the process Σ^{δ} defined by the space-time transform*

$$\frac{1}{R_{3,\alpha}(t)} = \Sigma^{\delta}\left(\int_t^{\infty} \frac{du}{(R_{3,\alpha}(u))^4}\right), \qquad \textit{for all } t > 0, \tag{2.4}$$

satisfies equation (2.1) with $\delta = 2(1-\alpha)$, and the local times of these processes are connected by;

$$l_{\infty}^a(R_{3,\alpha}) = a l_{\infty}^{1/a}(\Sigma^{\delta}) \textit{ for all } a \geq 0.$$

Moreover the process $X_{3,\alpha}$ defined from $R_{3,\alpha}$ via the space-time change

$$\frac{R_{3,\alpha}(t)}{1+R_{3,\alpha}(t)} = X_{3,\alpha}\left(\int_0^t \frac{ds}{(1+R_{3,\alpha}(s))^4}\right). \tag{2.5}$$

is a PBES(3,α) process, starting from zero, run until it first hits one, and it is related to the process D^{δ} defined from Σ^{δ} by equation (2.2) by time-reversal, i.e.

$$D_t^{\delta} = 1 - X_{3,\alpha}(T_1 - t) \qquad \textit{for all } 0 \leq t \leq T_1 = T_1^{X_{3,\alpha}}. \tag{2.6}$$

Finally it holds that

$$l_{T_1}^a(X_{3,\alpha}) = l_{T_1}^{1-a}(D^{\delta}) \qquad \textit{for all } 0 \leq a \leq 1.$$

Proof. Given a process $R = R_{3,\alpha}$ which is a solution of (1.6) with $d = 3$, we use (1.9) to define a BES(3) process $\tilde{R}$. Since the case $\alpha = 0$ of (2.4), when both $R_{3,\alpha}$ and Σ^δ become BES(3) processes, is a special case of representation results in [9] and [4], so also is the process $\hat{R}$ defined by

$$\frac{1}{\tilde{R}(t)} = \hat{R}\left(\int_t^\infty \frac{du}{(\tilde{R}(u))^4}\right).$$

From this it follows, using (1.9) again, that

$$\frac{\{M_t^R\}^\alpha}{R(t)} = \hat{R}\left(\int_t^\infty \frac{\{M_u^R\}^{2\alpha} du}{(R(u))^4}\right),$$

so that if Σ^δ is defined by (2.4) we have

$$\Sigma^\delta\left(\int_t^\infty \frac{du}{(R(u))^4}\right) \cdot \{M_t^R\}^\alpha = \hat{R}\left(\int_t^\infty \frac{\{M_u^R\}^{2\alpha} du}{(R(u))^4}\right).$$

Using the relation between M_t^R and $J_t^\Sigma = \inf\{\Sigma_u^\delta : u \geq t\}$ which follows from (2.4), we obtain the first in the following equivalent pair of representations

$$\text{(2.7)} \qquad \frac{\Sigma_t^\delta}{(J_t^\Sigma)^\alpha} = \hat{R}\left(\int_0^t \frac{ds}{(J_s^\Sigma)^{2\alpha}}\right), \qquad \hat{R}_t(\hat{J}_t)^{\alpha *} = \Sigma^\delta\left(\int_0^t \{\hat{J}_s\}^{2\alpha^*} ds\right),$$

where $\hat{J}_t = \inf\{\hat{R}_u : u \geq t\}$, and $\alpha^* = \alpha/(1-\alpha)$. The second follows by inverting the time change. Further, recalling that $2 - \alpha = 1 + \delta/2$, an application of Itô's formula shows that (2.7) is equivalent to the existence of a Brownian motion $\hat{B}$ such that (2.3) holds, and we have seen this is equivalent to (2.1). If we apply Itô's formula to $R_t/(1+R_t)$, and then make the time change, we see easily that the first assertion of the theorem about $X_{3,\alpha}$ is correct. To see that the relation (2.6) holds, we write $\hat{X}_t = 1 - X_{3,\alpha}(T_1 - t)$ for $t < T_1$ and note that $T_1 = \int_0^\infty \frac{du}{(1+R_u)^4}$, so that

$$\frac{1}{1+R_t} = \hat{X}\left\{\int_t^\infty \frac{du}{(1+R_u)^4}\right\}.$$

From (2.4) we see that

$$\int_t^\infty \frac{du}{(1+R_u)^4} = \int_0^{A_t} \frac{ds}{(1+\Sigma_s^\delta)^4},$$

where $A_t = \int_t^\infty \frac{du}{(R_u)^4}$. It follows that

$$\frac{\Sigma_t^\delta}{1+\Sigma_t^\delta} = \hat{X}\left\{\int_0^t \frac{ds}{(1+\Sigma_s^\delta)^4}\right\},$$

and comparing this to (2.2), we conclude that $X_{3,\alpha}$ and D^δ are related by time-reversal, as claimed. The results about the local times follow easily. ∎

Theorem 2.3. *The laws of the local times of $R_{3,\alpha}$ when it starts from zero, at times $T_1 = \inf\{t : R_{3,\alpha}(t) \geq 1\}$ and infinity, are respectively $\mathbb{Q}_{0\to 0}^\delta$ and $\mathbb{Q}_0^\delta$.*

Proof. These assertions follow from the statements about local times in Theorem 2.2, using the familiar properties of time inversion (for squared Bessel processes) and time reversal (for bridges of squared Bessel processes). ∎

We remark that in [18] there is presented the following additive decomposition,

$$\mathbb{Q}_0^\delta = \mathbb{Q}_{0\to 0}^\delta * R^\delta. \tag{2.8}$$

(We keep the notation R^δ from [18], hoping that it does not lead to any confusion with the various Bessel processes R_λ involved in our discussion.) The identification of the law of R^δ, except for the case $\delta = 2$, is not entirely satisfactory, involving a reweighting of the local times of the three dimensional Bessel process. We can now clarify this result by noting that if $R_{3,\alpha}$ starts from 1 and we define R^δ as the law of $(l_\infty^a(R_{3,\alpha}); a \geq 0)$, then (2.8) follows from the 'strong Markov' property of $R_{d,\alpha}$ at T_1.

Theorems 2.1 and 2.3 can be reformulated as statements about the *unperturbed* 3-dimensional Bessel process. These alternative presentations involve the local times of semimartingales whose martingale parts are not Brownian motions, and we stress that, if Y is such a semimartingale, then $l^a(Y)$ is an occupation density with respect to $d\langle Y\rangle_s$.

Theorem 2.4. *Let $\hat{R}$ be a BES(3) process starting from zero, put $\hat{M}_t = \sup_{s\leq t}\hat{R}_s$, $\hat{J}_t = \inf_{s\geq t}\hat{R}_s$, and define $Y_t^{(1)} = \{\hat{M}_t\}^{\alpha^*}\hat{R}_t$ and $Y_t^{(2)} = \{\hat{J}_t\}^{\alpha^*}\hat{R}_t$.*

Then for $i = 1, 2$

$$(l_\infty^a(Y^{(i)}); a \geq 0) \qquad \textit{has law} \qquad \mathbb{Q}_0^\delta, \tag{2.9}$$

and

$$(l_\infty^a((1+Y^{(i)})^{-1}); 0 \leq a \leq 1) \qquad \textit{has law} \qquad \mathbb{Q}_{0\to 0}^\delta \tag{2.10}$$

Furthermore, for $i = 1$, (2.10) is equivalent to

$$(l_{T_1}^a(Y^{(1)}); 0 \leq a \leq 1) \qquad \textit{has law} \qquad \mathbb{Q}_{0\to 0}^\delta. \tag{2.11}$$

Proof. From Theorem 1.1 we have the representation $Y_t^{(1)} = R(\Gamma_t)$, where R is a PBES$(3,\alpha)$ and $\Gamma_t = \int_0^t \hat{M}_s^{2\alpha^*}ds$. It follows that $l_t^a(Y^{(1)}) \equiv l_{\Gamma_t}^a(R)$, and since $T_1^R = \Gamma(T_1^{Y^{(1)}})$, statements (2.9) and (2.11) for $i = 1$ follow from Theorem 2.3. Also, by Theorem 2.2, we can write $1 - Y_t^{(1)} = R^*(\Theta(\Gamma_t))$,where R^* is a PBES$(3,\alpha)$ and $\Theta_t = \int_0^t \frac{ds}{(1+R_s)^4}$. It follows that $l_\infty^a(1 - Y^{(1)}) \equiv l_{\Gamma_t}^a(R^*)$, and (2.10) for $i = 1$ also follows from Theorem 2.3. For $i = 2$ we start with the representation $Y_t^{(2)} = \Sigma^\delta(\Phi_t)$ of (2.7), where $\Phi_t = \int_0^t\{\hat{J}_s\}^{2\alpha^*}ds$, and appeal to Theorem 2.1. But note that there is no analogue of (2.11), because $l_{T_1}^a(Y^{(2)}) \equiv l_{T_1}^a(\Sigma^\delta)$, and this is not $\mathbb{Q}_0^\delta$ distributed. ∎

3. Perturbed Bessel processes of dimension $0 < d < 1$

The problem of extending the definition of the perturbed Bessel processes to dimension $0 < d < 1$ can, as in the unperturbed case, be solved by defining first the *perturbed squared Bessel processes.* Note first that if $R_{d,\alpha}$ is a PBES(d,α) with $d \geq 1$, starting from $a \geq 0$, then one deduces easily from (1.6) and (1.7) that $Y = \{R_{d,\alpha}\}^2$ satisfies

$$Y_t = a^2 + 2\int_0^t \sqrt{Y_s}dB_s + (d.t) + \alpha(M_t^Y - a^2). \tag{3.1}$$

Of course, this equation makes sense for any $d > 0$, and in fact has a solution which is unique in law. This is a consequence of the following analogue of our basic representation result, Theorem 1.1.

Theorem 3.1. *Let $d > 0$ and $\alpha < 1$. Suppose that Y is defined from a given squared Bessel process $\tilde{Y}$ of dimension d starting at $\tilde{a}^2 \geq 0$ via the time-change*

$$\{\tilde{M}_u\}^{\alpha^*}\tilde{Y}_u = Y\left(\int_0^u dv\, \tilde{M}_v^{\alpha^*}\right), \tag{3.2}$$

where $\tilde{M}_u = \sup_{0\leq s\leq u}\tilde{R}_s$ and $\alpha^* = \alpha/1-\alpha$. Then Y satisfies (3.1) with $a = \{\tilde{a}\}^{1/1-\alpha}$. Conversely if Y solves (3.1) then the process $\tilde{Y}$ defined via the time-change

$$\frac{Y(t)}{M_t^\alpha} = \tilde{Y}\left(\int_0^t \frac{ds}{M_s^\alpha}\right), \tag{3.3}$$

where $M_t = \sup_{0\leq s\leq t} Y(s)$, is a squared Bessel process of dimension d starting from $\tilde{a}^2 = a^{2(1-\alpha)}$.

Proof. This follows the same lines as the proof of Theorem 1.1. Note that there is an analogue of (1.10), so that (3.2) and (3.3) are actually equivalent. ∎

We now define, for $0 < d < 1$ and $\alpha < 1$, the α-perturbed Bessel process of dimension d starting at $a \geq 0$ as the square root of an α-perturbed squared Bessel process of dimension d starting at a^2. Just as in the case $\alpha = 0$, for $0 < d < 1$ the perturbed Bessel process is not a semimartingale, although it is clear from (3.1) that its square is. However our next result shows that its expression as a Dirichlet process is exactly the α-perturbed version of the corresponding expression for BES(d), which is discussed in [1] and Chap. X of [19].

Theorem 3.2. *If $0 < d < 1$ and R is a PBES(d,α) process starting from $a \geq 0$, then it satisfies the equation*

$$R_t = a + B_t + (\frac{d-1}{2})K_t + \alpha(M_t^R - a), \qquad t \geq 0. \tag{3.4}$$

Here B is a Brownian motion and the drift term K is defined by

$$K_t = p.v.\int_0^t \frac{ds}{R_s} = \int_0^\infty a^{d-2}\{\lambda_t(a) - \lambda_t(0)\}da, \tag{3.5}$$

where the occupation density λ satisfies, for any Borel function $\phi \geq 0$,

$$\int_0^t \phi(R_s)ds = \int_0^\infty a^{d-1}\phi(a)\lambda_t(a)da. \tag{3.6}$$

Proof. Applying the Itô-Tanaka formula to $\phi_\varepsilon(Y)$, where $\varepsilon > 0$ and $\phi_\varepsilon(y) = \sqrt{y_\wedge\varepsilon}$ and $Y = R^2$ solves (3.1), gives

(3.7)
$$\phi_\varepsilon(Y_t) = \sqrt{a_\wedge\varepsilon} + \int_0^t 1_{\{Y_s\geq\varepsilon\}}dB_s + \frac{\alpha}{2}\int_0^t 1_{\{Y_s\geq\varepsilon\}}Y_s^{-\frac{1}{2}}dM_s^Y + \frac{d-1}{2}\int_0^t 1_{\{Y_s\geq\varepsilon\}}Y_s^{-\frac{1}{2}}ds + \frac{1}{4}\varepsilon^{-\frac{1}{2}}l_t^\varepsilon(Y).$$

From (3.6) we see that

$$l_t^a(Y) = 2a^{\frac{d}{2}}\lambda_t(\sqrt{a}), \tag{3.8}$$

and hence we have

$$\int_0^t 1_{\{Y_s\geq\varepsilon\}}Y_s^{-\frac{1}{2}}ds = \int_0^t 1_{\{Y_s\geq\varepsilon\}}\frac{d\langle Y\rangle_s}{4Y_s^{\frac{3}{2}}} = \int_\varepsilon^\infty \frac{l_t^x(Y)}{4x^{\frac{3}{2}}}dx = \frac{1}{2}\int_\varepsilon^\infty x^{\frac{d-3}{2}}\lambda_t(\sqrt{x})dx. \tag{3.9}$$

Hence

$$
\begin{aligned}
\int_\varepsilon^\infty x^{\frac{d-3}{2}}\lambda_t(\sqrt{x})dx + \frac{1}{2d-2}\varepsilon^{\frac{d-1}{2}}\lambda_t(\sqrt{\varepsilon}) &= \frac{1}{2}\int_\varepsilon^\infty x^{\frac{d-3}{2}}\{\lambda_t(\sqrt{x}) - \lambda_t(\sqrt{\varepsilon})dx \\
(3.10) \qquad &= \int_{\sqrt{\varepsilon}}^\infty a^{d-2}\{\lambda_t(a) - \lambda_t(\sqrt{\varepsilon})da.
\end{aligned}
$$

Now it is not difficult to see, as in the proof of Theorem 4.2 below, that the the process defined by $W_t = \{R_t\}^{2-d} = \{Y_t\}^{1-\frac{d}{2}}$ is a semimartingale whose local time satisfies

$$
\int_0^\infty \phi(a)l_t^a(W)da = (2-d)^2\int_0^t R_s^{2-2d}\phi(R_s^{2-d})ds.
$$

Comparing this to (3.6) yields the identity $\lambda_t(a) \equiv l_t^{a^{2-d}}(W)$. As a consequence of Kolmogorov's criterion we deduce that for any $\gamma \in (0, \frac{1}{2})$ we have $|\lambda_t(a) - \lambda_t(0)| \leq ca^{\gamma(2-d)}$ for some positive constant c. This implies both that the final expression in (3.5) is finite, and that (3.4) results by letting $\varepsilon \downarrow 0$ in (3.7). ∎

Remark 3.1. *We can now see that Theorem 1.1, and its consequences, extends immediately to the case* $0 < d < 1$.

4. More time changes

In Theorem 1.1 we saw that for any fixed $d > 0$ we can represent a perturbed Bessel process of dimension d with any perturbation parameter $\alpha < 1$ in terms of another Bessel process of dimension d . It is therefore not surprising that there is a similar link between perturbed Bessel processes of dimension d with different perturbation parameters.

Theorem 4.1. *Suppose that R is an α-perturbed Bessel process of dimension $d > 0$, $M_t = \sup_{s\leq t} R_s$, and $\beta < 1$.Then there exists a γ-perturbed Bessel process $\hat{R}$ of dimension d, with $\gamma = \frac{\alpha-\beta}{1-\beta}$, such that*

$$
\frac{R(t)}{M_t^\beta} = \hat{R}\left(\int_0^t \frac{ds}{\{M_s\}^{2\beta}}\right).
$$

Proof. Just use Theorem 1.1 twice, first to define a BES(3) process $\tilde{R}$ from R, and then to define a PBES(d,γ) process $\hat{R}$ from $\tilde{R}$. ∎

Perhaps more importantly, the fact that a power of a Bessel process is a time-change of a Bessel process of a different dimension (see e.g. Proposition 1.11, chap. XI of [15]) has an analogue for perturbed processes; note that the perturbation parameter is unchanged.

Theorem 4.2. *Suppose that R is a PBES(d,α) process with $d > 0$, and β is such that $d_\beta := 2 + \frac{d-2}{\beta} > 0$. Then there exists a PBES(d_β,α) process $R^\#$ such that*

$$
\{R(t)\}^\beta = R^\#\left(\beta^2\int_0^t ds\{R(s)\}^{2(\beta-1)}\right).
$$

Proof. If $\beta = 1$ there is nothing to prove. If $\beta > 1$ it is straightforward to apply Itô's formula to Y^β, where $Y = R^2$, to deduce from (3.1) that $Y^\# = \{R^\#\}^2$ satisfies (3.1) with d replaced by d_β, and the result follows. If $\beta < 1$ we note that $d = 2 + \beta(d_\beta - 2)$, and repeat the previous argument with β replaced by $1/\beta$ and d and d_β interchanged. ∎

Remark 4.1. *The important cases of this result are when $d = 1$ and $d = 3$, since it allows us to express any PBES(d, α) process in terms of a PBES$(1, \alpha)$ process if $0 < d < 2$, and in terms of a PBES$(3, \alpha)$ process if $2 < d < \infty$.*

We also mention that just as a Bessel process of dimension d is given by Lamperti's representation as a time change of the exponential of Brownian motion with drift, so can a perturbed Bessel process be expressed in terms of a *perturbed Brownian motion with drift.*

Theorem 4.3. *Define the index $\nu = (d/2) - 1$, and let $\left(B_t^{(\nu,\alpha)}, t \geq 0\right)$ be equal in law to the solution of*

$$X_t = B_t + \nu t + \alpha M_t^X, t \geq 0,$$

where B is a Brownian motion. Then, for $d \geq 0$ and $\alpha < 1$ we have the representation

$$\exp\left(B_t^{(\nu,\alpha)}\right) = R_{d,\alpha}\left(\int_0^t ds \exp(2B_s^{(\nu,\alpha)})\right),$$

where $R_{d,\alpha}$ is a PBES(d, α) process.

Proof. Apply Itô's formula to $\exp(2B_t^{(\nu,\alpha)})$ to see that a time-change of this satisfies (3.1). ∎

Recalling that $\mathbb{P}_a^{d,\alpha}$ stands for the law of a PBES(d, α) starting from a, we now give an absolute continuity relationship between $\mathbb{P}_a^{d,\alpha}$ and $\mathbb{P}_a^{2,\alpha}$ which extends a result for BES(d) processes given as Exercise 1.22, Chap. XI of [15].

Theorem 4.4. *For $d \geq 2$ and $a > 0$ it holds that*

$$\mathbb{P}_a^{d,\alpha} \mid \mathcal{F}_t = \left(\frac{R_t}{a^{1-\alpha} M_t^\alpha}\right)^\nu \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right) . \mathbb{P}_a^{2,\alpha} \mid \mathcal{F}_t.$$

Proof. This is a consequence of Girsanov's theorem, and the fact that, under $\mathbb{P}_a^{2,\alpha}$, $\log(R_t/M_t^\alpha)$ is a local martingale. Alternatively it could be deduced from the result for $\alpha = 0$, using the relationship (1.9). ∎

It is well-known that Spitzer's theorem on the asymptotics of planar Brownian windings (see, e.g., Thm. 4.1, Chap. X in [15]) may be deduced from some asymptotics for the BES(2) process. We now extend these results to PBES processes.

Theorem 4.5. *1. Assume $R_{2,\alpha}(0) > 0$. Then*

$$\frac{4}{(\log t)^2}\int_0^t \frac{ds}{R_{2,\alpha}^2(s)} \overset{(law)}{\longrightarrow} \sigma_\alpha \equiv \inf\{t : B_t^{0,\alpha} = 1\} \text{ as } t \to \infty,$$

and, moreover, σ_α is equal in law to $\sigma = \inf\{t : B_t = 1\}$.
2. Assume $d > 2$ and $R_{d,\alpha}(0) > 0$. Then

$$\frac{1}{\log t}\int_0^t \frac{ds}{R_{d,\alpha}^2(s)} \overset{a.s.}{\longrightarrow} E_0^{d,\alpha}\left(\frac{1}{R_1^2}\right) = \frac{1-\alpha}{d-2} \text{ as } t \to \infty.$$

Proof. The first result follows easily from the Lamperti relationship obtained in Theorem 4.3 above. (This deduction for the case $\alpha = 0$ is presented in Exercise 4.11, Chap. X in [15].)

The second result may be deduced from Birkhoff's theorem on path-space. (Again, for the case $\alpha = 0$, see Exercise 3.20, Chap. X in[15].) ∎

5. An extension of the Ciesielski-Taylor Theorem.

The classical version of the Ciesielski-Taylor theorem is the case $d = 3, \alpha = 0$, of the identity

$$\int_0^\infty ds 1\{R_{d,\alpha} \leq 1\} =^{(\text{law})} T_1\{R_{d-2,\alpha}\}, \tag{5.1}$$

which we will show below to be valid for any $d > 2, \alpha < 1$. For the case $\alpha = 0$ this has been established by several authors; see e.g. [2], [8] and [17]. The method used in [17] was to write both sides of (5.1) as integrals with respect to the local times $l^a_\infty\{R_{d,0}\}$ and $l^a_{T_1}\{R_{d-2,0}\}$ respectively, and then exploit the $\alpha = 0$ case of Theorem 4.2 to express these local times in terms of squares of Bessel processes. The same method can be applied to the case $\alpha \neq 0$, once we know that (5.1) holds with $d = 3$. This can be seen from the description of $l^a_\infty\{R_{3,\alpha}\}$ given in Theorem 2.3, together with the Ray-Knight theorems for perturbed Brownian motion (see [3], [16],or [7]), and the observation that a time change of the positive part of an α-perturbed Brownian motion is an α-perturbed Bessel process of dimension 1. (See [5]).

Alternatively we can express the integrals with respect to the local times in terms of integrals with respect to $\mathbf{n}^s_{d,\alpha} ds$, the intensity measure of the excursions away from 0 of the strong Markov process $M^{R_{d,\alpha}} - R_{d,\alpha}$. Then the validity of (5.1) for $\alpha \neq 0$ and $d > 2$ follows from its validity when $\alpha = 0$, the Lévy-Khintchine representations of $\mathbb{Q}^\delta_0$ and $\mathbb{Q}^\delta_{0\to 0}$ (see [18]), and the following result.

Lemma 5.1. *For any $\alpha < 1, d > 0$ it holds that*

$$\mathbf{n}^s_{d,\alpha} = (1-\alpha)\mathbf{n}^s_{d,0}.$$

Combining this with Theorem 4.1 of [18] gives the following result, in which $\{q^\delta(a), a \geq 0\}$ [respectively $\{\tilde{q}^\delta(a), 0 \leq a \leq 1\}$] denotes a process with the law $\mathbb{Q}^\delta_0$ [$\mathbb{Q}^\delta_{0\to 0}$], and again $\delta = 2(1-\alpha)$.

Theorem 5.2. *1. The Ciesielski-Taylor identity (5.1) is valid for any $\alpha < 1, d > 2$.*

2. For $d > 2$ we have

$$\left(l^a_\infty(R_{d,\alpha}), a \geq 0\right) =^{(law)} \left(\frac{a^{3-d}}{d-2} q^\delta(a^{d-2}), a \geq 0\right),$$

and

$$\left(l^a_{T_1}(R_{d,\alpha}), 0 \leq a \leq 1\right) =^{(law)} \left(\frac{a^{3-d}}{d-2} \tilde{q}^\delta(a^{d-2}), 0 \leq a \leq 1\right).$$

3. For $d = 2$ we have $\left(l^a_{T_1}(R_{2,\alpha}), 0 < a \leq 1\right) =^{(law)} \left(a\tilde{q}^\delta(\log 1/a), 0 < a \leq 1\right)$.

4. For $0 < d < 2$ we have

$$\left(l^a_{T_1}(R_{d,\alpha}), 0 < a \leq 1\right) =^{(law)} \left(\frac{1}{2-d} \tilde{q}^\delta(1 - a^{2-d}), 0 < a \leq 1\right).$$

6. Variable Perturbations

A process $(\Sigma_t^\Delta, t \geq 0)$ has been considered by Le-Gall and Yor, and others; see [19], section 18.3, and the references therein, and [6]. It is a simple generalization of the process Σ^δ defined via equation (2.1), where the constant δ has been replaced by a strictly increasing C^1 function $\Delta : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\Delta(0) = 0$ and $\Delta(\infty) = \infty$. More precisely the process Σ^Δ satisfies

$$\Sigma_t^\Delta = |B_t| + \Delta^{-1}(2l_t^0(B)), \tag{6.1}$$

for $t \geq 0$, where Δ^{-1} is the inverse of the function Δ. By taking $\Delta(t) = \delta t$ we recover Σ^δ.It was shown in [11] that the local time process $\big(l_\infty^a(\Sigma^\Delta); a \geq 0\big)$ has law, denoted by $\mathbb{Q}_0^\Delta$, which is that of a process $(Z_t; t \geq 0)$ satisfying

$$Z_t = 2\int_0^t \sqrt{Z_s}d\beta_s + \Delta(t), \qquad t \geq 0, \tag{6.2}$$

for some Brownian motion β.

We now consider an analogous generalization of the perturbed Bessel process of dimension three. Suppose Δ is as above, and additionally Δ' is bounded away from zero and infinity (this condition could be weakened). The process $(R_{3,h}(t); t \geq 0)$ obtained from Σ^Δ by the space-time inversion

$$\frac{1}{R_{3,h}(t)} = \Sigma^\Delta\left(\int_t^\infty \frac{du}{(R_{3,h}(u))^4}\right), \qquad \text{for all } t > 0, \tag{6.3}$$

satisfies

$$R_{3,h}(t) = \beta_t + \int_0^t \frac{ds}{R_{3,h}(s)} + h(M_t), \tag{6.4}$$

where β is a Brownian motion and $M_t = \sup_{s \leq t} R_{3,h}(s)$. The C^1 function $h : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$h(0) = 0, \quad 2(1 - h'(y)) = \Delta'(1/y), \qquad 0 < y < \infty, \tag{6.5}$$

which generalizes the relation between α and δ. The proof of this assertion follows a now familiar course. The process Σ^Δ can be obtained from a three-dimensional Bessel process R by a space-time change

$$\frac{\Sigma_t^\Delta}{\sigma(J_t)} = R\left(\int_0^t \frac{ds}{(\sigma(J_s))^2}\right), \tag{6.6}$$

where $J_t = \inf_{u \geq t} \Sigma_u^\Delta$ and the function σ satisfies

$$\frac{1}{2}\Delta'(y) = 1 - \frac{\sigma'(y)}{\sigma(y)}y, \qquad 0 < y < \infty. \tag{6.7}$$

Likewise the process $R_{3,h}$ satisfies (6.4) if and only if it is obtained from some three-dimensional Bessel process $\tilde{R}$ via

$$\frac{R_{3,h}(t)}{k(M_t)} = \tilde{R}\left(\int_0^t \frac{ds}{\{k(M_s)\}^2}\right), \tag{6.8}$$

where the function k satisfies

$$h'(x) = \frac{k'(x)}{k(x)}x, \qquad 0 < x < \infty. \tag{6.9}$$

Next, we observe that the space-time inversion (6.3) which connects $R_{3,h}$ and Σ^{Δ} corresponds exactly to

$$\frac{1}{R(t)} = \hat{R}\left(\int_t^{\infty} \frac{du}{(R(u))^4}\right). \tag{6.10}$$

Moreover the relation (6.5) is now a simple consequence of combining (6.7) and (6.9).

The local time process $(l^a_{\infty}(R_{3,h}); a \geq 0)$ is equal in law to $(\hat{Z}_t \equiv t^2 Z_{1/t}; t \geq 0)$ where Z satisfies (6.2). However it can be shown, although we do not pursue this here, that the law of $\hat{Z}$ is only of the form $\mathbb{Q}_0^{\hat{\Delta}}$ when $\Delta(y) = \hat{\Delta}(y) = \delta y$.

7. Perturbations at the future minimum.

Inspection of (2.3), rewritten as

$$\Sigma_t^{\delta} = B_t + (2-\alpha) J_t^{\Sigma}, \tag{7.1}$$

shows that a Σ^{δ} process can be thought of as a version of a BES(3) process, perturbed at its future minimum. In order to obtain analogous generalisations of BES(d), $d > 2$, we remark first that if R is a BES(d) process, starting from zero, and $d > 2$, then R satisfies

$$R_t = B_t + 2J_t^R + \frac{1}{2}(3-d)\int_0^t \frac{ds}{R_s}.$$

(See, e.g. [19], Chap. XII, p46, [13], [14], and [12].) This motivates the following definition. We say that Σ is a Bessel process of dimension $d > 2$, α-perturbed at its future minimum, and write Σ is a JBES(d, α), if it satisfies

$$\Sigma_t = B_t + (2-\alpha) J_t^{\Sigma} + \frac{1}{2}(3-d)\int_0^t \frac{ds}{\Sigma_s}. \tag{7.2}$$

It is easy to see that uniqueness of law of solutions of (7.2) holds, by establishing that the equivalent pair of relations given in (2.7) extend to the situation where $\hat{R}$ is a BES(d) process and Σ is a JBES(d, α) process. This is the "JBES version" of Theorem 1.1, and in fact many of our results have extensions to this situation. We conclude by giving some of these, without proofs.

First, the mapping given in (2.4) which maps a PBES($3, \alpha$) into a JBES($3, \alpha$) also maps a PBES(d, α) into a JBES(d, α) for any $d > 2$. Consequently, using Theorem 5.2, one can deduce the law of $\{l^a_{\infty}(\Sigma_{d,\alpha}), a \geq 0\}$. Moreover the mapping that extends (2.5) by mapping a PBES(d, α) R into a PBES(d, α) X killed on hitting 1 is given by

$$\frac{R_t}{\{1 + R_t^{1/\beta}\}^{\beta}} = X\left(\int_0^t \frac{du}{\{1 + R_u^{1/\beta}\}^{2+2\beta}}\right). \tag{7.3}$$

However it can not be true, for $d \neq 3$, that the process we get by applying this same transformation to Σ, the JBES(d, α) process which is the image under (2.4) of R, is related to X by time-reversal, as this is not true when $\alpha = 0$.

Finally the extension of Theorem 4.2 is the assertion that, if $d > 2$ and $\beta = 1/(d-2)$, then a JBES(3,α) process Σ and a JBES(d, α) process $\Sigma^{\#}$ are related by

$$\{\Sigma(t)\}^{\beta} = \Sigma^{\#}\left(\beta^2 \int_0^t ds \{\Sigma(s)\}^{2(\beta-1)}\right). \tag{7.4}$$

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