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SOME CALCULATIONS FOR PERTURBED BROWNIAN MOTION

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1. Introduction

If B is a standard Brownian motion starting from zero, and $\bar{B}_t = \sup_{0\le s\le t} B_s$, then the process X defined by

$$X_t = B_t + \frac{\alpha}{1-\alpha}\bar{B}_t, \tag{1}$$

where $\alpha < 1$ is called an α-perturbed Brownian motion. It is immediate from (1) that if $\overline{X_t}$ $=\sup_{0\le s\le t} X_s$ then

$$\overline{X}_t = \frac{1}{1-\alpha}\bar{B}_t, \tag{2}$$

so that (1) shows that X is the unique pathwise solution of the functional equation

$$X_t = B_t + \alpha\bar{X}_t. \tag{3}$$

This is a special case of the equation

$$X_t = B_t + \alpha\bar{X}_t + \beta\underline{X}_t\,,$$

where $\underline{X}_t = \inf_{0\le s\le t} X_s$,which has been studied by a number of authors; see ([3], [5], [4], and [8]). It should also be mentioned that, by the Lévy equivalence, (1) can be written as

$$-X_t = W_t - (1-\alpha)^{-1}L_t,$$

where W is a reflected Brownian motion whose local time at zero is L, so X is often referred to as "reflected Brownian motion perturbed by its local time". (See e.g.[11].)

From (1) it is clear that X is a non-Markov process which moves like Brownian motion except when it is at its maximum, and, moreover, X has the Brownian scaling property. Many other results known for Brownian motion have analogues for perturbed Brownian motion, including Lévy's Arc-sine law, the Ray-Knight theorems, and the solution to the two-sided exit problem. (See [7] , [2], [10], and [11].)

In this note we give an excursion theory approach, based on the excursions of X away from its maximum, which leads to simple proofs of some of these results, and to new ones. In particular, we give new proofs of the Ray-Knight theorems and extend the results known about the two-sided exit problem by computing the transition density of the bivariate Markov process $(X,\bar{X})$, killed when X exits the interval, at an exponential time. From these results we are able to deduce some information about "X conditioned to stay positive".

The basis for our calculations is the following observation; write $P^{(\alpha)}$ for the measure of X and $n^{(\alpha)}$ for the characteristic measure, under $P^{(\alpha)}$, of the excursions away from zero of $\bar{X}-X$. Note that $n = n^{(0)}$ coincides with the characteristic measure of excursions away from zero of reflected Brownian motion.

Proposition 1.1. *The measures $n^{(\alpha)}$ and n are related by*

$$n^{(\alpha)} = (1-\alpha)n. \tag{4}$$

Proof. From (1)and(3) we have

$$\bar{X}_t - X_t = (1-\alpha)^{-1}\bar{B}_t - \{B_t + \alpha(1-\alpha)^{-1}\bar{B}_t\} = \bar{B}_t - B_t,$$

which tells us that $n^{(\alpha)}$ is a multiple of n. But (2) tells us that the local times at zero of $\bar{X} - X$ and $\bar{B} - B$ are related by $l^{(\bar{X}-X)} = (1-\alpha)^{-1} l^{(\bar{B}-B)}$, and this identifies the constant. ∎

2. Ray-Knight Theorems

Let L_t^xdenote a jointly continuous version of the local time at level x and time t of X, and write Q_x^δ for the law of the square of a Bessel process of dimension δ starting from x.

Theorem 2.1. *For fixed $b > 0$ let $Z = \{Z(x), 0 \le x \le b\}$, where $Z_x = L_{T_b}^{b-x}$.Then the law of Z is the restriction to $[0,b]$ of $Q_0^{2\bar{\alpha}}$, where $\bar{\alpha} = 1-\alpha$.*

Proof. Since the result is classical for $\alpha = 0$, it follows from the Lévy- Khintchine representation of Q_0^δ (see Theorem 3.2, p30 of [11]) that it suffices to show that for any Borel function $f \ge 0$

$$P^{(\alpha)}\{\exp - \int_0^b f(x)Z(x)dx\} = [P^{(0)}\{\exp - \int_0^b f(x)Z(x)dx\}]^{\bar{\alpha}}.$$

However, if we write $g(.) = f(b - .)$, the occupation density theorem gives

$$\int_0^b f(x)Z(x)dx = \int_0^{T_b} g(X_s)ds = \int_0^{\tau_b} g(l_s - Y_s)ds,$$

where $Y = \bar{X} - X$ and τ is the inverse of $l = l^{(Y)}$. Applying the master formula of excursion theory gives, with $\zeta = \zeta(\varepsilon)$ standing for the lifetime of a generic excursion ε,

$$\begin{aligned} P^{(\alpha)}\{\exp - \int_0^b f(x)Z(x)dx\} &= \exp -\{\int_0^b dt \int_\Omega n^{(\alpha)}(d\varepsilon)[1 - \exp - \int_0^\zeta g(t - \varepsilon(u))du]\} \\ &= [P^{(0)}\{\exp - \int_0^b f(x)Z(x)dx\}]^{\bar{\alpha}} \end{aligned}$$

by virtue of (4), and the result follows. ∎

Next, we deduce the second Ray-Knight theorem. We write σ for the inverse of L^0 and $\tilde{Q}_x^\delta$ for the measure of the square of a Bessel process of dimension δ, starting from x and killed on hitting zero.

Theorem 2.2. *For fixed $t > 0$ let $U^{(t)} = \{U_x^{(t)}, x \ge 0\}$, where $U_x^{(t)} = L^x(\sigma_t)$. Then under $P^{(\alpha)}$ the law of $U^{(t)}$ is $\tilde{Q}_t^{2\alpha}$.*

Proof. For $x_0 > 0$ it is clear that, given $L^{x_0}(\sigma_t) = t_0$, $\{L^{x_0+x}(\sigma_t), x \ge 0\}$ is independent of $\{L^y(\sigma_t), 0 \le y < x_0\}$, and is distributed as $U^{(t_0)}$. Thus $\{U_x^{(t)}, x \ge 0\}$ is Markov, and the result will follow if we can show that, for all Borel subsets A of $[0,\infty)$ and any $t > 0, x > 0$,

$$P^{(\alpha)}\{U_x^{(t)} \in A\} = \tilde{Q}_t^{2\alpha}\{X_x \in A\}. \tag{5}$$

Now by Theorem 2,

$$P^{(\alpha)}\{U_x^{(t)}=0\}=P^{(\alpha)}\{L^0(T_x)>t\}=Q_0^{2\bar{\alpha}}\{X_x>t\},$$

whereas, writing $\lambda_t=\sup\{s:X_s=t\}$, it follows by time reversal(see e.g.Ex.1.23, p420 of [9]) that

$$\tilde{Q}_t^{2\alpha}\{X_x=0\}=Q_t^{2\alpha}\{T_0\le x\}=Q_0^{2+2\bar{\alpha}}\{\lambda_t\le x\}.$$

Finally, using the scaling property and the fact that the $Q_0^{2+2\bar{\alpha}}$ distribution of λ_1 coincides with the $Q_0^{2\bar{\alpha}}$ distribution of $\{X_1\}^{-2}$ (see Ex 1.18, p418 of [9]), we see that (5) holds for $A=\{0\}$. Next, on $\{U_x^{(t)}>0\}$, we set $\tilde{T}=\inf\{s>T_x:X(s)=0\}$,and write

$$U_x^{(t)}=L^x(\tilde{T})+\{L^x(\sigma_t)-L^x(\tilde{T})\}. \tag{6}$$

Since the excursions of X below x after time T_x have the same structure as the excursions below zero of a Brownian motion, it is clear that, given $L^0(T_x)=s$, the terms on the RHS of (6) are independent and, by the Ray-Knight theorems for Brownian motion, have the distribution of X_x under Q_0^2 and Q_{t-s}^0 respectively. Using the composition law for squares of Bessel processes (Theorem 1.2, p410 of [9]) and appealing again to Theorem 2 gives

$$\begin{aligned}
P^{(\alpha)}\{U_x^{(t)} &\in dy\}=\int_0^t Q_0^{2\bar{\alpha}}\{X_x\in ds\}Q_{t-s}^2\{X_x\in dy\}\\
&= \frac{dy}{dt}\int_0^t Q_0^{2\bar{\alpha}}\{X_x\in ds\}Q_y\{X_x\in dt-s\}\\
&= \frac{dy}{dt}Q_y^{2+2\bar{\alpha}}\{X_x\in dt\}.
\end{aligned}$$

Finally, time reversal gives

$$\frac{1}{dt}Q_y^{2+2\bar{\alpha}}\{X_x\in dt\}=\frac{1}{dy}Q_t^{2\alpha}\{X_x\in dy;T_0>x\}=\frac{1}{dy}\tilde{Q}_t^{2\alpha}\{X_x\in dy\},$$

which completes the proof of (5). ∎

3. The process killed on leaving $[-a,b]$.

We will write $S=S(a,b)=T_{-a}\wedge T_b$ for the first exit time of $[-a,b]$, and V_{θ^*} for an independent, exponentially distributed random variable with parameter $\theta^*=\theta^2/2$.

Theorem 3.1. *It holds that, for $a>0,b>0$,*

$$P^{(\alpha)}\{S\le V_{\theta^*};X(S)=b\}=\left(\frac{\sinh a\theta}{\sinh(a+b)\theta}\right)^{\bar{\alpha}}, \tag{7}$$

for $0<y<b$,

$$P^{(\alpha)}\{S\le V_{\theta^*};X(S)=-a,\bar{X}(S)\in dy\}=\frac{\bar{\alpha}\theta(\sinh a\theta)^{\bar{\alpha}}}{\{\sinh(a+y)\theta\}^{\bar{\alpha}+1}}dy, \tag{8}$$

and for $-a<z<y, 0<y<\infty$,

$$P^{(\alpha)}\{T_{-a}>V_{\theta^*};X(V_{\theta^*})\in dz,\bar{X}(V_{\theta^*})\in dy\}=\frac{\bar{\alpha}\theta^2(\sinh a\theta)^{\bar{\alpha}}\sinh(a+z)\theta}{\{\sinh(a+y)\theta\}^{\bar{\alpha}+1}}dydz. \tag{9}$$

Proof. Write $A(\theta^*, c)$ for $\{\varepsilon : \zeta(\varepsilon) > V_{\theta^*}\} \cup \{\varepsilon : \zeta(\varepsilon) \leq V_{\theta^*}, \bar{\varepsilon}(\zeta) > c\}$, and recall that

$$n(A(\theta^*, c)) = \theta \coth c\theta. \tag{10}$$

Then $P^{(\alpha)}\{S \leq V_{\theta^*}; X(S) = b\} = P^{(\alpha)}\{\phi > b\}$, where $\phi = \inf\{s : \varepsilon_s \in A(\theta^*, a+s)\}$. Thus

$$\begin{aligned} P^{(\alpha)}\{S &\leq V_{\theta^*}; X(S) = b\} = \exp\{-\int_0^b n^{(\alpha)}(A(\theta^*, a+s))ds\} \\ &= \exp\{-\bar{\alpha}\int_0^b \theta \coth(a+s)\theta ds\}, \end{aligned}$$

from (4) and (10), and (7) follows. Also

$$\begin{aligned} P^{(\alpha)}\{S &\leq V_{\theta^*}; X(S) = -a,\} = P^{(\alpha)}\{T_{-a} < T_b \wedge V_{\theta^*}\} \\ &= \int_0^b P^{(\alpha)}\{\phi > y\} n^{(\alpha)}\{\varepsilon : T_{a+y} < \zeta(\varepsilon) \wedge V_{\theta^*}\} dy \\ &= \int_0^b \left\{\frac{\sinh a\theta}{\sinh(a+y)\theta}\right\}^{\bar{\alpha}} \frac{\bar{\alpha}\theta}{\sinh(a+y)\theta} dy, \end{aligned}$$

where we have used another standard result for Brownian motion, and this is equivalent to (8). Similarly

$$\begin{aligned} P^{(\alpha)}\{S &> V_{\theta^*}; X(V_{\theta^*}) \in dz\} \\ &= \int_{z+}^b P^{(\alpha)}\{\phi > y\} n^{(\alpha)}\{\varepsilon : \zeta(\varepsilon) > V_{\theta^*}, \bar{\varepsilon}(V_{\theta^*}) \leq a+y, \varepsilon(V_{\theta^*}) \in y - dz\} dy \end{aligned}$$

and since, for $0 < u < v$

$$\begin{aligned} n\{\varepsilon &: \zeta(\varepsilon) > V_{\theta^*}, \bar{\varepsilon}(V_{\theta^*}) \leq v, \varepsilon(V_{\theta^*}) \in du\} \\ &= n\{\varepsilon : T_u < \zeta(\varepsilon) \wedge V_{\theta^*}\} P_u^{(0)}\{X(V_{\theta^*}) \in du, T_0 \wedge T_v > V_{\theta^*}\} \\ &= \left\{\frac{\theta}{\sinh u\theta}\right\} \cdot \left\{\frac{\theta \sinh u\theta \sinh(v-u)\theta du}{\sinh v\theta}\right\} = \frac{\theta^2 \sinh(v-u)\theta}{\sinh v\theta} du, \end{aligned}$$

(9) is also immediate. ∎

From this some known results in [2] and [8] follow immediately.

Corollary 3.2. *For α-perturbed Brownian motion we have*

$$P^{(\alpha)}\{X \textit{ exits } [-a, b] \textit{ at } b\} = \left(\frac{a}{a+b}\right)^{\bar{\alpha}}, \tag{11}$$

$$E^{(\alpha)}\{e^{-\theta^* T_b}\} = e^{-\bar{\alpha}b\theta},$$

and

$$E^{(\alpha)}\{e^{-\theta^* T_{-a}}\} = \int_0^\infty \frac{\bar{\alpha}\theta(\sinh a\theta)^{\bar{\alpha}}}{\{\sinh(a+y)\theta\}^{\bar{\alpha}+1}} dy.$$

We can also deduce some facts about X conditioned "to stay positive";

Corollary 3.3. *It holds that*

$$\lim_{a\downarrow 0}\lim_{k\uparrow\infty} E^{(\alpha)}\{e^{-\theta^* T_b} \mid X\, exits\ [-a,k]\ at\ k\} = \left(\frac{b\theta}{\sinh b\theta}\right)^{\bar\alpha}, \tag{12}$$

and

$$\begin{aligned}\lim_{a\downarrow 0}\lim_{k\uparrow\infty} P^{(\alpha)}\{X(V_{\theta^*}) &\in dz \mid X\, exits\ [-a,k]\ at\ k\} \\ &= \theta^{1+\bar\alpha} z\sinh z\theta.\int_z^\infty \frac{\bar\alpha\theta}{y^\alpha\{\sinh y\theta\}^{\bar\alpha+1}}dydz. \end{aligned}\tag{13}$$

Proof. Note that for $k > b$,

$$\begin{aligned} P^{(\alpha)}\{T_b &\le V_{\theta^*} \mid X\text{exits } [-a,k] \text{ at } k\} \\ &= \frac{P^{(\alpha)}\{S \le V_{\theta^*}, X(S) = b\}P^{(\alpha)}\{X \text{ exits } [-(a+b), k-b] \text{ at } k-\ b\}\}}{P^{(\alpha)}\{X \text{ exits } [-a,b] \text{ at } b\}} \\ &= \{\frac{\sinh a\theta}{\sinh(a+b)\theta}\}^{\bar\alpha}\{\frac{a+b}{a+k}\}^{\bar\alpha}\{\frac{a}{a+k}\}^{\bar\alpha}, \end{aligned}$$

which does not depend on k. So (12) follows by letting $a\downarrow 0$.
Similarly, we see that for $z < y < k$

$$\begin{aligned} &P^{(\alpha)}\{X(V_{\theta^*}) \in dz, \bar X(V_{\theta^*}) \in dy \mid \text{ exits } [-a,k] \text{ at } k\} \\ =\ &P^{(\alpha)}\{T_{-a} > V_{\theta^*}, X(V_{\theta^*}) \in dz, \bar X(V_{\theta^*}) \in dy\}P^{(0)}\{X \text{ exits } [-(a+z), y-z] \text{ at } y-z\} \\ &\times \left\{\frac{P^{(\alpha)}\{X \text{ exits } [-(a+y), k-y] \text{ at } k-\ y\}}{P^{(\alpha)}\{X \text{ exits } [-a,k] \text{ at } k\}}\right\} \\ =\ &\frac{\bar\alpha\theta^2(\sinh a\theta)^{\bar\alpha}\sinh(a+z)\theta}{\{\sinh(a+y)\theta\}^{\bar\alpha+1}}dydz.\frac{z+a}{y+a}.\left(\frac{a+y}{a=k}\right)^{\bar\alpha}\cdot\left(\frac{a+k}{a}\right)^{\bar\alpha}, \end{aligned}$$

and this leads to (13). ∎

REMARK Using (13), it is not difficult to show that there is a probability measure $R^{(\alpha)}$ say, which is the weak limit of $P^{(\alpha)}(.\mid X \text{ exits } [-a,k])$ as $k\uparrow\infty$ and $a\downarrow 0$, and it would be interesting to describe X under $R^{(\alpha)}$. Of course $R^{(0)}$ corresponds to the BES(3) process, and one way to realize that is as $|B_t| + L_t$,where L is the local time at zero of $|B|$. This suggests the process $\Sigma^{(\delta)} = |B| + \frac{2}{\delta}L$, which has been studied in [11],chapter 4, as a candidate to have the $R^{(\alpha)}$ measure, for some suitable δ . Furthermore, when $\delta = 2(1-\alpha)$, one can check that, under $P^{(\alpha)}$, the time-reversed process $\{1 - X_{T_1 - t}, 0\le t\le T_1\}$ has the same measure as $\{\Sigma_t^{(\delta)}, 0 \le t \le \lambda_1^{(\delta)}\}$, where $\lambda_1^{(\delta)} = \sup\{s : \Sigma_s^{(\delta)} = 1\}$. (I owe this observation, which extends a well-known connection between Brownian motion and BES(3), to Loic Chaumont.) However it follows from results in [1] that if $T^{(\delta)}$ is the hitting time process of Σ^δ, then

$$E\{e^{-\theta^* T_b^{(\delta)}}\} = \frac{\bar\alpha\theta}{(\sinh b\theta)^{\bar\alpha}}\int_0^b \frac{dy}{(\sinh y\theta)^\alpha}.$$

Since this disagrees with (12),we conclude that $\Sigma^{(\delta)}$ does not have $R^{(\alpha)}$ as its measure. This question is discussed further in [6].

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