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Normalized Stochastic Integrals in Topological Vector Spaces

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1 Introduction

Stochastic integration in infinite dimensional spaces is a mature area. Several important classes of stochastic integrals were introduced and studied in depth by Kunita [12], Métivier and Pistone [15], Meyer [14], Métivier and Pellaumail [16], Gyöngi and Krylov [7], Grigelionis and Mikulevicius [5], Walsh [22], Korezlioglu [9], Kunita [11], etc. Not surprisingly, the approaches to infinite dimensional stochastic integration proposed in these works have some similarities but also some distinct features. The latter are mainly related to the specifics of the spaces and processes involved. For example, the integral with respect to a stochastic flow (see Kunita [11], and also Gihman, Skorohod [4]) and the integrals with respect to orthogonal martingale measures (see Gyöngy, Krylov [7], [6], Walsh [22]) seem to have very little in common. In fact, the relation between these two integrals as well as others mentioned above is stronger than it might appear. More specifically, it will be shown below that all these integrals and some others are particular cases of one stochastic integral with respect to a locally square integrable cylindrical martingale in a topological vector space.

Let E be a quasicomplete locally convex topological vector space with weakly separable dual space E', i.e. E is a locally convex topological vector space so that all its bounded closed subsets are complete. Let $(Q_s)_{s\geq 0}$ be a predictable family of symmetric non-negative linear forms from E' into $\bar{E}$ and λ_s be a predictable increasing process. By a locally square integrable cylindrical martingale in E

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(with covariance operator function Q_s and quadratic variation $\int_0^t Q_s d\lambda_s$) we understand a family of real valued locally square integrable martingales $M_t(y)$, $y \in E'$, such that

$$\langle M(y), M(y')\rangle_t = \int_0^t \langle Q_s y, y'\rangle_{E,E'}\, d\lambda_s.$$

The stochastic integral is constructed in three steps. To begin with, we define an Ito integral for integrands from the set $\tilde{L}^2_{loc}(Q)$ consisting of E'-valued predictable functions f_s such that

$$\int_0^1 \langle Q_s f_s,\, f_s\rangle_{E,E'}\, d\lambda_s < \infty \qquad \mathbf{P}\text{-a.e.}$$

(see Proposition 9). Below this integral is denoted $\int_0^t f_s dM_s$ or $I_t(f)$. In our approach, the set S_b of simple (elementary) functions consists of all finite linear combinations $\sum_1^N f_s^k y_k$, $y_k \in E'$, of real valued predictable functions so that

$$\int_0^1 \sum_{k,j=1}^N f_s^k f_s^j \langle Q_s y_k,\, y_j\rangle_{E,E'}\, d\lambda_s < \infty \qquad \mathbf{P}\text{-a.e.}$$

The choice of the set of simple functions is almost the only nonstandard feature of the first part of our construction.

Unfortunately, the above integral is not quite satisfactory in that the space of integrands, $\tilde{L}^2_{loc}(Q)$, is not complete. So the next important step is to find a natural completion of this space. To address this problem we rely on the L. Schwartz theory of reproducing kernels [21]. The results in [21] allow to construct a family of Hilbert subspaces $H_s \subset E$ naturally associated with the covariance operator function Q_s; below these spaces are referred to as *covariance spaces.* The covariance space H_s is defined as the completion of $Q_s E'$ with respect to the inner product

$$(Q_s y,\, Q_s y')_{H_s} := \langle Q_s y,\, y'\rangle_{E,E'} \tag{1}$$

Using these results we demonstrate (Proposition 10) that the closure of $\tilde{L}^2_{loc}(Q)$ is isometric to the space $L^2_{loc}(Q) := \{$predictable E-valued g : $\int_0^1 |g_s|^2_{H_s} d\lambda_s < \infty$ $\mathbf{P}$-a.s.$\}$.

The third and final step of our construction is to extend the stochastic integral from $\tilde{L}^2_{loc}(Q)$ onto $L^2_{loc}(Q)$. To achieve this goal, we introduce a *normalized* stochastic integral for E-valued integrands. We denote this integral $\int_0^t g_s * dM_s$ or $\mathcal{R}_t(g)$. Loosely speaking, the integral is defined by the equality

$$\mathcal{R}_t(g) = \int_0^t g_s * dM_s := \int_0^t g_s d(M_s/Q_s)\,. \tag{2}$$

Of course, this "definition" is formal; it explains the origins of the term "normalized" rather than defines the integral. However, if $g_s = Q_s f_s$ and $f \in \tilde{L}^2_{loc}(Q)$,

(2) can be made meaningful by setting

$$\int_0^t g_s * dM_s := \int_0^t Q_s f_s d(M_s/Q_s) := \int_0^t f_s dM_s,$$

(see section 4.1 as well).

Since $Q_s f_s \in L^2_{loc}(Q)$, the idea now is to extend the Ito stochastic integral $\int_0^t f_s dM_s$ by extending the normalized integral $\int_0^t Q_s f_s * dM_s$ to all integrands belonging to $L^2_{loc}(Q)$. We prove that this is indeed possible (Proposition 11), and for every $g \in L^2_{loc}(Q)$, $\mathcal{R}_t(g)$ is a local square integrable martingale such that

$$\langle \mathcal{R}(g) \rangle_t = \int_0^t |g_s|^2_{H_s} d\lambda_s \ .$$

In addition, we show (Proposition 11) that the range of Ito stochastic integrals, $R(\mathcal{I}(f),\ f \in \tilde{L}^2_{loc}(Q))$ is a dense subset of the range of normalized stochastic integrals, $R(\mathcal{R}.(f),\ f \in L^2_{loc}(Q))$ in the topology generated by uniform in t convergence in probability.

The linkage between the normalized integral and other extensions of Ito stochastic integral is considered in detail in Sections 3.3 and 4.1.

The normalized integrals arise naturally in many problems of stochastic analysis. Indeed, their utility is quite evident in the characterization of measures that are absolutely continuous with respect to the measure generated by a given martingale. For example,consider the pair of 1-dimensional processes:

$$\begin{cases} dX_t = a_t dt + \sigma_t dW_t \\ dM_t = \sigma_t dW_t \end{cases}$$

Then

$$\begin{aligned} &dP_X/dP_M = \exp\{\textstyle\int_0^t a_s \sigma_s^{-2} dM_s - \frac{1}{2}\int_0^t a_s^2 \sigma_s^{-2} ds\} \\ &= \exp\{\textstyle\int_0^t a_s * dM_s - \frac{1}{2}\int_0^t |a_s|^2_{H_s} ds\}, \\ &\text{where } |f_s|_{H_s} := |\sigma_s^{-2} f_s|. \end{aligned}$$

In the forthcoming paper [19] we prove that all absolutely continuous shifts of a local square integrable cylindrical martingale M_t introduced above are of the form $\int_0^t g_s d\lambda_s, g \in L^2_{loc}(Q)$, and the corresponding Radon-Nikodym derivative is given by $\exp\{\int_0^t g_s * dM_s - \frac{1}{2}\int_0^t |g_s|^2_{H_s} d\lambda_s\}$.

Another interesting example arises in the characterization of the stable subspaces of local martingales. It is well known that this problem is of central importance for the representation theorem in martingale problems (see e.g. [8]). In Section 3 (Proposition 12) we prove that the stable space of a locally square integrable continuous cylindrical martingale $\{M_t(y'), y' \in E'\}$ coincides with the set of normalized integrals

$$\mathcal{L}^1(M) = \{\mathcal{R}(f) : E[(\int_0^1 |f_s|^2_{H_s} d\lambda_s)^{1/2}]\} < \infty \,.$$

In Section 4 of the paper we discuss various particular cases of the normalized integral. These include Hilbert-valued stochastic integrals, stochastic integrals with respect to orthogonal martingale measures, stochastic integrals with respect to stochastic flows, etc.

In Section 5 we apply the same ideas for integration of vector valued functions with respect to martingale measures.

Our construction obviously does not cover the more difficult case of Banach space valued integrands with respect to one-dimensional Brownian motion where the geometry of Banach space is involved (see [1], [2], etc.). Also, we leave aside the complicated problem of the existence of the factorization $Q_s d\lambda_{s_x}$ in the most general case. In many particular cases this factorization is known. It was established in [15], [14] in the Hilbert space setting, in [9] for nuclear space valued square integrable martingales, in [22], [6] and [7] for orthogonal martingale measures, etc. The stochastic integral for Banach space valued square integrable martingales constructed by Métivier-Pellaumail [16] is based on an a priori estimate of simple integrals. In the Appendix we show that this estimate actually implies the existence of the factorization $Qd\lambda$.

2 Ito Stochastic Integrals

Suppose we have a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the right-continuous filtration of σ-algebras $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le 1}$. Let $\mathcal{P}(\mathbb{F})$ be the $\mathbb{F}$-predictable σ-algebra. Let E be a quasi-complete locally convex topological vector space, i.e. E is locally convex topological vector space so that all its bounded closed subsets are complete. Let E' be its topological dual. Denote by $\langle x, y\rangle \, (x \in E', \quad y \in E)$ the canonical bilinear form. We suppose that there exists a countable weakly dense subset of E'. Let $\mathcal{L}^+(E)$ be the space of symmetric non-negative definite forms Q from E' to E, i.e:

$$\langle y', Qy''\rangle = \langle y'', Qy'\rangle, \langle y', Qy'\rangle \ge 0 \quad \forall y', y'' \in E'$$

Definition 1. We say that a family of real valued random processes $M_t = (M_t(y'))_{y' \in E'}$ is a locally square integrable cylindrical martingale in E with covariance operator function Q_s and quadratic variation $\int_0^{\cdot} Q_s d\lambda_s$ if for each $y' \in E'$ $M_t(y') \in \mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$ and

$$M_t(y')M_t(y'') - \int_0^t \langle y', Q_s y''\rangle \, d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}), \tag{3}$$

where $Q : [0,1] \times \Omega \to \mathcal{L}^+(E)$ is a $\mathcal{P}(\mathbb{F})$-measurable function (i.e. $\forall y', y'' \in E'$, $\langle y', Q_s y''\rangle$ is $\mathcal{P}(\mathbb{F})$-measurable), and λ_t is an increasing $\mathcal{P}(\mathbb{F})$-measurable process.

Here and below $\mathcal{M}_{loc}(\mathbb{F}, \mathbf{P})$ is the space of real-valued local $(\mathbb{F}, \mathbf{P})$-martingales and $\mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$ is the space of locally square integrable real valued $(\mathbb{F}, \mathbf{P})$- martingales.

Our next step is to construct a $\mathcal{P}(\mathbb{F})$-measurable family of Hilbert subspaces of E generated by the covariance operator of the cylindrical martingale M.

According to [21], for any $K \in \mathcal{L}^+(E)$, one can define an inner product in KE' by the formula $(Ky', Ky'')_K = \langle y', Ky'' \rangle \quad \forall y', y'' \in E'$.

The following statements hold true (see Appendix for the proofs).

Proposition 2. *(See Proposition 10 in [21]). There exists a completion H_K of KE' with respect to the inner product $(\cdot,\cdot)_K$ such that $H_K \subset E$ and the natural imbedding is continuous.*

Corollary 3. *(cf. Corollary to Proposition 7 in [21]). Let T be a countable weakly dense subset of E'. Then KT is strongly dense in H_K, i.e., H_K is a separable Hilbert space.*

Denoting $H_s = H_{Q_s}$, we can rewrite (3) as

$$M_t(y')M_t(y'') - \int_0^t (Q_s y', Q_s y'')_{H_s}\, d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}), \tag{4}$$

Definition 4. We say that $(H_s) = (H_{Q_s})$ is the family of covariance spaces of M.

Let $L(Q)$ be the set of all vector fields $f = f_s = f(s,\omega)$ such that $f_s \in H_s$ and $(f_s, Q_s y')_{H_s}$ are $\mathcal{P}(\mathbb{F})$-measurable for each $y' \in E'$. Denote $L^2_{loc}(Q) = \{f \in L(Q) : \int_0^1 |f|^2_{H_s}\, d\lambda_s < \infty \quad \mathbf{P}\text{-a.s.}\} = L^2_{loc}(Q, \mathbf{P})$.

Let $T = \{e'_1, \ldots\}$ be a countable weakly dense subset of E'. We define a sequence of E-valued $\mathcal{P}(\mathbb{F})$-measurable functions:

$$\begin{aligned}
e^1_s &= \begin{cases} Q_s e'_1 / |Q_s e'_1|_{H_s}, & \text{if } Q_s e'_1 \neq 0 \\ 0, & \text{if } Q_s e'_1 = 0 \end{cases}, \\
&\ldots \\
e^{k+1}_s &= \begin{cases} \left(Q_s e'_{k+1} - \sum_{i=1}^k (Q_s e'_{k+1}, e^i_s) e^i_s\right) / d^{k+1}_s, & \text{if } d^{k+1}_s \neq 0 \\ 0, & \text{if } d^{k+1}_s = 0 \end{cases}, \\
&\ldots
\end{aligned} \tag{5}$$

where $d^{k+1}_s = |Q_s e'_{k+1} - \sum_{i=1}^k (Q_s e'_{k+1}, e^i_s)_{H_s} e^i_s|_{H_s}$. It follows from the definition of the sequence (e^n_s) that for each n, there exists an E'-valued $\mathcal{P}(\mathbb{F})$-measurable function $\tilde{e}^n_s$ such that

$$e^n_s = Q_s \tilde{e}^n_s\,. \tag{6}$$

According to Corollary 3, $Q_s T$ is a dense subset of H_s, then (5) is the Hilbert-Schmidt orthogonalization procedure. This yields that for each s, the vectors (e^k_s) form a basis in H_s. Thus we arrive at the following statement.

Corollary 5. *Let $f \in L(Q)$. Then for each s, we have the expansion in $H_{Q_s} = H_s$*

$$f_s = \sum_k (f_s, e^k_s)_{H_s} e^k_s, \quad \text{and } |f_s|^2_{H_s} = \sum_k (f_s, e^k_s)^2_{H_s}, \tag{7}$$

In particular, this expansion implies that $|f_s|_{H_s}$ is a predictable function.

Remark 6. Assume that $\mathbf{P}$-a.s. for each $y \in E'$,

$$\int_0^1 \langle Q_s y, y\rangle d\lambda_s < \infty .$$

Then it follows that $\mathbf{P}$-a.s. for each $y \in E'$, $\int_0^1 \langle Q_s y, \cdot\rangle d\lambda_s \in E'^*$, where E'^* is the algebraic dual of E'.

Remark 7. Assume that $\mathbf{P}$-a.s. $\int_0^1 \langle Q_s y, \cdot\rangle d\lambda_s \in E$ for each $y \in E'$. Let $f \in L^2_{loc}(Q)$. Then $\int_0^t f_s d\lambda_s \in E$ $\mathbf{P}$-a.s. for each t.

Indeed, for each $y \in E'$,

$$|\int_0^t \langle f_s, y\rangle d\lambda_s| \le (\int_0^t |f_s|^2_{H_s} d\lambda_s)^{1/2} (\int_0^1 \langle Q_s y, y\rangle d\lambda_s)^{1/2}$$

and the statement follows.

Remark 8. We remark that for a predictable increasing process A_t, $t \in [0, 1]$, (we assume $A_0 = 0$) the condition $A_1 < \infty$ $\mathbf{P}$-a.s. is equivalent to the existence of a sequence of stopping times (τ_n) such that $\mathbf{P}(\tau_n < 1) \to 0$, and $\mathbf{E}A_{\tau_n} < \infty$ for each n (see Lemma 1.37 in [8]).

Now we can construct the Ito stochastic integral for the class $\tilde{L}^2_{loc}(Q)$ of all $\mathcal{P}(\mathbb{F})$-measurable E'-valued functions f such that

$$\int_0^1 \langle Q_s f_s, f_s\rangle \, d\lambda_s < \infty \qquad \mathbf{P}\text{-a.s.}$$

We start with the set of simple functions $S_b = \{f \in \tilde{L}^2_{loc}(Q) : f = \sum_1^n f_s^k h_k,\ f^k$ are $\mathcal{P}(\mathbb{F})$-measurable bounded scalar functions, $h_k \in E', k = 1, \ldots, n, \quad n \ge 1\}$. For $f = \sum_1^n f_s^k h_k \in S_b$, we define the Ito integral by

$$\mathcal{I}_t(f) = \int_0^t f_s dM_s = \sum_1^n \int_0^t f_s^k dM_s(h_k) .$$

We see immediately that the map $f \rightsquigarrow \mathcal{I}(f)$ defined on S_b (with values in $\mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$) is linear up to evanescence and for each $f \in S_b$,

$$\langle \mathcal{I}(f)\rangle_t = \int_0^t \langle Q_s f_s, f_s\rangle \, d\lambda_s . \tag{8}$$

Proposition 9. *The map $f \rightsquigarrow \mathcal{I}(f)$ defined on S_b has a further extension to the set $\tilde{L}^2_{loc}(Q)$ (still denoted $f \rightsquigarrow \mathcal{I}_t(f) = \int_0^t f_s dM_s$) such that:*

1. $\mathcal{I}(f) \in \mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$ and (8) holds;

2. $f \rightsquigarrow \mathcal{I}(f)$ is linear up to evanescence;

3. If $f^n, f \in \tilde{L}^2_{loc}(Q)$ and $\int_0^1 \langle Q_s(f_s^n - f_s), f_s^n - f_s\rangle d\lambda_s \to 0$ in probability, then $\sup_{s \le 1} |\mathcal{I}_s(f^n) - \mathcal{I}_s(f)| \to 0$ in probability, as $n \to \infty$.

Moreover, this extension is unique (up to evanescence).

Proof. 1^0. Firstly, we extend the Ito integral to

$$\tilde{L}^2(Q) = \left\{ f \in \tilde{L}^2_{\mathrm{loc}} : E \int_0^1 \langle Q_s f_s, f_s \rangle d\lambda_s < \infty \right\} .$$

Let $S = \left\{ f \in \tilde{L}^2(Q) : f = \sum_1^n f_s^k h_k \right.$, f^k are real valued $\mathcal{P}(\mathbb{F})$-measurable functions, $h_k \in E'$, $k = 1, \dots, n, n \geq 1\}$. Fix $f_s = \sum_1^p f_s^k h_k \in S$ and define

$$g_s^n = f_s 1_{\{\max_{1\leq k\leq p} |f_s^k| \leq n\}} .$$

Obviously $g^n \in S_b \cap \tilde{L}^2(Q)$ and by the Lebesgue dominated convergence theorem

$$\mathbf{E} \int_0^1 \langle Q_s(f_s - g_s^n), f_s - g_s^n \rangle d\lambda_s \to 0, \quad \text{as } n \to \infty .$$

Now it follows from (8) that

$$\mathbf{E} \sup_t |\mathcal{I}_t(g^n) - \mathcal{I}_t(g^m)|^2 \to 0, \quad \text{as } n, m \to \infty .$$

Thus we can extend $\mathcal{I}$ to S linearly so that for each $f \in \mathcal{S}$, $\mathcal{I}_t(f) \in \mathcal{M}^2(\mathbb{F}, P)$ and (8) holds.

Now fix $f \in \tilde{L}^2(Q)$. Then $Q_s f_s \in L^2_{\mathrm{loc}}(Q)$ and by Corollary 5 (see (7)) $Q_s f_s = \sum_k (Q_s f_s, e_s^k)_{H_s} e_s^k$. Let $g_s^N = \sum_1^N (Q_s f_s, e_s^k)_{H_s} e_s^k$. By the Lebesgue dominated convergence theorem

$$\mathbf{E} \int_0^1 |Q_s f_s - g_s^N|^2_{H_s} d\lambda_s \to 0 \tag{9}$$

as $N \to \infty$. By the definition of g_s^N and e_s^k (see (5), (6)) it follows that there exists $f^N \in S$ such that $g_s^N = Q_s f_s^N$. Therefore we can write (9) as

$$\mathbf{E} \int_0^1 |Q_s f_s - Q_s f_s^N|^2_{H_s} d\lambda_s = \mathbf{E} \int_0^1 \langle Q_s(f_s - f_s^N), f_s - f_s^N \rangle d\lambda_s \to 0, \text{ as } N \to \infty .$$

Thus $\mathcal{I}_t(f^N)$ is a Cauchy sequence and we can find $\mathcal{I}(f) \in M^2(\mathbb{F}, \mathbf{P})$ such that (8) holds and

$$\mathbf{E} \sup_t |\mathcal{I}_t(f) - \mathcal{I}_t(f^N)|^2 \to 0, \text{ as } N \to \infty .$$

Obviously this extension is linear (up to evanescence) and unique by the Property 3.

2^o. In order to extend $\mathcal{I}$ to $\tilde{L}^2_{\mathrm{loc}}(Q)$ we apply the standard localization procedure. Fix $f \in \tilde{L}^2_{\mathrm{loc}}(Q)$, then there exists a sequence of stopping times $\tau_m \uparrow 1$ such that $f_s 1_{\{s \leq \tau_m\}} \in \tilde{L}^2(Q)$ for each m and we can find $\mathcal{I}(f) \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{F}, \mathbf{P})$ such that $\mathcal{I}_{t\wedge\tau_m}(f) = \mathcal{I}_t\left(f 1_{\{\cdot \leq \tau_m\}}\right)$. Properties 1,2 of the extension are obvious.

3^o. Finally, we prove that property 3 holds for $\mathcal{I}(f), f \in \tilde{L}^2_{\text{loc}}(Q)$. Let $\int_0^1 \langle Q_s(f^n_s - f_s), f^n_s - f_s\rangle d\lambda_s \to 0$ in probability as $n \to \infty$. There exists a sequence (τ_n) of stopping times such that

$$\mathbf{E}\int_0^{\tau_n} \langle Q_s(f^n_s - f_s), f^n_s - f_s\rangle d\lambda_s + P(\tau_n < 1) \to 0$$

as $n \to \infty$, and for each n $\mathbf{E}\int_0^{\tau_n} \langle Q_s f_s, f_s\rangle d\lambda_s < \infty$. Thus $\mathbf{E}\sup_t |\mathcal{I}_{t\wedge\tau_n}(f_n) - \mathcal{I}_{t\wedge\tau_n}(f)|^2 \to 0$, as $n \to \infty$. Since $P(\tau_n < 1) \overset{n\to\infty}{\to} 0$, we derive easily that $\sup_t |\mathcal{I}_t(f_n) - \mathcal{I}_t(f)| \to 0$ in probability as $n \to \infty$. Then the statement follows.

3 Normalized and Ito Stochastic Integrals

3.1 Normalized stochastic integrals

If $f^n \in \tilde{L}^2_{loc}(Q)$ and $\int_0^1 \langle Q_s(f^n_s - f^m_s,), f^m_s - f^n_s\rangle \, d\lambda_s \to 0$ in probability as $n, m \to \infty$, then there exists $\mathcal{I}_t \in \mathcal{M}^2_{loc}(\mathbb{F}, P)$ such that $\sup_t |\mathcal{I}_t - \mathcal{I}_t(f_n)| \to 0$ in probability, as $n \to \infty$. In order to describe $\mathcal{I}_t$ we need to complete $\tilde{L}^2_{loc}(Q)$.

Let $\tilde{\mathcal{O}} = \left\{f \in \tilde{L}^2_{loc}(Q) : \int_0^1 \langle Q_s f_s, f_s\rangle \, d\lambda_s = 0 \quad P\text{-a.s.}\right\}$ and $\tilde{\mathcal{L}}^2_{loc}(Q) = \tilde{L}^2_{loc}(Q)/\tilde{\mathcal{O}}$. For $f \in \tilde{L}^2_{loc}(Q)$ we denote $\hat{f} = f + \tilde{\mathcal{O}}$ and define the distance of the convergence in probability.

$$\tilde{d}(\hat{f}, \hat{g}) = E[\int_0^1 \langle Q_s(f_s - g_s), f_s - g_s\rangle \, d\lambda_s^{1/2} \wedge 1] \, .$$

Let $\mathcal{O} = \left\{f \in L^2_{loc}(Q) : \int_0^1 |f_s|^2_{H_s} \, d\lambda_s = 0 \quad P\text{-a.s.}\right\}$, $\mathcal{L}^2_{loc}(Q) = L^2_{loc}(Q)/\mathcal{O}$. For $f \in L^2_{loc}(Q)$ we denote $\hat{f} = f + \mathcal{O}$ and define the distance

$$d(\hat{f}, \hat{g}) = E[\int_0^1 |f_s - g_s|^2_{H_s} d\lambda_s^{1/2} \wedge 1], f, g \in L^2_{loc}(Q) \, .$$

It is easy to see that these definitions do not depend on the particular representative of the equivalence class and $\mathcal{L}_{loc}(Q)$ is a complete metric space.

Proposition 10. *(see [18, 19]) The map $\hat{\mathcal{G}} : \hat{f}_s \rightsquigarrow \widehat{Q_s f_s}$ is an isometric imbedding of $\tilde{\mathcal{L}}^2_{loc}(Q)$ into $\mathcal{L}^2_{loc}(Q)$ and $\hat{\mathcal{G}}(\tilde{\mathcal{L}}^2_{loc}(Q))$ is a dense subset of $\mathcal{L}^2_{loc}(Q)$, i.e., $\mathcal{L}^2_{loc}(Q)$ is the completion of $\tilde{\mathcal{L}}^2_{loc}(Q)$.*

Proof. For each $y \in E'$, $\langle Q_s y, y\rangle = 0$ if and only if $Q_s y = 0$ and the first part of the statement follows from the definitions.

Let $f_s \in L^2_{loc}(Q), f^N_s = \sum_1^N (f_s, e^k_s)e^k_s$. Then by Corollary 5

$$\int_0^1 |f_s - f^N_s|^2_{H_s} d\lambda_s \to 0, \qquad \text{as } N \to \infty, \qquad \mathbf{P}\text{-a.s.} \tag{10}$$

From the definition of (e^k_s) (see (5), (6)) it follows that there exists a sequence $\tilde{f}^N \in \tilde{L}^2_{loc}(Q)$ such that $f^N_s = Q_s\tilde{f}^N_s$. We can rewrite (10) as

$$\int_0^1 |f_s - Q_s\tilde{f}^N_s|^2_{H_s} d\lambda_s \to 0, \qquad \text{as } N \to \infty, \qquad \mathbf{P}\text{-a.s.}$$

Now the second part of the statement follows.

Let $\mathcal{G}$ be the map $f_s \rightsquigarrow Q_s f_s$ from $\tilde{L}^2_{loc}(Q)$ to $L^2_{loc}(Q)$. If $f, g \in \tilde{L}^2_{loc}(Q)$ and $f - g \in \tilde{\mathcal{O}}$, we have $\mathcal{T}(f) = \mathcal{T}(g)$. Thus according to Propositions 9 and 10, we can define the stochastic integral on $\mathcal{G}(\tilde{L}^2_{loc}(Q)) \subset L^2_{loc}(Q)$ by

$$\mathcal{R}_t(\hat{g}) = \mathcal{R}_t(g) = \int_0^t g_s * dM_s = \int_0^t f_s dM_s = \mathcal{I}_t(f) = \mathcal{I}_t(\hat{f}) \ , \tag{11}$$

where $g_s = Q_s f_s$, $f_s \in \tilde{L}^2_{loc}(Q)$. Obviously $\langle \mathcal{R}(g)\rangle_t = \int_0^t |g_s|^2_{H_s} \, d\lambda_s$.

Proposition 11. *(see [18, 19])The map $f \rightsquigarrow \mathcal{R}(f)$ defined on $\mathcal{G}(\tilde{L}^2_{loc}(Q))$ has a unique extension to the set $L^2_{loc}(Q)$, still denoted $f \rightsquigarrow \mathcal{R}_t(f) = \int_0^t f_s * dM_s$, with these properties:*

1. $\mathcal{R}(f) \in \mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$, $\quad \langle \mathcal{R}(f)\rangle_t = \int_0^t |f_s|^2_{H_s} \, d\lambda_s$;

2. $\mathcal{R}(f)$ is linear up to evanescence;

3. If $f_n, f \in L^2_{loc}(Q)$ and $\int_0^1 |f^n_s - f_s|^2_{H_s} \, d\lambda_s \to \infty$ in probability, as $n \to \infty$, then $\sup_{s\leq 1} |\mathcal{R}_s(f_n) - \mathcal{R}_s(f)| \to 0$ in probability, as $n \to \infty$.

Proof. 1°. Let $f \in L^2_{loc}(Q), f^N = \sum_1^N (f_s, e^k_s)_{H_s} e^k_s$. By the definition of (e^k_s) (see (5), (6)), it follows that $f^N \in \mathcal{G}(\tilde{L}^2_{loc}(Q))$. By Corollary 5, we have P-a.e.

$$\int_0^1 |f_s - f^N_s|^2_{H_s} d\lambda_s \to 0, \quad \text{as } N \to \infty \, .$$

Thus there exists increasing sequences of stopping times $(\tau_{N,p}), (\tau_p)$ such that $\tau_{N,p} \leq \tau_p \leq 1$ for each N, p, and

$$\mathbf{P}(\tau_p < 1) \overset{p\to\infty}{\longrightarrow} 0, \quad \mathbf{P}(\tau_{N,p} < \tau_p) \overset{n\to\infty}{\longrightarrow} 0,$$

$$\mathbf{E} \int_0^{\tau_p} |f_s|^2_{H_0} d\lambda_s < \infty, \quad \int_0^{\tau_{N,p}} |f^N_s|^2_{H_s} d\lambda_s \leq \int_0^{\tau_p} |f_s|^2 d\lambda_s + 1 \, .$$

Then for each p,

$$\mathbf{E} \sup_t |\mathcal{T}_{t\wedge\tau_{N,p}}(f^N) - \mathcal{T}_{t\wedge\tau_{M,p}}(f^M)|^2 \to 0 \ ,$$

as $N, M \to \infty$. Thus the existence of an extension satisfying 1,2 follows immediately.

2°. Now we prove Property 3 of the extension. Let $f^n, f \in L^2_{loc}(Q)$ and

$$\int_0^1 |f^n_s - f_s|^2_{H_s} d\lambda_s \to 0, \quad \text{as} \quad n \to \infty \, .$$

Then there exists a sequence of stopping times (τ_n) such that $\mathbf{P}(\tau_n < 1) + \mathbf{E}\int_0^{\tau_n} |f_s^n - f_s|^2_{H_s} d\lambda_s \overset{n\to\infty}{\longrightarrow} 0$.

Hence,

$$\mathbf{E}\sup_t |\mathcal{R}_{t\wedge\tau_n}(f^n) - \mathcal{R}_{t\wedge\tau_n}(f)|^2 \leq$$
$$\leq C\mathbf{E}\int_0^{\tau_n} |f_s^n - f_s|^2_{H_s} d\lambda_s \to 0, \text{ in probability, as } n \to \infty .$$

Thus the property 3 holds for the extension which is obviously unique.

For the martingale representation theorem, it is important to describe the stable subspace generated by $M(y)$, $y \in E'$. According to the definition (see [8]), this is the smallest subspace, $\mathcal{L}^1(M)$, of the closure of $H^1 = \{M \in \mathcal{M}_{loc}(\mathbb{F}, P) : |M|_1 = \mathbf{E}\sup_t |M_t| < \infty\}$ with respect to the norm $|\cdot|_1$ that contains all the integrals $\int_0^t h_s\, dM_s(y)$ where h_s is a real valued predictable function such that

$$\mathbf{E}[\int_0^1 h_s^2\, d < M(y) >_s]^{1/2} < \infty, y \in E'.$$

Proposition 12. *Let $M(y) \in \mathcal{M}^c_{loc}(\mathbb{F}, \mathbf{P})$ for each $y \in E$. Then the stable subspace of H^1 generated by $M(y)$ is*

$$\mathcal{L}^1(M) = \Big\{\mathcal{R}_t(f) = \int_0^t f_s * dM_s : \quad f \in L^2_{loc}(Q)$$
$$\textit{and} \quad \mathbf{E}\left[\left(\int_0^1 |f_s|^2_{H_s} d\lambda_s\right)^{1/2}\right] < \infty\Big\} .$$

Proof. From Burkholder's inequality (see [8]) it follows that $\mathcal{L}^1(M)$ is a closed subspace of H^1. Now the statement follows by the definition of the basis (e^i) and the normalized integrals $\mathcal{R}(f)$.

3.2 Linear transformations of integrands and covariance spaces

Let F be a quasicomplete locally convex topological vector space and F' its topological dual. Denote $\mathcal{L}_w(E, F)$ the set of weakly continuous linear forms from E to F. Let $L^2_{loc}(Q, \mathcal{L}_w(E, F))$ be the set of all predictable $\mathcal{L}_w(E, F)$-valued functions u_s such that for each $f' \in F$,

$$\int_0^1 \langle u'_s f', Q_s u'_s f'\rangle d\lambda_s < \infty \quad \mathbf{P}\text{-a.s.}$$

where $u'_s : F' \to E'$ is an adjoint linear form.

Definition 13. We define the stochastic integral $\int_0^t u'_s dM_s$ as the cylindrical locally square integrable martingale $\bar{M}_t = (\bar{M}_t(f'))$ in F such that

$$\bar{M}_t(f') = \int_0^t u'_s f' dM_s, f' \in F' .$$

Remark 14. It follows immediately by the definition that the covariance operator function of $\bar{M}$

$$\bar{Q}_s = u_s Q_s u_s' .$$

Obviously, $f_s' \in \tilde{L}^2_{loc}(\bar{Q})$ if and only if $u_s' f_s' \in \tilde{L}^2_{loc}(Q)$.

Let E_1be a quasicomplete locally convex topological vector space and E_1'be its topological dual with weakly dense countable subset. Let u be a weakly continuous linear form from $E_1$to E, (i.e., $u \in \mathcal{L}_w(E_1, E)$), $K \in \mathcal{L}^+(E_1)$, $H = H_K \subset E_1$. We define a Hilbert structure on $u(H) \subset E$ by (see [21])

$$|f|_G = \inf_{u(y)=f} |y|_H .$$

We shall need the following statement from [21] (see Appendix for the proof).

Proposition 15. *(see [21], Proposition 21).*
1. The set $Ku'E'$ is a dense subset of the orthogonal complement $\mathcal{K}$ to $\mathcal{N} = u^{-1}(0)$ =Ker u in H and u is an isometry between $\mathcal{K}$ and $u(H)$;
2. $u(H) = H_{\bar{K}}$, where $\bar{K} = uKu' \in \mathcal{L}^+(E)$.

Corollary 16. *For each $y \in H$, $|u(y)|_{u(H)} \leq |y|_H$.*

Proof. The statement follows obviously from part 1) of Proposition 15.

These statements can be generalized a little. Consider a finite number of quasicomplete locally convex topological vector spaces $E_i, (i = 1, \ldots, N)$ with topological duals E_i' having weakly dense countable subsets. Let $K^i \in \mathcal{L}^+(E_i)$, $H^i = H_{K^i} \subset E_i$, $u_i \in \mathcal{L}_w(E_i, E)$. We define a Hilbert structure on $G = \sum_1^N u_i(H^i)$ by (see [21])

$$|f|_G^2 = \inf_{f=\sum_1^N u_i(y_i)} \sum_1^N |y_i|^2_{H^i} .$$

This setting can be reduced to the previous one by setting $E = E_1 \oplus \ldots \oplus E_N$, $H = H^1 \oplus \ldots \oplus H^N$, and $u(y_1 \oplus \ldots \oplus y_N) = u_1(y_1) + \ldots + u_N(y_N)$.

Hence we obtained the following result.

Corollary 17. *1. $G = H_{\bar{K}}$ where $\bar{K} = \sum_1^N u^i K^i u^{i\prime}$.*
2. $|u_1(y_1) + \ldots + u_N(y_N)|_G^2 \leq |y_1|^2_{H^1} + \ldots + |y_N|^2_{H^N}$.

Let Q^1 be a predictable $\mathcal{L}^+(E_1)$-valued function and u_s be a predictable $\mathcal{L}_w(E_1, E)$-valued function. Denote $H_s^1 = H_{Q_s^1}$, $H_s = H_{Q_s}$.

Proposition 18. *Let $Q_s = u_s Q_s^1 u_s'$. Then*
a) $H_s = u_s(H_s^1)$;
b) $f_s \in L^2(Q)$ if and only if there exists $g_s \in L^2(Q^1)$ such that $f_s = u_s(g_s)$.

Proof. Part a) follows immediately from part 2) of Proposition 15. Since by Corollary 5 $|f_s|_{H_s} \le |g_s|$, one of the implications in b) is obvious. Assume now that $f_s \in L^2(Q)$. Let $f_s^n = \sum_1^n (f_s, e_s^k)_{H_s} e_s^k$. Then $\int_0^1 |f_s - f_s^n|_{H_s}^2 d\lambda_s \overset{n\to\infty}{\to} 0$. By the definition of e_s^k, we see that $f_s^n = Q_s \bar{f}_s^n$ for some predictable E'-valued function $\bar{f}^n$. Thus $\int_0^1 |Q_s \bar{f}_s^n - f_s|_{H_s}^2 d\lambda_s \overset{n\to\infty}{\to} 0$. Let $g_s^n := Q_s^1 u_s' \bar{f}_s^n$. Then $Q_s \bar{f}_s^n = u_s(g_s^n)$. By Proposition 15, g_s^n takes values in the orthogonal complement of $u_s^{-1}\{0\}$, and

$$|Q_s \bar{f}_s^n|_{H_s}^2 = |g_s^n|_{H_s^1}^2, \ |Q_s \bar{f}_s^n - Q_s \bar{f}_s^m|_{H_s}^2 = |g_s^n - g_s^m|_{H_s^1}^2 \overset{m,n\to\infty}{\longrightarrow} 0 \, .$$

This completes the proof.

It is readily checked that Corollary 17 and Proposition 15 yield the following statement.

Corollary 19. *Let $E_1, \dots, E_n$ be quasicomplete topological vector spaces with topological duals $E_1', \dots E_N'$, respectively, having weakly dense countable subsets. Let $Q_s = \sum_1^N u_s^i Q_s^i u_s^{i\prime}$ for some predictable $\mathcal{L}^+(E_i)$-valued functions Q_s^i and some predictable $\mathcal{L}_w(E_i, E)$- valued functions u^i. Then*

a) $H_s = \sum_1^N u_s^i(H_s^i)$ $(H_s = H_{Q_s}, H_s^i = H_{Q_s^i})$,

b) $f_s \in L^2(Q)$ if and only if there exists $g_s^i \in L^2(Q^i)$ such that $f_s = \sum_1^N u_s^i(g_s^i)$.

Remark 20. By Corollary 17,

$$|f_s|_{H_s}^2 \le \sum_1^N |g_s^i|_{H_s^i}^2, \text{ if } f_s = u_s^i(g_s^i) \, .$$

Let $M^i (i = 1, \dots, N)$ be cylindrical locally square integrable martingales in E_i with covariance operator functions Q_s^i and quadratic variations $\int_0^\cdot Q_s^i d\lambda_s$. Assume that for each $y_i' \in E_i', y_j' \in E_j'$,

$$\langle M^i(y_i'), M^j(y_j') \rangle = 0, \text{ if } i \ne j \, .$$

Proposition 21. *Let $u^i \in L_{loc}^2(Q^i, \mathcal{L}_w(E_i, E))$. Then $M_t = \sum_1^N \int_0^t u_s^{i\prime} dM_s^i$ is a cylindrical locally square integrable martingale in E with covariance operator function $Q_s = \sum_1^N u_s^i Q_s^i u_s^{i\prime}$ and quadratic variation $\int_0^\cdot Q_s d\lambda_s$.*

Proof. By the definition for each $f' \in E'$ $M_t(f') = \sum_1^N \int_0^t u_s^{i\prime} f' dM_s^i$. By our assumption

$$\langle \int_0^1 u_s^{i\prime} f' dM_s^i, \int_0^i u_s^{j\prime} g' dM_s^j \rangle = 0,$$

if $i \ne j$, for each $f', g' \in E'$. Now the statement follows.

3.3 Linkage between normalized and Ito integrals

Now we shall discuss the relation of the normalized and Ito integrals. Let $T = \{e'_1, e'_2, \ldots\}$ be a countable weakly dense subset of E'. For any $K \in \mathcal{L}^+(E)$, using the Hilbert- Schmidt orthogonalization procedure, we obtain an orthogonal basis (e^n) in H_K :

$$e^1 = \begin{cases} Ke'_1/|Ke'_1|_{H_K}, \text{ if } |Ke'_1|_{H_K} \neq 0, \\ 0, \text{ otherwise,} \end{cases}$$

$$\ldots$$

$$e^{k+1} = \begin{cases} (Ke'_{k+1} - \sum_{i=1}^k (Ke'_{k+1}, e^i)_{H_K} e^i)/d^{k+1}, \text{ if } d^{k+1} \neq 0, \\ 0, \text{ otherwise,} \end{cases}$$

$$\ldots$$

where $d^{k+1} = |Ke'_{k+1} - \sum_{i=1}^k (Ke'_{k+1}, e^i)_{H_K} e^i|_{H_K}$.

Remark 22. From the definition of (e^k), it follows that for each n there exists $\tilde{e}^k \in E'$ such that

$$e^n = K\tilde{e}^n, \; n = 1, 2, \ldots \tag{12}$$

Lemma 23. *a) If $h \in H_K$, then there exists a unique $F \in H'_K$ such that $h = F \circ K$ (here $F \circ K(e') = F(Ke')$, $e' \in E'$);*

b) Let $F \in H'_K$. Then $F \circ K \in H_K$ and

$$|F \circ K|^2_{H_K} = \sup_{e'} |F \circ K(e')|^2 / \langle Ke', e' \rangle = \textstyle\sum_n F \circ K(\tilde{e}^n)^2 = |F|^2_{H'_K},$$
$$(F \circ K, Ke')_{H_K} = F \circ K(e') \text{ for all } e' \in E'.$$

Proof. a) If $h \in H_K$, then for all $e' \in E'$,

$$\langle h, e' \rangle = (h, Ke')_{H_K}. \tag{13}$$

Define $F(Ke') = \langle h, e' \rangle$. Since (13) holds, $F \in (KE')^*$ and $h = F \circ K$.

b) Let $F \in H'_K$. Then by Riesz theorem, there exists $h \in H_K$ such that for all $e' \in E'$,

$$F(Ke') = F \circ K(e') = (h, Ke')_{H_K} = \langle h, e' \rangle .$$

Thus $h = F \circ K$ and

$$|h|^2_{H_K} = \sup_{e'} |F \circ K(e')|^2 / \langle Ke', e' \rangle = \sum_n F \circ K(\tilde{e}^n)^2 < +\infty.$$

The statement is proved.

Remark 24. Since the imbedding of H_K into E is continuous, a continuous form on E is continuous on H_K, i.e. $E' \subset H'_K$.

Corollary 25. *a) If $f' \in E'$, we have $f' \circ K = Kf'$ and $|f'|_{H'_K} = |Kf'|_{H_K}$. Also,*

$$f' \circ K(e') = \langle f', Ke' \rangle_{E',E} = \langle e', Kf' \rangle_{E',E} = (Ke', Kf')_{H_K} \qquad (14)$$
$$= \langle f', Ke' \rangle_{H'_K, H_K} .$$

b) the sequence $(\tilde{e}^n)$ defined by (12) is an orthogonal basis of H'_K, i.e. for $F \in H'_K$ we have an expansion $F = \sum_n F \circ K(\tilde{e}^n)\tilde{e}^n$ in H'_K, and E' is a dense subset of H'_K.

c) the kernel K on E' can be continuously extended from E' to H'_K, and it defines a canonical isometry from H'_K onto H_K. Also, for all $F \in H'_K$, $e' \in E'$, and $h \in H_K$,

$$\langle e', KF \rangle_{E',E} = \langle F, Ke' \rangle_{H'_K, H_K} = (KF, Ke')_{H_K} = F \circ K(e'), \qquad (15)$$
$$(KF, h)_{H_K} = \langle F, h \rangle_{H'_K, H_K}, \ \langle F, KF \rangle_{H'_K, H_K} = |KF|^2_{H_K} = |F|^2_{H'_K} .$$

Proof. a) Indeed, for each $e' \in E'$ we have

$$f' \circ K(e') = \langle f', Ke' \rangle = \langle e', Kf' \rangle .$$

Thus $f' \circ K = Kf'$ and obviously $|f'|_{H'_K} = |Kf'|_{H_K}$.

b) By Lemma 23 for each $e' \in E'$, we have $(F \circ K, Ke')_{H_K} = F \circ K(e')$. Therefore

$$F \circ K = \sum_n (F \circ K, K\tilde{e}^n)_{H_K} K\tilde{e}^n = \sum_n F \circ K(\tilde{e}^n)\, \tilde{e}^n \circ K,$$

i.e. $F = \sum_n F \circ K(\tilde{e}^n)\tilde{e}^n$ in H'_K. Since $\tilde{e}^n \in E'$, it follows obviously that E' is dense in H'_K.

c) By (14),

$$\langle e', Kf' \rangle_{E',E} = (Ke', Kf')_{H_K} = \langle f', Ke' \rangle_{H'_K, H_K} .$$

By a), a sequence $f'_n \in E'$ is a Cauchy sequence in H'_K if and only if Kf'_n is Cauchy in H_K. If $f'_n \to F$ in H'_K then (Kf'_n) converges to an element $g \in H_K$. We denote $g = KF$ and obtain (15) by continuity.

Remark 26. We note that the set of restrictions $E'|_{H_K} = \{e'|_{H_K} : e' \in E'\}$ is isomorphic to $E'/\text{Ker}\, K$ and $K : E'|_{H_K} \to KE'$ is an algebraic isomorphism. It follows from the Corollary 25 that K can be continuously extended to a canonical isomorphism from H'_K onto H_K.

Let $\hat{L}(Q)$ be the set of all functions $F_s = F_s(\omega)$ such that $F_s \in H'_{Q_s}$ and $F_s \circ Q_s(e') = \langle e', Q_s F_s \rangle$ is $\mathcal{P}(\mathbb{F})$-measurable for all $e' \in E'$. Define

$$\hat{L}^2_{loc}(Q) = \{F \in \hat{L}(Q) : \int_0^1 |F_s|^2_{H'_s}\, d\lambda_s < \infty \ \mathbf{P}\text{-a.s.}\}.$$

Lemma 27. *The inclusion $\tilde{L}^2_{loc}(Q) \subset \hat{L}^2_{loc}(Q)$ holds, and for each $F \in \hat{L}^2_{loc}(Q)$, there exists a sequence $F^n \in \tilde{L}^2_{loc}(Q)$ such that*

$$\int_0^1 |F^n_s - F_s|^2_{H'_s} \, d\lambda_s \to 0$$

in probability, as $n \to \infty$.

Proof. Obviously, $\tilde{L}^2_{loc}(Q) \subset \hat{L}^2_{loc}(Q)$ by Corollary 25. Let (e^n_s) and $(\tilde{e}^n_s)$ be sequences defined by (5) and (6), i.e. (e^n_s) is a basis in $H_s = H_{Q_s}$, and $(\tilde{e}^n_s)$ is a sequence of E'-valued predictable processes such that $Q_s \tilde{e}^n_s = e^n_s$. Let

$$F^n_s = \sum_{k=1}^{n} F \circ Q_s(\tilde{e}^k_s)\, \tilde{e}^k_s \, .$$

Then the statement follows immediately by Corollary 25.

Now we show that there exists a natural extension of the Ito integral related to the normalized integral.

Proposition 28. *The map $F \rightsquigarrow \mathcal{I}_t(F)$ defined on $\tilde{L}^2_{loc}(Q)$ has a further extension to the set $\hat{L}^2_{loc}(Q)$ (still denoted $F \rightsquigarrow \mathcal{I}_t(F) = \int_0^t F_s dM_s$) with these properties:*

1. $\mathcal{I}(F) \in \mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$ and $\langle \mathcal{I}(F) \rangle_t = \int_0^t |F_s|^2_{H'_s} \, d\lambda_s$ holds;

2. $F \rightsquigarrow \mathcal{I}_t(F)$ is linear up to evanescence;

3. If $F^n, F \in \hat{L}^2_{loc}(Q)$ and $\int_0^1 |F^n_s - F_s|^2_{H'_s} \, d\lambda_s \to 0$ in probability, then $\sup_{s \le 1} |\mathcal{I}_s(F^n) - \mathcal{I}_s(F)| \to 0$ in probability, as $n \to \infty$.

Moreover, this extension is unique (up to evanescence), and for all $F \in \hat{L}^2_{loc}(Q)$, $g \in L^2_{loc}(Q)$,

$$\int_0^t F_s dM_s = \int_0^t (F_s \circ Q_s) * dM_s = \int_0^t Q_s F_s * dM_s,$$

$$\int_0^t g_s * dMs = \int_0^t G_s dM_s,$$

where $G_s \circ Q_s = g_s$ (recall that if $F \in \tilde{L}^2_{loc}(Q)$, then $F_s \circ Q_s = Q_s F_s$ and the first equality is simply (11)).

Proof. We claim that the extension with the properties specified above is given by

$$\mathcal{I}_t(F) = \int_0^t F_s \circ Q_s * dM_s.$$

This fact is an obvious consequence of the isometry of H'_s and H_s induced by Q_s (see Corollary 25, (c), definition of the normalized integral and Lemma 27).

Example 29. Let Y be a separable Hilbert space. Consider a Y-valued locally square integrable martingale with a deterministic covariance operator Q and $\lambda_t = t$. In this case there exists a CONS (e_k) of eigenvectors of Q, and

$$Q = \sum_k \lambda_k (e_k, \cdot)_Y \, e_k \text{ (see [14]).}$$

Then $H_Q = Q^{1/2}H$ is the set of all $h = \sum_k \lambda_k^{1/2} h_k e_k$ such that $\sum_k h_k^2 < \infty$. The dual space H'_Q is the set of all $g = \sum_{\lambda_k > 0} \lambda_k^{-1/2} g_k \, e_k$ such that $\sum_k g_k^2 < \infty$. We have an obvious duality here:

$$g(h) = \sum_{\lambda_k > 0} h_k g_k$$

where $h = \sum_k \lambda_k^{1/2} h_k e_k \in H_Q$, $g = \sum_{\lambda_k>0} \lambda_k^{-1/2} g_k e_k \in H'_Q$.

Let $g^n = \sum_{\lambda_k>0,\, k \le n} \lambda_k^{-1/2} g_k e_k$. Then

$$\langle Q(g^n - g^m), g^n - g^m \rangle = \sum_{\lambda_k > 0,\, m<k\le n} g_k^2 \to 0,$$

as $n, m \to \infty$. It was explained above that in this case $g \circ Q = Qg = \sum_k \lambda_k^{1/2} g_k \in H_Q$.

Remark 30. A stochastic integral with respect to a Hilbert space-valued martingale with $\lambda_t = t$, and $Q = \sum_k 2^{-k} (e_k, \cdot)_Y \, e_k$ was constructed in [16] (p. 171).It was shown in there that the unbounded deterministic linear operator $g = \sum_k k e_k$ is an admissible integrand.

Obviously the aforementioned integral fits into the setting of Example 29 In particular, it follows from Example 29 that in the case of [16] (p. 171) the set of all deterministic integrands coincides with

$$H'_Q = \{g = \sum_k 2^{k/2} g_k e_k : \sum_k g_k^2 < \infty\}.$$

4 Some Particular Cases

4.1 Case when E' is a subspace of E.

In this subsection it is assumed that we are given a symmetric injection $I : E' \to E$. So we identify E' by I to a subspace of E and call I an identity. Obviously I is weakly continuous. Since I is injective, then $I' = I$ has a dense image, i.e. in this case E' is necessarily a dense subset of E . We think here most frequently about the example $E = \mathcal{D}'(X)$, the space of distributions on an open set $X \subset \mathbf{R}^n$, $E' = \mathcal{D}(X)$, the space of test functions on X.

Remark 31. If E is an arbitrary quasicomplete locally convex space, $K \in \mathcal{L}^+(E)$, and $H = H_K$ is a dense Hilbert subspace of E, then K is an injection and can play the role of I.

Also, in this case $f' \circ K = Kf'$ $(\langle Kf', e'\rangle = \langle Ke', f'\rangle)$.

We introduce now an important class of Hilbert subspaces of E.

Definition 32. We say that a Hilbert subspace $H \subset E$ with weakly continuous injection $H \to E$ is *normal* if E' is a dense subspace of H.

Remark 33. If $H \subset E$ is a normal Hilbert subspace, then H' is a normal Hilbert subspace of E.

Proof. Indeed, if H is normal we have weakly continuous dense injections $E' \xrightarrow{i} H \xrightarrow{j} E$. Passing to the adjoints we get weakly continuous dense injections $E' \xrightarrow{j'} H' \xrightarrow{i'} E$. Thus we identify H' with a subspace of E which is normal as well: for each e', $f' \in E'$.

$$\langle e', f'\rangle_{H,H'} = \langle e', f'\rangle_{E,E'} \, .$$

One can say here (see [13]) that H' is identified with the subspace of elements $e \in E$ such that the linear form $e' \to \langle e, e'\rangle_{E,E'}$ is continuous on E' with respect to the topology induced by H.

Example 34. Let $X \subset \mathbf{R}^n$ be a bounded open subset. Let H^s be a completion of $E' = \mathcal{D}(X)$ with respect to the norm

$$|\varphi|_s^2 = \int_X (-\Delta)^s \varphi(x)\varphi(x)dx, s \geq 0 \, .$$

Define $H^{-s} = (H^s)', s \geq 0$. All the $H^s, s \in \mathbf{R}$, are normal subspaces of $E = \mathcal{D}'(X)$. Obviously $H^{-s} = H_{(-\Delta)^{-s}}$ and $(-\Delta)^{-s} : (H^s)' = H^{-s} \to H^s$ is a canonical isomorphism between $(H^s)'$ and H^s.

The following statement is true (Propositions 28 bis, 29 in [21], see Appendix for the proof).

Proposition 35. *For $K \in \mathcal{L}^+(E)$, assume that $H = H_K$ is a normal Hilbert subspace and $H' = H_{\hat{K}}$ is its dual. Then*

a) we can extend the E'- restrictions of K and $\hat{K}$ to the continuous linear forms $K : H' \to H$, $\hat{K} : H \to H'$ which are canonical isomorphisms between H and H', i.e., $\hat{K} = K^{-1}$, $KH' = H$, $K^{-1}H = H'$;

b) for each e', $f' \in E'$

$$(e', f')_{H'} = (Ke', Kf')_H = \langle Ke', f'\rangle_{E,E'} = \langle Kf', e'\rangle_{E,E'}$$
$$(e', f')_H = (\hat{K}e', \hat{K}f')_{H'} = \langle \hat{K}e', f'\rangle_{E,E'} = \langle \hat{K}f', e'\rangle_{E,E'} .$$

Remark 36. Let $H = H_K$ be a normal Hilbert subspace. If $F \in H'_K$, then $F \circ K = KF \in H_K$; if $h \in H_K$, then $K^{-1}h \in H'_K$.

Denote $\mathcal{L}_n^+(E) = \{K \in \mathcal{L}^+(E) : H_K$ is a normal subspace$\}$. By Proposition 28 and the previous remark we get the following statement.

Proposition 37. *Let Q be a predictable $\mathcal{L}_n^+(E)$-valued function. In this case $F_s \in \hat{L}^2_{loc}(Q)$ if and only if $Q_sF_s \in L^2_{loc}(Q)$. Also, $h_s \in L^2_{loc}(Q)$ if and only if $Q_s^{-1}h_s \in \hat{L}^2_{loc}(Q)$, and for each $f \in \hat{L}^2_{loc}(Q)$, $g \in L^2_{loc}(Q)$,*

$$\int_0^t f_s dM_s = \int_0^t Q_s f_s * dM_s, \quad \int_0^t g_s * dMs = \int_0^t Q_s^{-1} g_s dM_s.$$

Now we shall discuss briefly some criterions of normality.

Proposition 38. *(see [21], p. 215). Assume that we can extend $K \in \mathcal{L}^+(E)$ to a weakly continuous linear form $K : E \to E$ and $E' \subset H \subset H_K$. Then H is normal.*

Proof. Since $K = K'$, we have $KE' \subset E' \subset H = H_K$ and HE' is dense in H. Then E' is dense in H_K and the statement follows.

Corollary 39. *Let $E_i (i = 1, \ldots, N)$ be quasi-complete locally convex topological vector spaces, let $u^i : E_i \to E$ be weakly continuous linear forms, $K^i \in \mathcal{L}_n^+(E_i)$ and $K = \sum_1^N u^i k^i u^{i'}$. Assume that $u^{i'}$ is extendable to a weakly continuous linear form $u^{i'} : E \to H^i = H_{K_i}$, and there exists i such that $E' \subset u^i(H^i)$. Then $K \in \mathcal{L}_n^+(E)$.*

Proof. By the assumptions K is extendable to a weakly continuous linear form $K : E \to E$ and $E' \subset \sum_1^N u^i(H^i) = H_K$. The statement follows by Proposition 38.

4.2 Orthogonal martingale measures.

Let $(U, \mathcal{B}(U))$ be a countable measurable space (i.e., $\mathcal{B}(U)$ is generated by a countable subset of $\mathcal{B}(U)$). Let $\mathcal{A}$ be an algebra generating $\mathcal{B}(U)$ and for each $A \in \mathcal{A}$ we have $M_t(A) \in \mathcal{M}^2_{loc}(\mathbb{F}, P)$. Suppose that there exists an increasing $\mathcal{P}(\mathbb{F})$-measurable process λ and $\mathcal{P}(\mathbb{F})$-measurable family $q_s(dx)$of non-negative measures on $(U, \mathcal{B}(U))$ such that

$$M_t(A)M_t(B) - \int_0^t \int_{A\cap B} q_s(dx)\, d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}) \quad \forall A, B \in \mathcal{A}.$$

Suppose that there exists an increasing sequence of $\mathcal{B}(U)$-measurable sets (U_n) such that $U_n \uparrow U$ and

$$\int_0^t \int_{U_n} q_s(dx)\, d\lambda_s < \infty \quad \mathbf{P}\text{-a.s.} \quad \forall n\ .$$

Let E' be the set of bounded $\mathcal{B}(U)$-measurable functions f such that supp $f \subset U_n$ for some n. Let E be the set of measures μ on $(U, \mathcal{B}(U))$ such that $\mu|_{U_n}$ is bounded for each n with the weak topology $\sigma(E, E')$. Now $Q_s f = f(x)q_s(dx)$ for each $f \in E'$, i.e., it is a measure on $(U, \mathcal{B}(U))$ from E.

Proposition 40. *(see [18]) For each s, H_s is the set of all measures on $(U, \mathcal{B}(U))$ of the form $f(x)q_s(dx)$ such that $\int_U (f(x))^2 q_s(dx) < \infty$, and H'_s is the set of all measurable functions f on U such that $\int_U (f(x))^2 q_s(dx) < \infty$.*

$L^2_{loc}(Q)$ is the set of all $\mathcal{P}(\mathbb{F})$-measurable measures on $(U, \mathcal{B}(U))$ of the form $f(s,x)q_s(dx)$ (f is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(U)$-measurable function) such that.

$$\int_0^t \int_U |f(s,x)|^2 q_s(dx)\, d\lambda_s < \infty \qquad \mathbf{P}\text{-}a.s., \forall t,$$

$$\textit{and} \qquad |f_s q_s|^2_{H_s} = \int_U |f(s,\, x)|^2 q_s(dx).$$

$\hat{L}^2_{loc}(Q)$ is the set of all $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(U)$-measurable functions f such that

$$\int_0^t \int_U |f(s,x)|^2 q_s(dx)\, d\lambda_s < \infty \ \mathbf{P}\text{-}a.s., \forall t,$$

$$|f_s|^2_{H'_s} = \int_U |f(s,\, x)|^2 q_s(dx).$$

Proof. The set $Q_s E'$ consists of all measures of the form $f(u)q_s(du)$, $f \in E'$. We will treat it as the space of classes of equivalent measures, two measures $f(u)q_s(du)$ and $g(u)q_s(du)$ are equivalent if $f = g$ q-a.s. This vector space endowed with the norm

$$|f(u)q_s(du)|^2_{H_s} = \int_\Omega f^2(u) q_s(du) = \langle Q_s f,\ f \rangle$$

becomes a Banach space isometrically isomorphic to $L_2(U, \mathcal{B}(U),\, q_s) = H'_s$, and hence complete. Now our statement follows simply from the definitions and Theorem IV:Q.4 in [3].

4.3 Integrals with respect to stochastic flows.

Here we generalize some results of H. Kunita in [11] concerning the integrals with respect to stochastic flows.

Let X be a locally compact metrisable space (there exists a countable dense subset of X). For each $x \in X$ we are given $M_t(x) \in \mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$. Suppose that there exists a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(X) \otimes \mathcal{B}(X)$ measurable function $Q_s(x,y)$ and an increasing $\mathcal{P}(\mathbb{F})$-measurable process λ_t such that

$$M_t(x) M_t(y) - \int_0^t Q_s(x,\, y)\, d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}) \tag{16}$$

We assume that Q_s is symmetric and non-negative definite, i.e. $\forall(\xi_n)$, $\forall(x_j)$,

$$\sum_{n,j} \xi_n \xi_j Q_s(x_n,\, x_j) \geq 0 \;.$$

Let E be the set of all real valued functions with the topology of simple convergence and E' its dual space, i.e., the space of finite combinations of Dirac measures. If $\mu \in E'$, we have $\mu = \sum_x c_x \delta_x$ and only a finite number of $c_x \neq 0$. For $\mu = \sum_x c_x \delta_x$ we can define

$$M_t(\mu) = \sum_x c_x M_t(\delta_x) = \sum_x c_x M_t(x) \;.$$

Obviously, (16) yields

$$M_t(\mu) M_t(\mu') - \int_0^t \sum_{x,y} c_x c'_y Q_s(x,y)\; d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}) \;,$$

where

$$\mu = \sum_x c_x \delta_x, \quad \mu' = \sum_x c'_x \delta_x \;.$$

For $\mu = \sum_x c_x \delta_x$, we define $\bar{Q}_s \mu(x) = \sum_y c_y Q_s(y,\, x)$, i.e., $\bar{Q}_s \in \mathcal{L}^+(E)$ and $\forall \mu$, $\mu' \in E'$

$$M_t(\mu) M_t(\mu') - \int_0^t \langle \mu', \bar{Q}_s \mu \rangle_{E',E}\; d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}).$$

In this case the corresponding Hilbert subspaces $H_s = H_{\bar{Q}_s}$ can be described using their reproducing kernels $Q_s(x,y)$ (see [21]). We consider two particular cases.

(a.) Let Q_s be separately continuous on $X \times X$, continuous on the diagonal and locally bounded.

Denote $\mathcal{E}^0(X)$ the space of continuous functions on X with the topology of uniform convergence on compact sets. Let $\mathcal{E}^0(X)'$ be its dual space which is the space of Radon measures with compact support. We can extend the kernel $\bar{Q}_s$ to $\mathcal{E}^0(X)'$. If $\nu \in \mathcal{E}^0(X)'$, we set

$$\bar{Q}_s \nu(x) = \int_X Q_s(x,y) \nu(dy) \;.$$

Lemma 41. *For each $\nu \in \mathcal{E}^0(X)'$, there exists a sequence (μ_n) from E' and $M_t(\nu) \in \mathcal{M}^2_{\rm loc}(\mathbb{F},\, \mathbf{P})$ such that*

$$\sup_t |M_t(\mu_n) - M_t(\nu)| + \int_0^1 \langle \bar{Q}_s(\mu_n - \nu),\, \mu_n - \nu \rangle\, d\lambda_s \overset{n\to\infty}{\longrightarrow} 0$$

in probability. Moreover, for each $\mu, \nu \in \mathcal{E}^0(X)'$,

$$M_t(\mu) M_t(\nu) - \int_0^t \langle \bar{Q}_s(\mu - \nu),\, (\mu - \nu) \rangle d\lambda s \in \mathcal{M}_{\rm loc}(\mathbb{F}, \mathbf{P}) \;.$$

Proof. For each n there exists a finite measurable partition (A_k^n) of $\operatorname{supp}\nu$ such that $\operatorname{diam}(A_k^n) \leq 1/n$. We choose arbitrary $x_k \in A_k^n$ and define $\mu_n = \sum_k \nu(A_k^n)\delta_{x_k}$. Obviously, we have inequality $|\mu_n^K| \leq |\nu|$ for their variations and for each continuous function f on X $\mu_n(f) \to \nu(f)$. Thus by our assumptions $\int_0^1 \langle \bar{Q}_s(\mu_n - \nu), \mu_n - \nu\rangle d\lambda_s \overset{n\to\infty}{\longrightarrow} 0$ in probability. Therefore there exists increasing sequences of stopping times $(\tau_{n,p}), (\tau_p)$ such that $\tau_{n,p} \leq \tau_p \leq 1$ and

$$\mathbf{P}(\tau_p < 1) \overset{p\to\infty}{\longrightarrow} 0,\ \mathbf{P}(\tau_{n,p} < \tau_p) \overset{n\to\infty}{\longrightarrow} 0,$$

$$\mathbf{E}\int_0^{\tau_p} \langle Q_s\nu, \nu\rangle d\lambda_s < \infty,\ \int_0^{\tau_{n,p}} \langle Q_s\mu_n, \mu_n\rangle d\lambda_s \leq$$

$$\leq \int_0^{\tau_p} \langle Q_s\nu, \nu\rangle d\lambda_s + 1\ .$$

Then for each p

$$\mathbf{E}\sup_t |M_{t\wedge\tau_{n,p}}(\mu_n) - M_{t\wedge\tau_{n',p}}(\mu_{n'})|^2 \to 0\ ,$$

as $n, n' \to \infty$. Thus the existence of a limit with required properties follows immediately.

Proposition 42. *Suppose that for each s, $\bar{Q}_s \in \mathcal{L}^+(\mathcal{E}^0(X))$, and $\int_0^1 \bar{Q}_s\, d\lambda_s \in \mathcal{L}^+(\mathcal{E}^0(X))$. If ν_s is a $\mathcal{E}^0(X)'$-valued $\mathcal{P}(\mathbb{F})$-measurable function such that*

$$\int_0^1 \int_X \nu_s(dx) \int_X Q_s(x,y)\nu_s(dy)\, d\lambda_s < \infty \quad \mathbf{P}\text{-}a.s.,$$

we can define the Ito integral $\int_0^t \nu_s dM_s = \mathcal{I}_t(\nu) \in \mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$ such that $\langle \mathcal{I}(\nu)\rangle_t = \int_0^t \int_X \nu_s(dx) \int_X Q_s(x,y)\nu_s(dy) d\lambda_s$.

Proof. The statement follows from Lemma 41, the definitions and Proposition 9.

Remark 43. In [11] the case $\nu_s = \delta_{f_s}$ was considered, where f_s is an X-valued $\mathcal{P}(\mathbb{F})$-measurable function.

(b.) In addition to the assumptions made in (a) let us suppose that X is an open subset of $\mathbf{R}^d$. Assume that Q_s has all derivatives up to order m and for $|p| \leq m, |q| \leq m$, $D_x^p D_y^q Q_s(x,y)$ are separately continuous, locally bounded and continuous on the diagonal. Let $\mathcal{E}^m(X)$ be the space of m times continuously differentiable functions with the topology of uniform convergence on compact sets of all derivatives up to order m. The dual space $\mathcal{E}^m(X)'$ will be the space of the generalized functions of order $\leq m$ with compact support. We can easily extend $\bar{Q}_s$ to $\mathcal{E}^m(X)'$ by $\bar{Q}_s T(x) = \int_X Q_s(x,y)T(y)dy$, where $T \in \mathcal{E}^m(X)'$ (obviously $\bar{Q}_s T \in \mathcal{E}^m(X)$).

Lemma 44. *For each $T \in \mathcal{E}^m(X)'$ there exist a sequence $\mu_n \in E'$ and a process $M_t(T) \in \mathcal{M}^2_{\rm loc}(\mathbb{F}, P)$ such that*

$$\int_0^1 \int_X \bar{Q}_s(T-\mu_n)(x)(T(x)-\mu_n(x))dx\, d\lambda_s +$$

$$\sup_t |M_t(T) - M_t(\mu_n)| \stackrel{n\to\infty}{\longrightarrow} 0 \quad \textit{in probability.}$$

Proof. Fix $T \in \mathcal{E}^m(X)'$. Then there is a family of Radon measures $\{\nu_p\}$ ($|p| \leq m$, $p \in \mathbf{N}^d$) with compact supports such that

$$T = \sum_{|p|\leq m} \left(\frac{\partial}{\partial x}\right)^p \nu_p \,.$$

It is enough, obviously, to prove the statement for $T = \left(\frac{\partial}{\partial x}\right)^p \nu_p$. As in the proof of Lemma 41, for each n we take a measurable partition (A^n_k) of the support of ν_p such that $\mathrm{diam}(A^n_k) \leq 1/n$. Then we choose an arbitrary $x_k \in A^n_k$ and define $\bar{\mu}_n = \sum_k \nu_p(A^n_k)\delta_{x_k}$. Obviously, the variations satisfy the inequality $|\bar{\mu}_n| \leq |\nu_p|$, and $(\bar{\mu}_n)$ converges weakly to ν_p. Thus,

$$\int_0^1 \langle \bar{Q}_s \left(\left(\frac{\partial}{\partial x}\right)^p \bar{\mu}_m - T\right), \quad \left(\frac{\partial}{\partial x}\right)^p \bar{\mu}_n - T\rangle d\lambda_s \stackrel{n\to\infty}{\longrightarrow} 0$$

in probability. Let $(e_k)_{1\leq k\leq d}$ be a canonical basis in $\mathbf{R}^d$, $p = (p_1, \ldots, p_d)$. For sufficiently small $h > 0$ and $x \in X$, define $\Delta^k_h \delta_x = \frac{1}{h}(\delta_{x+he_k} - \delta_x)$. Consider $\bar{\mu}^h_n = (\Delta^1_h)^{p_1} \ldots (\Delta^d_h)^{p_d}\bar{\mu}_n = \sum_k \nu_p(A^n_k)(\Delta^1_h)^{p_n} \ldots (\Delta^d_h)^{p_d}\bar{\delta}_{x_k}$. It is well defined for small $h > 0$ and is an element of E'. By our assumptions, for each n,

$$\int_0^1 \langle \bar{Q}_s \left(\mu^h_n - \left(\frac{\partial}{\partial x}\right)^p \bar{\mu}_n\right), \bar{\mu}^h_n - \left(\frac{\partial}{\partial x}\right)^p \bar{\mu}_n\rangle\, d\lambda_s \stackrel{h\downarrow 0}{\longrightarrow} 0\,.$$

Thus we can find a sequence (h_n) such that

$$\int_0^1 \langle \bar{Q}\left(\bar{\mu}^{h_n}_n - T\right), \left(\bar{\mu}^{h_n}_n - T\right)\rangle\, d\lambda s \stackrel{n\to\infty}{\longrightarrow} 0$$

in probability, and $\mu_n = \bar{\mu}^{h_n}_n \in E'$. We complete the proof as in the case of Lemma 41. There exist sequences of stopping times $(\tau_{n,p}), (\tau_p)$ such that $\tau_{n,p} \leq \tau_p \leq 1$ and

$$\mathbf{P}(\tau_p < 1) \stackrel{p\to\infty}{\longrightarrow} 0, \quad \mathbf{P}(\tau_{n,p} < \tau_p) \stackrel{n\to\infty}{\longrightarrow} 0,$$

$$\mathbf{E}\int_0^{\tau_p} \langle \bar{Q}_s, T, T\rangle d\lambda_s < \infty, \quad \int_0^{\tau_{n,p}} \langle \bar{Q}_s \mu_n, \mu_n\rangle d\lambda_s \leq$$

$$\leq \int_0^{\tau_p} \langle \bar{Q}_s T, T\rangle d\lambda_s + 1$$

Then for each $p \sup_t |M_{t\wedge\tau_{n,p}}(\mu_n) - M_{t\wedge\tau_{n',p}}(\mu_{n'})|^2 \to 0$, as $n, n' \to \infty$. The existence of a limit with required properties follows immediately.

Proposition 45. *For each s, $\bar{Q}_s \in \mathcal{L}^+(\mathcal{E}^m(X))$, $\int_0^1 \bar{Q}_s \, d\lambda_s \in \mathcal{L}^+(\mathcal{E}^m(X))$. If T_s is an $\mathcal{E}^m(X)'$-valued $\mathcal{P}(\mathbb{F})$-measurable function such that*

$$\int_0^1 \int_X T_s(x)dx \int_X Q_s(x,y)T_s(y)\, dy\, d\lambda_s < \infty \ \mathbf{P}\text{-a.s.},$$

we can define the integral $\mathcal{I}_t(T) = \int_0^t T_s dM_s \in \mathcal{M}^2_{loc}(\mathbb{F}, \mathbf{P})$ such that

$$\langle \mathcal{I}(T) \rangle_t = \int_0^t \int_X T_s(x)dx \int_X Q_s(x,y)T_s(y)dy\; d\lambda_s \, .$$

Proof. The statement follows from Lemma 44, the definitions and Proposition 9.

Remark 46. Let $T_s = \sum_{|\alpha| \le m} c_s^\alpha \partial^\alpha \mu_s^\alpha$ be such that

$$\int_0^1 \int_X \int_X \sum_{\alpha,\alpha'} (-1)^{|\alpha|+|\alpha'|} c_s^\alpha c_s^{\alpha'} \partial_y^{\alpha'} \partial_x^\alpha Q_s(x,y) \mu_s^\alpha(dx) \mu_s^{\alpha'}(dy)\; ds < \infty \, ,$$

where c_s^α are one dimensional predictable functions and μ_s^α are predictable functions with values in the space $\mathcal{E}^0(X)$ of Radon measures on X with compact support. Then the function T is M-integrable.

4.4 Integrals with respect to purely discontinuous martingales

In this Section we apply the ideas discussed above to integration of infinite-dimensional vector functions with respect to purely discontinuous martingales.

Let $(U, \mathcal{U})$ be a measurable space and $p(dt, du)$ be a non-negative point measure on $([0,1] \times U, \mathcal{B}([0,1]) \otimes \mathcal{U})$. Assume that there exists a $\mathcal{P}(\mathbb{F})$-measurable family of measures $\pi_s(dx)$ and an increasing continuous process λ_t such that $q(dt, du) = p(dt, du) - \pi_t(du)d\lambda_t$ is a martingale measure. Let (U_n) be an increasing sequence of measurable subsets of U such that $\cup_n U_n = U$ and for every n,

$$\int_0^1 \int_{U_n} \pi_s(du)d\lambda_s < \infty \qquad \mathbf{P}\text{-a.s.}$$

Let $f(s,u)$ be a $\mathcal{P}(\mathbb{F})$-measurable E-valued function such that

$$\int_0^1 \int \langle f(s,u), y \rangle^2 \wedge |\langle f(s,u), y \rangle| \pi_s(du) d\lambda_s < \infty$$

for each $y \in E'$, P-a.e.

For each $y \in E'$, define

$$M_t(y) = \int_0^t \int_u \langle f(s,u), y \rangle q(ds, du) \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}) \, .$$

The function $f(s,u)$ defines a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{U}$-measurable family $\bar{Q}_{s,u}$ of kernels from $\mathcal{L}^+(E)$ such that

$$\langle \bar{Q}_{s,u} y, y' \rangle = \langle f(s,u), y \rangle \langle f(s,u), y' \rangle \text{ for all } y, y' \in E$$

Obviously, the corresponding Hilbert space $\bar{H}_{s,u}$ is one dimensional:

$$\bar{H}_{s,u} = \begin{cases} \mathbf{R} f(s,u), & \text{if } f(s,u) \neq 0 \\ 0, & \text{if } f(s,u) = 0 . \end{cases}$$

In this case we can integrate functions of the variables (ω, s, u). Let $\tilde{D}^1 = \tilde{D}^1(\mathbf{P})$ be the set of all E'-valued $\mathcal{P}(\mathbb{F}) \otimes \mathcal{U}$-measurable functions g such that

$$\mathbf{E}[\int_0^1 \int_U \langle g(s,u), f(s,u) \rangle^2 p(ds,du)^{1/2}] < \infty .$$

Define $\bar{D}^1 = \bar{D}^1(\mathbf{P})$ as the set of all $\mathcal{P}(\mathbb{F}) \otimes \mathcal{U}$-measurable scalar functions ρ such that

$$\rho = \rho 1_{\{f \neq 0\}} \quad \text{and} \quad \mathbf{E}[\int_0^1 \int_U \rho(s,u)^2 p(ds,du)^{1/2}] < \infty .$$

Let $D^1 = D^1(\mathbf{P}) = \{ g = \rho f : \rho \in \bar{D}^1 \}$. Since $\bar{H}_{s,u}$ is one dimensional and $q(ds,du)$ is a scalar measure, the integration is elementary. Now let $\{y_1, y_2, \ldots\}$ be a weakly dense subset of E'. Write

$$\begin{aligned} \bar{N} = \bar{n}(s,u) &= \begin{cases} \inf(n : \langle f(s,u), y_n \rangle \neq 0, & \text{if } f(s,u) \neq 0 \\ 1, & \text{if } f(s,u) = 0 \end{cases} \\ \bar{e}'(s,u) &= \begin{cases} y_{\bar{n}(s,u)} \langle f(s,u), y_{\bar{n}(s,u)} \rangle^{-2}, & \text{if } f(s,u) \neq 0 \\ 0, & \text{if } f(s,u) = 0 . \end{cases} \end{aligned} \tag{17}$$

Then $\langle f(s,u), \bar{e}'(s,u) \rangle f(s,u)$ is a measurable basis in $\bar{H}_{s,u}$. We define the integrals

$$\begin{aligned} \mathcal{T}_t(\tilde{g}) &= \textstyle\int_0^t \tilde{g}_s dM_s = \int_0^t \int_U \langle f(s,u), \tilde{g}(s,u) \rangle q(ds,du), \tilde{g} \in \tilde{D}^1 , \\ \mathcal{R}_t(g) &= \textstyle\int_0^t g_s * dM_s = \int_0^t \int_U \rho(s,u) q(ds,du), g = \rho f \in D^1, \end{aligned}$$

(see [8] for the definition of the right sides). Now we write

$$\begin{aligned} \tilde{\mathcal{O}} &= \left\{ g \in \tilde{D}^1 : \langle f, g \rangle = 0 \quad \pi_s(du) d\lambda_s d\mathbf{P}\text{-a.s} \right\} \\ \mathcal{O} &= \left\{ g = \rho f \in D^1 : \rho = 0 \quad \pi_s(du) d\lambda_s d\mathbf{P}\text{-a.s.} \right\} , \\ \bar{\mathcal{O}} &= \left\{ \rho \in \bar{D}^1 : \rho = 0 \quad \pi_s(du) d\lambda_s d\mathbf{P}\text{-a.s.} \right\} , \\ \tilde{\mathcal{D}}^1 &= \tilde{D}^1 / \tilde{\mathcal{O}}, \quad \mathcal{D}^1 = D^1 / \mathcal{O}, \bar{\mathcal{D}}^1 = \bar{D}^1 / \bar{\mathcal{O}} . \end{aligned}$$

For $g \in \tilde{D}^1$, we denote $\hat{g} = g + \tilde{\mathcal{O}}$ and define the distance

$$\tilde{d}(\hat{g}, \hat{g'}) = \mathbf{E} \left[\int_0^1 \langle f(s,u), g(s,u) - g'(s,u) \rangle^2 p(ds,du)^{1/2} \right] .$$

For $g \in D^1$, we denote $\hat{g} = g + \mathcal{O}$ and define the distance ($g = \rho f, g' = \rho' f \in D^1$) $d(\hat{g}, \hat{g'}) = \mathbf{E}\left[\int_0^1 \int_U (\rho(s,u) - \rho'(s,u))^2 p(ds,du)^{1/2}\right]$. For $\rho \in \bar{D}^1$, we write $\hat{\rho} = \rho + \hat{\mathcal{O}}$ and define the distance

$$\bar{d}(\hat{\rho}, \hat{\rho'}) = \mathbf{E}[\int_0^1 \int_U (\rho(s,u) - \rho'(s,u))^2 p(ds,du)^{1/2}] .$$

Proposition 47. *The maps* $\mathcal{G}_1 : \widehat{g_s} \rightsquigarrow \widehat{\langle f_s, g_s \rangle f_s}$, *and* $\mathcal{G}_2 : \widehat{\rho_s f_s} \rightsquigarrow \widehat{\rho_s}$ *are isometries from* $\tilde{\mathcal{D}}^1$ *to* $\bar{\mathcal{D}}^1$ *and from* $\mathcal{D}^1$ *to* $\bar{\mathcal{D}}^1$, *respectively, i.e., all the spaces are complete. Moreover,*

$$\mathcal{T}_t(g) = \mathcal{R}_t(\langle f, g \rangle f) = \int_0^t \int_U \langle f, g \rangle q(ds,du),$$
$$\mathcal{R}_t(\rho f) = \int_0^t \int_U \rho q(ds,du), g \in \tilde{D}^1, \rho f \in D^1 .$$

Proof. Let $g = \rho f \in D^1$ and $g' = \rho(s,u)\bar{e}'(s,u)$ where $\bar{e}'(s,u)$ is defined by (17). Then $\langle f, g' \rangle = \rho$, i.e., $g' \in \tilde{D}^1$ and the statement follows from the definitions.

Remark 48. We can localize the definitions and integrate the functions from D^1_{loc}, $\tilde{D}^1_{loc}$ and $\bar{D}^1_{loc}$ (a function $g \in \mathcal{A}_{loc}$, if there exists a sequence of stopping times (τ_n) such that $\tau_n \uparrow 1$ and $g 1_{[0,\tau_n]} \in \mathcal{A}$ where $\mathcal{A} = D^1, \tilde{D}^1, \bar{D}^1$).

Let $G^1_{loc} = G^1_{loc}(\pi d\lambda, \mathbf{P})$ be the set of all $\mathcal{P}(\mathbb{F}) \otimes U$-measurable scalar functions ρ such that $\mathbf{P}$-a.e.

$$\int_0^1 \int_U |\rho(s,u)|^2 \wedge |\rho(s,u)| \pi_s(du) d\lambda_s < \infty .$$

Remark 49. By Proposition (3.71) in [8] we have the following predictable characterizations of the classes D^1_{loc}, $\bar{D}^1_{loc}$, $\tilde{D}^1_{loc}$:

a) $\rho \in \bar{D}^1_{loc} \Leftrightarrow \rho \in G^1_{loc}$,
b) $g \in \tilde{D}^1_{loc} \Leftrightarrow \langle g, f \rangle \in G^1_{loc}$,
c) $g = \rho f \in D^1_{loc} \Leftrightarrow \rho \in G^1_{loc} \Leftrightarrow \rho \in \bar{D}^1_{loc}$.

5 Appendix

Proof of Proposition 2. Let $\hat{\mathcal{H}}_0$ be a completion of $\mathcal{H}_0 = KE$. Since the natural embedding $j : \mathcal{H}_o \to E$ is continuous we can extend it continuously to a linear continuous mapping $\hat{j} : \hat{\mathcal{H}}_0 \to \hat{E}$ where $\hat{E}$ is the completion of E and $\hat{j}|_{\mathcal{H}_0} = j$. Let $x_n \in \mathcal{H}_0$ and $x_n \to x$ in $\hat{\mathcal{H}}_0$. Then $\hat{j}(x_n) = j(x_n) \to \hat{j}(x)$ in $\hat{E}$. Therefore $\hat{j}(x_n) = j(x_n)$ is a bounded Cauchy sequence in E. From the quasi-completeness of E it follows that $\hat{j}(x) \in E$, i.e. $\hat{j}$ is a continuous linear map from $\hat{\mathcal{H}}_0$ to E. We shall prove that $\hat{j}$ is an injection. Let $k \in \hat{\mathcal{H}}_0$ and $\hat{j}(k) = 0$. Then for each $y \in E'$,

$$0 = \langle \hat{j}(k), y \rangle = (k, Ky)_{\hat{\mathcal{H}}_0} .$$

This equality is obvious for $k \in \mathcal{H}_0$ and the general case follows by continuity of both sides. Thus $k \in \hat{\mathcal{H}}_0$ is orthogonal to $\mathcal{H}_0$, i.e., $k = 0$. Thus $\hat{j} : \hat{\mathcal{H}}_0 \to E$ is a continuous injection. Let $H_K = \hat{j}(\hat{\mathcal{H}}_0) \subset E$ and define $(x,y)_K = (\hat{j}^{-1}(x), \hat{j}^{-1}(y))_{\hat{\mathcal{H}}_0}$ for every $x, y \in H_K$. Then $H_K \in E$ is the completion of $\mathcal{H}_0$ and the natural imbedding $H_K \to E$ is continuous by the continuity of $\hat{j}$. The statement is proved.

Proof of Corollary 3. Let $\tilde{L}$ be a linear subspace generated by T. It is weakly dense in E'. Denote by L the closure of $K\tilde{L}$ in KE. If $L \neq KE$, we can find $x_0 = Ky_0 \in KE \setminus L$ and a continuous linear form $l : KE \to \mathbf{R}$ such that $l(x_0) = l(Ky_0) > 0$ and $l|_L = 0$. By Riesz theorem there exists $h \in H_K$ such that $l(Ky) = (h, Ky)_K$ for each $y \in E'$. On the other hand $(h, Ky)_K = \langle h, y \rangle$. Indeed, this equality is obvious for $h \in KE$ and follows from continuity of both sides in the general case. Thus $\langle h, y_0 \rangle > 0$ and $\langle h, y \rangle = 0$ for each $y \in \tilde{L}$, and we get a contradiction. It means that $L = KE$. Since KE is obviously strongly dense in H_K, our statement is proved.

Proof of Proposition 15. For each $f' \in E'$ and $h \in H = H_K \subset E_1$,

$$(h, Ku'f')_H = \langle h, u'f' \rangle_{E_1, E_1'} = \langle uh, f' \rangle_{E,E'} \,.$$

So, h is orthogonal to $Ku'E'$ if and only if $uh = 0$, i.e., $h \in u^{-1}\{0\}$ =Ker u. Then the orthogonal complement to $\mathcal{N}$ in H is the closure in H of the set $Ku'E'$. Then the scalar product $(h, Ku'f')_H$ is always equal to the scalar product of the corresponding images in $u(H)$:

$$\langle uh, f' \rangle_{E,E'} = (h, Ku'f')_H = (uh, uKu'f')_{u(H)}.$$

This proves that uKu' is a reproducing kernel of $u(H)$, i.e., $u(H) = H_{uKu'}$.

Proof of Proposition 35. If $H = H_K$, then $K \in \mathcal{L}^+(E)$ is the composition of linear forms: $K : E' \xrightarrow{j'} H' \xrightarrow{\theta} H \xrightarrow{j} E$, where j is the injection from H to E, j' is the injection from E' to H' and θ is the canonical isomorphism between Hilbert space H and its dual H'. Similarly with the identifications discussed in the proof of Remark 33, $\hat{K}$ is the composition of the linear forms j', θ^{-1}, and j:

$$\hat{K} : E' \xrightarrow{j'} H \xrightarrow{\theta^{-1}} H' \xrightarrow{j} E \,.$$

Now we shall discuss briefly the stochastic integral for Banach space valued martingales (see [16]). The Métivier-Pellaumail construction is based on an a priori estimate for simple integrals. It will be shown below that this estimate guarantees the existence of the factorization $Qd\lambda$. For the sake of simplicity, we consider a particular situation. Let L be a separable Banach-space with its dual L'. Let X be a square integrable L-valued martingale and $\tilde{\mathcal{P}}$ be a boolean ring generated by the sets of the form $]s,t] \times F$, $F \in \mathcal{F}_s$. $\mathcal{E}(L')$ will denote the vector space of the L'-valued and $\tilde{\mathcal{P}}$-simple processes, i.e., the processes Y such that $Y = \sum_i a_i 1_{A_i}$, where (a_i) is a finite family of L' elements and (A_i) is a finite family of $\tilde{\mathcal{P}}$ elements. If $Y \in \mathcal{E}(L')$ the definition of the stochastic integral $\int_0^t Y dX$ is obvious (see [16]). In [16], p. 20, the following assumption is made .

[i] there exists a finite positive measure α on predictable sets, vanishing on evanescent sets and such that for every L-valued $\tilde{\mathcal{P}}$-simple process Y,

$$\mathbf{E}(\int_0^1 Y_s dX_s)^2 \leq \int_{\Omega'} |Y_s|_{L'}^2 d\alpha < \infty ,$$

where $\Omega' = [0,1] \times \Omega$.

Note that this assumption is always satisfied if L is a Hilbert space.

Let $\mathcal{L}(L', L'')$ be the space of continuous linear operators from L' to its dual L'' and

$$\mathcal{L}^+(L', L'') = \{A \in \mathcal{L}(L', L'') : (Ay)y \geq 0, (Ay)z = (Az)y \ \forall y, z \in L'\} .$$

Proposition 50. *Let L' be separable and the assumption (i) be satisfied. Then there exists an increasing $\mathcal{P}(\mathbb{F})$-measurable process λ and $\mathcal{L}^+(L', L'')$-valued $\mathcal{P}(\mathbb{F})$-measurable function Q such that for each $z, y \in L'$*

$$z(X_t)y(X_t) - \int_0^t (Q_s y) z d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}) .$$

Proof. Let $\{y_1, y_2, \ldots\}$ be a countable dense subset in L'. Denote $\lambda_t^i = \langle y_i(X) \rangle_t$. Choose a sequence (c_i), $c_i > 0$ such that $E \sum_i c_i \lambda_1^i < \infty$ and define $\lambda_t = \sum_i c_i \lambda_t^i$. Then by assumption (i), we have that for each $y \in L'$,

$$d\langle y(X)\rangle_s d\mathbf{P} \ll d\lambda_s d\mathbf{P} \qquad \text{on } \mathcal{P}(\mathbb{F}).$$

Thus for each $y, z \in L'$, there exists a $\mathcal{P}(\mathbb{F})$-measurable function $C_s(y,z)$ such that

$$y(X_t)z(X_t) - \int_0^t C_s(y,z) d\lambda_s \in \mathcal{M}_{loc}(\mathbb{F}, \mathbf{P}) .$$

Obviously, for each $y, z, u \in L'$ and $a, b \in \mathbf{R}$

$C_s(y,z) = C_s(z,y), C_s(y,y) \geq 0, C_s(ay + bz, u) = aC_s(y,u) + bC_s(z,u)$
$d\lambda_s d\mathbf{P}$-a.s.

Let $\mathcal{J}$ be the vector space generated by $\{y_1, y_2, \ldots\}$. Then it is easy to find a bilinear form $\tilde{Q}_s$ on $\mathcal{J} \times \mathcal{J}$ such that $d\lambda_s d\mathbf{P}$-a.e. for each $y, z \in \mathcal{J}$, $\tilde{Q}(y,z) = \tilde{Q}_s(z,y)$, $\tilde{Q}_s(y,y) \geq 0$ and, moreover, $\tilde{Q}_s(y,z) = C_s(y,z)$.

By the Lebesgue theorem, there exists a $\mathcal{P}(\mathbb{F})$-measurable function $f \geq 0$ and a finite measure $\bar{\alpha}$ on $\mathcal{P}(\mathbb{F})$ orthogonal to $d\lambda d\mathbf{P}$ such that

$$d\alpha = f d\lambda d\mathbf{P} + d\bar{\alpha} .$$

Let $l \in \mathcal{J}$, $A \in \mathcal{P}(\mathbb{F})$, $y = l 1_A$. Then

$$\mathbf{E}(\int_0^1 l 1_A dX_s)^2 = \mathbf{E}\int_0^1 1_A \tilde{Q}(l,l) d\lambda_s \leq \mathbf{E}\int_0^1 1_A |l|_{L'}^2 f s d\lambda_s .$$

Since A is arbitrary, $\tilde{Q}_s(l,l) \leq f_s |l|_{L'}^2$ $d\lambda_s d\mathbf{P}$-a.e. Thus we can find a $d\lambda d\mathbf{P}$-modification of $\tilde{Q}_s$ such that

$$0 \leq \tilde{Q}_s(y,y) \leq f_s |y|_{L'}^2, \forall y \in \mathcal{J} ,$$

everywhere and we can extend $\tilde{Q}_s$ continuously as a bilinear form on the whole $L' \times L'$. So there exists $Q_s : L' \to L''$ such that $\tilde{Q}_s(y,z) = (Q_s y)z$ for every $y, z \in L'$ and we are done.

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