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A NOTE ON CRAMER'S THEOREM

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Let X be a locally convex vectorial space, Polish with a metric ρ. Let $(\xi_n)_{n\geq 1}$ be a sequence of X-valued i.i.d.r.v., defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the empirical means

$$L_n := S_n/n = \sum_{k=1}^{n} \xi_k / n \ , \ n\geq 1 \ .$$

The Cramer functional of ξ_1 is given by

(1) $\Lambda(y) := \log \mathbb{E} \exp \langle\xi_1, y\rangle \in (-\infty, +\infty]$, for $y\in X'$,

where X' is the topological dual space of X with the dual relation denoted by <x,y>. The Legendre transformation is defined by

(2) $\Lambda^*(x) = \sup\{ \langle x,y\rangle - \Lambda(y) \mid y\in X'\}$ *for all* $x\in X$.

The purpose of this Note is to prove

Theorem 1: *As* $n\to+\infty$, $\mathbb{P}(L_n\in \bullet)$ *satisfies the large deviation principle (in abridge: LDP) on* (X,ρ) *(i.e.,*

(i) $\exists\ I : X \to [0, +\infty]$ *such that* $[I\leq L]$ *is compact for any* $0\leq L<+\infty$ *;*

(ii) for any Borel subset A in (X,ρ),

(3) $$-\inf_{x\in A^0} I(x) \leq \lim_{n\to+\infty} \binom{\inf}{\sup} \frac{1}{n} \log \mathbb{P}(L_n\in A) \leq -\inf_{x\in \bar{A}} I(x) \ ;$$

where A^0 *and* $\bar{A}$ *are respectively the interior and the closure of A), if and only if there is a compact convex balanced subset* $\mathbf{K}$ *in X such that*

(4) $$\mathbb{E} \exp q_{\mathbf{K}}(\xi_1) < +\infty \ ,$$

where $q_{\mathbf{K}}(x) = \inf\{\lambda>0 \mid x/\lambda \in \mathbf{K}\}$ *is the Minkovski functional of* $\mathbf{K}$.

In this case, $\Lambda^*(x) = I(x)$ *over X.*

Before giving its proof, let us make some remarks.

(a) If $\dim(X)<+\infty$, the condition (4) becomes

(5) $\exists\ \delta>0$ *such that* $\mathbb{E} \exp (\delta\|\xi_1\|) < +\infty$.

The sufficiency of (5) to the LDP is the well-known (improved) Cramer theorem, contained in Azencott [Az, 1980]. The necessity of (5) is already noted in [W].

(b) If $\dim(X)=+\infty$, and X is a separable Banach space, Donsker & Varadhan [DV, 1976] proved that the condition

(6) $\forall\ \lambda>0$, $\mathbb{E} \exp (\lambda\|\xi_1\|) < +\infty$,

is sufficient to the Cramer theorem (the LDP above).

(c) de Acosta gave another proof of the Cramer theorem due to Donsker and Varadhan by showing that (6) implies (4). One of his further remark is that (5) does not imply (4) in the infinite dimensional case in the following sense: for any separable Banach space $(X,\|\cdot\|)$ with $\dim(X)=+\infty$, there is always a X-valued r.v. ξ_1 which satisfies (5), but not (4).

Hence by Theorem 1 above, (5) is not enough to the Cramer theorem, illustrating an essential difference between the finite and infinite dimensional situations.

Proof of Theorem 1. The sufficiency. That the condition (4) implies the LDP with $I=\Lambda^*$ is a direct consequence of [St, Corollary 3.27], because (4) implies the exponential tightness of $\mathbb{P}(L_n\in\bullet)$.

The necessity. If the LDP holds, by [LS, Lemma 2.6], $\mathbb{P}(L_n\in\bullet)$ is exponentially tight. In particular, there is a compact subset $\mathbf{K}'$ in (X,ρ) such that

$$(7) \qquad \limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}(L_n \notin \mathbf{K}') \le -5 .$$

Let $\mathbf{K}$ be the closed, convex, and balanced hull of $\mathbf{K}'$, which is still compact ([Sc, p50]). We have,

$$\begin{aligned}[\xi_1/n \notin 3\mathbf{K}] &\subseteq [S_n/n \notin \mathbf{K}] \cup [(\xi_2 +\cdots+ \xi_n)/n \notin 2\mathbf{K}] \\ &\subseteq [S_n/n \notin \mathbf{K}] \cup [(\xi_2 +\cdots+ \xi_n)/(n-1) \notin 2\mathbf{K}] .\end{aligned}$$

Hence by (7), $\exists$ $N\ge 0$ such that for all $n\ge N$,

$$\mathbb{P}[\xi_1/n \notin 3\mathbf{K}] \le 2e^{-4n} , \ or \ \ \mathbb{P}(q_{\mathbf{K}}(\xi_1)>3n) \le 2\, e^{-4n} .$$

This last estimation implies

$$\mathbb{E}\, \exp(q_{\mathbf{K}}(\xi_1)) \le \sum_{n=0}^{\infty} e^{3n+3}\, \mathbb{P}(3n\le q_{\mathbf{K}}(\xi_1)<3(n+1)) < +\infty ,$$

the desired condition (4). ▌

Additional notes (due to the referee):

1) That the LDP implies the exponential tightness (due to [LS, Lemma 2.6]) holds in any Polish space.

2) Instead of the Polish property of the global space X, we assume that ξ_1 takes values in a convex **Polish** subspace Z of a locally convex quasi-complete vector space X (see Stroock [St]). Theorem 1 still holds in this situation. In fact, only the necessity requires a little more attention. By the previous note 1), we can always find a compact $\mathbf{K}'\subseteq Z$ such that (7) holds. By the quasi-completeness of X, the convex balanced closed hull $\mathbf{K}$ of $\mathbf{K}'$ is compact ([Sc,p50]). The rest is the same.

In this situation, it is in further known that $[\Lambda^*<+\infty]\subseteq Z$ (see [W]).

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