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KENNETH DAVID ELWORTHY
XU-MEI LI
MARC YOR

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On the tails of the supremum and the quadratic variation of strictly local martingales

K.D. Elworthy[1], *X.M. Li*[1], *M. Yor*[2]

(1) Mathematics Department, University of Warwick, Coventry CV4 7AL, UK

(2) Laboratoire de Probabilités, Université Pierre et Marie Curie, Tour 56, 3ème Etage, 4, Place Jussieu, F - 75252 PARIS CEDEX 05

Introduction and main results.

In this paper, we study some properties of continuous strictly local martingales, i.e : local martingales which are not martingales.
Our interest for this class of local martingales stems from the fact, under some mild additional conditions on such a process $(M_t, t \geq 0)$, the tails of the distributions of $\sup_{t\geq 0} M_t$ and $\langle M\rangle_\infty^{1/2}$ are equivalent to $\frac{c}{x}$, as $x \longrightarrow \infty$, for two related constants c_1 and c_2 (depending on M). Precisely, one of our main results, which has a number of applications, is the

Theorem 1 : *Let* $(M_t, t \geq 0)$ *be a continuous local martingale taking its values in* $\mathbb{R}_+$, *and satisfying* $E[M_0] < \infty$.
Then, both :

$$\ell = \lim_{x\to\infty}\left\{x\, P\left(\sup_{t\geq 0} M_t \geq x\right)\right\} \quad \text{and} \quad \sigma = \lim_{y\to\infty}\left\{y\, P\left(\langle M\rangle_\infty^{1/2} \geq y\right)\right\}$$

exist in $\mathbb{R}_+$, *and satisfy :*

$$\ell = \sqrt{\frac{\pi}{2}}\, \sigma = E[M_0 - M_\infty]. \tag{1}$$

It is particularly easy to prove this theorem if $M_\infty = 0$, and $M_0 = c > 0$, for simplicity. In this case, using the Dubins-Schwarz representation of $(M_t, t \geq 0)$ as :

$M_t = \beta_{\langle M\rangle_t}$, where $(\beta_u, u \geq 0)$ denotes a Brownian motion starting from c, we obtain :

$$\sup_{t\geq 0} M_t = \sup_{u\leq T_o} \beta_u \quad \text{and} \quad \langle M\rangle_\infty = T_o \equiv \inf\{u : \beta_u = 0\}.$$

It is now easy to show that :

$$\sup_{t\geq 0} M_t \overset{(law)}{=} \frac{c}{U}, \text{ and } \langle M\rangle_\infty^{1/2} \overset{(law)}{=} \frac{c}{|N|},$$

where U is uniform on $[0,1]$, and N is a standard reduced gaussian r.v. The double equality *(1)* now follows easily.

In fact, in the first paragraph below, we shall prove a more general result than that of Theorem 1 ; indeed, we shall consider a general $\mathbb{R}$-valued continuous local martingale M and we shall prove the following

Theorem 1' : *Let* $(M_t, t \geq 0)$ *be a continuous local martingale, with* $M_o = 0$. *Assume that :*

(i) $\{M_V^-$; V *finite stopping time*$\}$ *is uniformly integrable.*

Then, $\{M_t,\ t \longrightarrow \infty\}$ *converges a.s. ; we denote this limit by* M_∞.

Assume furthermore that :

(ii) *there exists* $\varepsilon > 0$ *such that :* $E[\exp(\varepsilon M_\infty^-)] < \infty$.

Then both :

$$\ell = \lim_{x\to\infty}\left\{x\, P\left(\sup_{t\geq 0} M_t \geq x\right)\right\} \text{ and } \sigma = \lim_{y\to\infty}\left\{y\, P(\langle M\rangle_\infty^{1/2} \geq y)\right\} \textit{ exist in } \mathbb{R}_+,$$

and satisfy :

$$(2) \qquad \ell = \sqrt{\frac{\pi}{2}}\,\sigma = -E(M_\infty).$$

In our second paragraph, we apply Theorem 1' to transient diffusions, in particular Bessel processes, and we show how the identity *(2)* translates into some remarkable identities involving Bessel functions.

A more general discussion of strictly local martingales and their relations with strong completeness of stochastic flows is made in [9].

Acknowledgment and priority :

The proof of Theorem 1', concerning σ, uses essentially the Tauberian theorem ; after writing a first draft of this paper in May 1995, we learnt that Galtchouk - Novikov [10] already went through a similar discussion.

Ron Doney (Manchester) and J. Warren (Bath) also convinced us that the argument, if not the result, was "well-known" (to some...).

1. Proof of Theorem 1'.

(1.1) We first show that $\{M_t, t \longrightarrow \infty\}$ converges a.s. ; indeed, we remark that (i) implies, from Fatou's lemma, that : $E[L_\infty] < \infty$, where $(L_t, t \geq 0)$ is the local time at 0 of M.
But, it is well-known that the sets : $\{M_t \xrightarrow[t\to\infty]{} \cdot\}$, $\{L_\infty < \infty\}$, and $\{\langle M\rangle_\infty < \infty\}$ are all a.s. equal ; in our situation, they all have probability 1.

(1.2) We first show that ℓ exists, and satisfies :

$$(1.a) \qquad \ell = E[-M_\infty].$$

To prove this (fairly well-known result), we apply the optional stopping theorem to $\tilde{M} = (M_{t\wedge T_x} \; ; \; t \geq 0)$, for $x > 0$; from (i) , $\tilde{M}$ is uniformly integrable ; hence, we obtain :

$$0 = E[M_{T_x}] = E[M_\infty 1_{(T_x=\infty)}] + x P(T_x < \infty)$$

Consequently :

$$x P\left(\sup_{t\geq 0} M_t \geq x\right) = E\left[(-M_\infty) 1_{(T_x=\infty)}\right].$$

The right-hand side converges, as $x \longrightarrow \infty$, to : $E[-M_\infty]$, thanks to the dominated convergence theorem, since $E[|M_\infty|] < \infty$. This integrability property follows from the equality : $E[M^+_{T_x}] = E[M^-_{T_x}]$, our hypothesis (i), and Fatou's lemma.

(1.3) The proof that σ exists, and satisfies :

$$(1.b) \qquad \sqrt{\frac{\pi}{2}}\, \sigma = E(-M_\infty)$$

hinges essentially on the following variant of the Tauberian theorem.

Lemma 1 (Feller [0], XIII.5 : Tauberian theorems, Example (c)).

Let X *be an* $\mathbb{R}_+$*-valued random variable, and* $L : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a slowly* *varying function at* ∞ *; finally, let* $0 < \alpha < 1$.

The following properties are equivalent :

i) $$\frac{1}{\lambda^{\alpha}}(1-E[\exp(-\lambda X)]) \underset{\lambda\to 0}{\sim} L(\tfrac{1}{\lambda})$$

ii) $$x^{\alpha} P(X \geq x) \underset{x\to\infty}{\sim} \frac{1}{\Gamma(1-\alpha)} L(x).$$

Proof of (1.b) : We write :

$$\frac{1}{\nu} E[1 - \exp(-\frac{\nu^2}{2} \langle M\rangle_{\infty})]$$

$$(*) \qquad = \frac{1}{\nu} E[\exp(\nu(-M_{\infty}) - \frac{\nu^2}{2} \langle M\rangle_{\infty}) - \exp(-\frac{\nu^2}{2} \langle M\rangle_{\infty})]$$

$$= E\left[\exp\left(-\frac{\nu^2}{2} \langle M\rangle_{\infty}\right)\left(\frac{\exp(\nu(-M_{\infty}))-1}{\nu}\right)\right]$$

It is then easily shown that, thanks to the hypothesis (ii) in Theorem 1', the last written expectation converges towards : $E(-M_{\infty})$ (precisely, we use dominated convergence, and the elementary fact :

$$\left|\frac{1}{\nu}(\exp(\nu x) - 1)\right| \leq \begin{cases} \exp(\nu x), & \text{if } x \geq 0 ; \\ x^{-}, & \text{if } x \leq 0 ; \end{cases} \qquad)$$

Thus, we see that Lemma 1 applies with $\lambda = \frac{\nu^2}{2}$, or equivalently : $\nu = \sqrt{2\lambda}$, $X = \langle M\rangle_{\infty}$, and $L(\mu) \equiv \sqrt{2}E(-M_{\infty})$. □

To be complete, we add the following justification of (*) : we need to show that :

$$(**) \qquad E[\exp(\nu(-M_{\infty}) - \frac{\nu^2}{2} \langle M\rangle_{\infty})] = 1,$$

which also follows from the hypotheses (i) and (ii) ; indeed, they imply that :

$$\exp(\nu M_V^-) \leq E[\exp(\nu M_{\infty}^-)|\mathcal{F}_V] \text{ , for } \nu \leq \varepsilon$$

hence the uniform integrability of

$$\{\exp(\nu M_V^-) \text{ ; } V \text{ finite stopping time, } \nu \leq \varepsilon\}$$

which yields (**).

(1.4) We now make some comments about the hypotheses and the conclusion of Theorem 1' :

- first, remark that (i) and (ii) imply, using both Jensen's and Doob's L^p inequalities, that :

$$\text{for } \varepsilon' < \varepsilon, \quad E[\exp(\varepsilon' \sup_{t\geq 0} M_t^-)] < \infty$$

- consequently, ℓ is also equal to :

$$\ell^* = \lim_{x\to\infty}\left\{x\, P\left(\sup_{t\geq 0} |M_t| \geq x\right)\right\} ;$$

likewise, σ is also equal to

$$\sigma^+ \stackrel{def}{=} \lim_{y\to\infty}\{y\, P((\langle M^+\rangle_\infty)^{1/2} \geq y)\}$$

$$\left(\text{recall that : } \langle M^+\rangle_t = \int_0^t 1_{(M_s>0)} d\langle M\rangle_s\right),$$

since, from (ii), $\langle M^-\rangle_\infty$ is integrable, and in fact admits moments of all orders.

To summarize, starting from an asymmetric hypothesis about a local martingale M, the conclusion of Theorem 1' may be presented in "symmetric" terms (i.e. involving only $|M|$).

The following variant of Theorem 1' seems to have a wider domain of applicability.

Theorem 1" : *Let* $X_t = M_t + A_t$ *be an* $\mathbb{R}_+$*-valued continuous local sub-martingale such that its increasing process* $(A_t, t \geq 0)$ *satisfies :*

(+) $$E[\exp(\varepsilon A_\infty)] < \infty, \text{ for some } \varepsilon > 0.$$

Then, the following limits exist in $\mathbb{R}_+$ *:*

$$\ell = \lim_{x\to\infty}\left\{x\, P\left(\sup_{t\geq 0} X_t > x\right)\right\} \quad \text{and} \quad \sigma = \lim_{y\to\infty}\left\{y\, P(\langle X\rangle_\infty^{1/2} \geq y)\right\}$$

and satisfy :

(3) $$\ell = \sqrt{\frac{\pi}{2}}\, \sigma = E[M_0 - M_\infty].$$

The proof of Theorem 1" is quite similar to that of Theorem 1' ; hence, it is left to the reader.

As an illustration, we remark that Theorem 1" applies to :

$$X_t^{(1)} = B^+_{t\wedge\tau_\lambda}\ , \text{ and } X_t^{(2)} = |B_{t\wedge\tau_\lambda}|\ , \text{ for some } \lambda > 0,$$

where : $\tau_\lambda = \inf\{u : \ell_u > \lambda\}$, with (ℓ_u) the local time at 0 of the Brownian motion B ; note that Theorem 1' does not apply to $(B_{t\wedge\tau_\lambda}, t \geq 0)$.
For these examples, *(3)* becomes :

$$\ell^{(1)} = \sqrt{\frac{\pi}{2}}\,\sigma^{(1)} = \frac{\lambda}{2} \quad\text{and}\quad \ell^{(2)} = \sqrt{\frac{\pi}{2}}\,\sigma^{(2)} = \lambda.$$

These results may also be checked directly, since it is well-known that :

$$\sup_{t\leq\tau_1} B_t \overset{(law)}{=} 1/_{2e} \quad\text{and}\quad \sup_{t\leq\tau_1} |B_t| \overset{(law)}{=} 1/_e\ ,$$

whereas : $\displaystyle\int_0^{\tau_1} ds\, 1_{(B_s>0)} \overset{(law)}{=} \frac{1}{4}\tau_1 \overset{(law)}{=} \frac{1}{4} T_1$, with

$T_1 = \inf\{t : B_t = 1\}$, and e is an exponential variable with mean 1.
Finally, concerning possible further generalizations, it would be most interesting to know whether one can avoid the Tauberian argument, and weaken the hypothesis (+) in Theorem 1".

2. Strictly local martingales, transient diffusions and some remarkable identities.

(2.1) Our examples will take place in the framework considered by Pitman-Yor ([3], [4] ; (1981, 1982)) and Le Gall ([1] ; (1986)) of a regular diffusion $(R_t\ , 0 \leq t < \zeta, P_r\ , 0 < r < \infty)$ on the interval $]0,\infty[$ of $\mathbb{R}$; let $s(\cdot)$ denote a scale function for the diffusion, and m the speed measure normalized so that the infinitesimal generator is $\frac{1}{2}\frac{d}{dm}\frac{d}{ds}$. We assume :

(i) $\quad \zeta = \inf\{t > 0 : X_{t-} = 0 \text{ or } \infty\}$

(ii) $\quad s(0) = -\infty\ , \ s(\infty) < \infty$

(iii) $\quad 0$ is an entrance point for the diffusion R.

In the sequel, we shall always take $s(\infty) = 0$.

We still need to introduce, for $0 < \rho < \infty$, the last passage time $L_\rho = \sup\{t \geq 0 : R_t = \rho\}$, and the expression of the semi-group

$$Q_t(r,d\rho) = p_t^*(r,\rho)m(d\rho).$$

Then, we have the following

Theorem 2 : *1. For* $0 \le r < \rho < \infty$,

(4) $$P_r(L_\rho \in dt) = -\frac{1}{2s(\rho)}\, p_t^*(r,\rho)dt$$

2. For every $r > 0$, *and* $t \ge 0$, *the limit*

(5) $$\lim_{\alpha\to\infty}\left\{\alpha\, P_r\left(\sup_{u\le t}\left(-\, s(R_u)\right) \ge \alpha\right)\right\}$$

exists, and is equal to :

(6) $$E_r\left[s(R_t) - s(r)\right] = \frac{1}{2}\int_0^t du\; p_u^*(0,r).$$

In particular, $(s(R_t),t \ge 0)$ *is a strictly local martingale.*

Proof : a) For the first statement, see Pitman-Yor (1981).

b) For the second statement, the existence of the limit and its equality to the left-hand side of *(6)* follows from *(1)* in Theorem 1, whereas

Le Gall ([1] (1986) ; Theorem 1.1, p. 1222)) expresses the limit *(5)* as the right- hand side of *(6)*. □

(2.2) The most standard example of a diffusion which satisfies the above hypothesis is the Bessel process with dimension $d = 2(1+\nu) > 2$, i.e : $\nu > 0$. We then have : $s(\rho) = -\dfrac{1}{\rho^{d-2}}$,

and the identity *(4)*, taken for $r = 0$, becomes :

(7) $$P_o(L_\rho \in dt) = \left(\frac{\rho^2}{2}\right)^\nu \frac{dt}{\Gamma(\nu)t^{\nu+1}} \exp\left(-\frac{\rho^2}{2t}\right).$$

Now, the identity *(6)* may be written as :

(6') $$E_r\left[\frac{s(R_t)}{s(r)}\right] = 1 - \left(\frac{-1}{2s(r)}\int_0^t du\; p_u^*(0,r)\right) = P_o(L_r \ge t)$$

which, as a consequence of *(7)*, becomes :

$$(8)\qquad \int_0^\infty \frac{d\rho}{\rho^{2\nu}}\, p_t(r,\rho) = \frac{1}{2^\nu \Gamma(\nu)} \int_t^\infty \frac{du}{u^{1+\nu}} \exp(-\frac{r^2}{2u}),$$

where
$$p_t(r,\rho) = \frac{1}{t}\, \rho(\frac{\rho}{r})^\nu \exp - \left(\frac{r^2+\rho^2}{2t}\right) I_\nu\left(\frac{r\rho}{t}\right)$$

is the density of the semi-group $Q_t(r,d\rho)$ with respect to $d\rho$.
Easy changes of variables then show that *(8)* is equivalent to

$$(8')\qquad \int_0^\infty \frac{\xi d\xi}{\xi^\nu a^{\nu-1}} \exp\left(-\frac{a^2}{2}(1+\xi^2)\right) I_\nu(a\xi) = \frac{1}{2^\nu\Gamma(\nu)} \int_0^1 \frac{du}{u}\, u^\nu \exp\left(-\frac{a^2 u}{2}\right)$$

and also to :

$$(8'')\qquad \int_0^\infty \eta\, d\eta \exp\left(-\frac{\eta^2}{2}\right) \frac{I_\nu(\sqrt{a}\eta)}{(\sqrt{a}\ \eta)^\nu} = \frac{1}{2^\nu\Gamma(\nu)} \int_0^1 dv(1-v)^{\nu-1} \exp\left(\frac{a}{2}\, v\right).$$

This identity *(8″)* may be verified by developing both sides as a series expansion in powers of a with the help, for the left-hand side, of the classical formula :

$$(9)\qquad I_\nu(z) = (\frac{z}{2})^\nu \sum_{n=0}^\infty \frac{1}{n!\Gamma(\nu+n+1)} (\frac{z}{2})^{2n} .$$

In any case, the identity *(8″)* is a particular case of the Lipschitz-Hankel integrals ; see, e.g., chap. XIII of Watson [6], formula 3, p. 394, which gives a formula for :

$$\int_0^\infty dt\ t^{\mu-1}\, e^{-p^2t^2}\, J_\nu(at) ,$$

with the help of the ${}_1F_1$ hypergeometric functions ; such formulae are also found in Lebedev ([2], p. 278, Exercise 12).

For clarity and future reference, we write again the equalities *(2)* and *(6)* in the particular case where $M_t = \frac{1}{R_t^{d-2}}$, under $P_r^{(\nu)}$, the law of R, starting from $r > 0$.

Proposition 1 : *The 4 following quantities are equal :*

$$\lim_{\alpha\to\infty}\left\{\alpha P_r^{(\nu)}\left(\frac{1}{(\inf_{s\le t} R_s^{d-2})} \ge \alpha\right)\right\}$$

$$\lim_{y\to\infty}\left\{\sqrt{\frac{\pi}{2}}\, y\, P_r^{(\nu)}\left((d-2)\left(\int_0^t \frac{ds}{R_s^{2(d-1)}}\right) \ge y\right)\right\}$$

$$E_r^{(\nu)}\left(\frac{1}{r^{d-2}} - \frac{1}{R_t^{d-2}}\right) = \frac{1}{2^{\nu}\Gamma(\nu)}\int_0^t \frac{du}{u^{1+\nu}}\exp\left(-\frac{r^2}{2u}\right). \tag{11}$$

(2.3) Associated with the 2-dimensional Bessel process $(R_t, t \ge 0)$, starting from $r > 0$, there is also the strictly local martingale $M_t = \log \frac{1}{R_t}$, which satisfies the hypothesis of our Theorem 1'.

In order to obtain the corresponding values of the quantities in *(1')* in the present case, it suffices to divide both sides of *(11)* by (2ν), and to let $\nu \to 0$.

Thus, we obtain the following

Proposition 2 : *Let* $P_r^{(o)}$ *be the law of* $(R_t, t \ge 0)$, *the* 2*-dimensional Bessel process starting from* $r > 0$. *Then, the following* 4 *quantities are equal :*

$$\lim_{\alpha\to\infty}\left\{\alpha\, P_r^{(o)}\left(\log\frac{1}{(\inf_{s\le t} R_s)} \ge \alpha\right)\right\}\ ;\ \lim_{y\to\infty}\left\{\sqrt{\frac{\pi}{2}}\, y\, P_r^{(o)}\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \ge y\right)\right\}$$

$$E_r^{(o)}\left[\log R_t - \log r\right] = \frac{1}{2}\int_0^t \frac{du}{u}\, e^{-\frac{r^2}{2u}}. \tag{12}$$

We again remark that the identity *(12)*, may be expressed as an integral identity involving the Bessel function I_o , i.e. see *(2.4)* below.

On the other hand, if we particularize our argument in the proof of Theorem 1' concerning the quadratic variation of M, we obtain, in the present case :

$$\frac{1}{\nu}\, E_r^{(o)}\left\{1 - \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right)\right\} \xrightarrow[\nu\to 0]{} E_r^{(o)}[\log R_t - \log r] \tag{13}$$

But, the following formula is known (see, e.g., Yor [8]) :

$$E_r^{(o)}\left\{\exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right) \Big| R_t\right\} = \frac{I_\nu}{I_o}\left(\frac{rR_t}{t}\right). \tag{14}$$

Hence, it is deduced from *(13)* that :

(15) $$E_r^{(o)}\left\{-\frac{\partial}{\partial\nu}\Big|_{\nu=0}\left(\frac{I_\nu}{I_o}\right)\left(\frac{rR_t}{t}\right)\right\} = E_r^{(o)}\Big[\log R_t - \log r\Big].$$

It now follows from a classical integral representation of $I_\nu(\xi)$ that :

$$-\frac{\partial}{\partial\lambda}\Big|_{\lambda=0} I_\lambda(\xi) = \int_0^\infty du\ e^{-\xi(\cosh u)} \equiv K_o(\xi)\ ^{(1)}.$$

Hence, we deduce from *(15)* and the explicit formula for the semi-group $Q_t^{(o)}(r,d\rho)$ that :

(16) $$\frac{1}{t}\int_0^\infty d\rho\ \rho\ e^{-\frac{r^2+\rho^2}{2t}}\int_0^\infty du\ e^{-\frac{r\rho}{t}(\cosh u)} = \frac{1}{2}\int_0^t \frac{du}{u}\ e^{-\frac{r^2}{2u}}.$$

(2.4) <u>Some remarks following Proposition 2, and formulae *(13)* through *(16)*.</u>

i) We can write formula *(12)* in the form :

(12') $$\int_0^\infty \frac{d\rho\ \rho}{t}\exp\left(-\frac{r^2+\rho^2}{2t}\right) I_o\left(\frac{r\rho}{t}\right)(\log\rho - \log r) = \frac{1}{2}\int_0^t \frac{du}{u}\ e^{-(r^2/2u)}$$

and, as above with *(11)* and *(12)* may deduce this identity *(12')* from the corresponding identity involving I_ν, and deduced from *(11)*.

ii) From *(14)*, we can obtain a result similar to, but deeper than, *(13)*, namely :

$$\lim_{\nu\to 0}\frac{1}{\nu}E_r^{(o)}\left[1-\exp\left(-\frac{\nu^2}{2}\int_0^t\frac{ds}{R_s^2}\right)\Big|R_t=\rho\right] = -\frac{\partial}{\partial\nu}\Big|_{\nu=0+}\frac{I_\nu(\xi)}{I_o(\xi)} = \frac{K_o(\xi)}{I_o(\xi)}$$

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(1) In fact, the equality : $K_o(\xi) = -\frac{\partial}{\partial\lambda}\Big|_{\lambda=0}(I_\lambda(\xi))$ also follows immediately from the formula : $K_\lambda(\xi) = \frac{\pi}{2}\frac{I_{-\lambda}(\xi)-I_\lambda(\xi)}{\sin(\pi\lambda)}$ (see, e.g., formula (5.7.2), p. 108 in Lebedev [2]). Also, the formula : $K_o(\xi) = \int_0^\infty du\ e^{-\xi(\cosh u)}$ is a particular case of : $K_\nu(\xi) = \int_0^\infty du\ e^{-\xi(\cosh u)}\cosh(\nu u)$.

where $\xi = \frac{r\rho}{t}$, on one hand, and, on the other hand, thanks to the Tauberian theorem, we find that the same quantity is equal to :

$$\sqrt{\frac{\pi}{2}}\lim_{y\to\infty}\left\{y\, P_r^o\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \geq y \,\Big|\, R_t = \rho\right)\right\}$$

(2.5) An important part of the results presented in Propositions 1 and 2 is well-known ; in fact, some of these results, namely those concerning

$$\lim_{\alpha\to\infty}\left\{\alpha\, P\left(\sup_{s\leq T} M_s \geq \alpha\right)\right\}$$

in their applications to Bessel processes, form the core of the arguments of the proof of the main asymptotics of the Wiener sausage, i.e : Le Gall [1], Theorem 1.1, and, in part, Spitzer [5].

The results about the asymptotics of $\sqrt{\frac{\pi}{2}}\, y\, P(\langle M\rangle_T^{1/2} \geq y)$ are perhaps less known, although they also appear in Werner [7].

An interesting consequence of Proposition 2 is the following

<u>Corollary</u> : *Let* $(\theta_u, u \geq 0)$ *be a continuous determination of the argument around* 0 *of the* 2-*dimensional Brownian motion* $(Z_u, u \geq 0)$, *starting from* $z_o \neq 0$. *Then, we have :*

$$\lim_{\alpha\to\infty}\{\alpha P(|\theta_t| \geq \alpha)\} = \lim_{\alpha\to\infty}\left\{\alpha\, P\left(\sup_{s\leq t}\theta_s \geq \alpha\right)\right\} = \frac{1}{\pi}\int_0^t \frac{du}{u}\exp\left(-\frac{r^2}{2u}\right)$$

$$\lim_{\alpha\to\infty}\left\{\alpha\, P\left(\sup_{s\leq t}|\theta_s| \geq \alpha\right)\right\} = \frac{1}{2}\int_0^t \frac{du}{u}\exp\left(-\frac{r^2}{2u}\right).$$

<u>Proof</u> : Thanks to the skew-product representation of Z, there exists a 1-dimensional Brownian motion $(\gamma_t, t \geq 0)$, independent of $(R_u, u \geq 0)$ such that :

$$\theta_t = \gamma\left(\int_0^t \frac{ds}{R_s^2}\right).$$

Hence, we have : $|\theta_t| \overset{(law)}{=} \sup_{s\leq t}\theta_s \overset{(law)}{=} \left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2}\left(\sup_{u\leq 1}\gamma_u\right) \overset{(law)}{=} \left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2}|\gamma_1|$

thanks to the reflection principle. Thus, we have :

$$\alpha P(|\theta_t| \geq \alpha) = \alpha P\left(\sup_{s\leq t} \theta_s \geq \alpha\right) = \alpha P\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \geq \frac{\alpha}{|\gamma_1|}\right),$$

so that, by dominated convergence, we find :

$$\lim_{\alpha\to\infty} \alpha P\left(\sup_{s\leq t} \theta_s \geq \alpha\right) = E(|\gamma_1|) \lim_{\beta\to\infty} \beta P\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \geq \beta\right)$$

and, since : $E(|\gamma_1|) = \sqrt{\frac{2}{\pi}}$, we obtain, from Proposition 2, that :

$$\lim_{\alpha\to\infty} \alpha P\left(\sup_{s\leq t} \theta_s \geq \alpha\right) = \frac{2}{\pi}\left(\frac{1}{2}\int_0^t \frac{du}{u}\, e^{-(r^2/2u)}\right).$$

Likewise, we obtain :

$$\lim_{\alpha\to\infty}\left\{\alpha P\left(\sup_{s\leq t} |\theta_s| \geq \alpha\right)\right\} = E[\gamma_1^*] \lim_{\beta\to\infty}\left\{\beta P\left(\left(\int_0^t \frac{ds}{R_s^2}\right)^{1/2} \geq \beta\right)\right\}$$

$$= \left(E[\gamma_1^*]\sqrt{\frac{2}{\pi}}\right)\left(\frac{1}{2}\int_0^t \frac{du}{u}\, e^{-r^2/2u}\right)$$

and the desired result follows from the next

Lemma 2 : *Define* $\gamma_1^* = \sup_{s\leq 1} |\gamma_s|$. *Then, one has :* $E[\gamma_1^*] = \sqrt{\frac{\pi}{2}}$.

Proof : Define $\hat{T}_1 = \inf\{t : |\gamma_t| = 1\}$. Then, from the scaling property of Brownian motion, we deduce : $\gamma_1^* \overset{(law)}{=} 1/(\hat{T}_1)^{1/2}$, so that :

$$E[\gamma_1^*] = E\left[\frac{1}{(\hat{T}_1)^{1/2}}\right] = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty du\, u^{-1/2}\, E\left[e^{-u\hat{T}_1}\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty dv\, E\left[e^{-\frac{v^2}{2}\hat{T}_1}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{dv}{(\operatorname{ch} v)}$$

Now, we have : $\displaystyle\int_0^\infty \frac{dv}{(\operatorname{ch} v)} = 2\int_0^\infty e^{-v}\frac{dv}{(1+e^{-2v})} = 2\int_0^1 \frac{dx}{1+x^2} = 2\,\mathrm{Arctg}(1) = \frac{\pi}{2}$, so that, finally : $E[\gamma_1^*] = \sqrt{\frac{\pi}{2}}$. □

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