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HOW LONG DOES IT TAKE A TRANSIENT BESSEL PROCESS TO REACH ITS FUTURE INFIMUM?

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Summary. We establish an iterated logarithm law for the location of the future infimum of a transient Bessel process.

1. Introduction.

Let $\{ R(t);\ t \geq 0 \}$ be a d-dimensional Bessel process, and let

$$\nu = \frac{d}{2} - 1, \tag{1.1}$$

be the "index" of R (see Revuz & Yor [R-Y] Chap. XI). When d is an integer, R can be realized as the radial part of an $\mathbb{R}^d$-valued Brownian motion. We refer to [R-Y] (Chap. XI) for a detailed account of general properties of Bessel processes. It is known ([R-Y] p.423) that R is transient (i.e. $\lim_{t\to\infty} R(t) = \infty$ almost surely) if and only if $d > 2$. Unless stated otherwise, this condition will be taken for granted throughout the note.

Define for $t > 0$,

$$\xi(t) = \inf\{ u \geq t :\ R(u) = \inf_{s \geq t} R(s) \}.$$

In words, for any given $t > 0$, $\xi(t)$ denotes the (almost surely unique) location of the infimum of R over $[t,\infty)$. Such random times have been first studied by Williams ([W1] and [W2]), who proved a path decomposition theorem at $\xi(t)$ respectively in case of Brownian motion and linear diffusions. Generalizations of Williams' result have since been established for Lévy and more general Markov processes. See for example Millar [M], Pitman [P], Bertoin [B] and Chaumont [C], and the references therein.

This note is concerned with $\xi(t)$ as a *process* of t, and more particularly, we are interested in the path property of $t \mapsto \xi(t)$. Of course, it is meaningless to study its liminf behaviour, since there are infinitely many large t's such that $\xi(t) = t$. Instead, we ask: what can be said about the limsup behaviour of $\xi(t)$?

Theorem 1. *For any non-decreasing function $f > 0$, we have*

$$\limsup_{t\to\infty} \frac{\xi(t)}{t\, f(t)} = 0 \quad \textit{or} \quad \infty, \quad \text{a.s.},$$

according as

$$\int^{\infty} \frac{dt}{t\, f^{\nu}(t)}$$

converges or diverges, where ν is defined in (1.1).

Remark. In case $R(0) = 0$, there is also a "local" version of Theorem 1 for small times t.

Theorem 1 is proved in Section 2. Some related problems are raised in Section 3.

2. Proof of Theorem 1.

Without loss of generality, we assume $R(0) = 0$. Throughout the note, $\{X(t);\ t \ge 0\}$ stands for a generic d-dimensional Bessel process starting from 1, independent of R, and we denote by V the (almost surely) unique time when X reaches the infimum over $(0,\infty)$. Observe that R and X almost have the same law, except that $R(0) = 0$ whereas $X(0) = 1$. The process X being a linear diffusion with scale function $-x^{-2\nu}$ (Revuz & Yor [R-Y] p.426), we obviously have

$$\mathbb{P}\Big(\inf_{u\ge 0} X(u) < x\Big) = x^{2\nu}, \quad 0 < x < 1. \tag{2.1}$$

In order to prove Theorem 1, some preliminary results are needed. In the sequel, $K > 1$, $K_1 > 1$ and $K_2 > 1$ denote unimportant finite constants. Their values, which may change from line to line, depend only on d.

Lemma 1. *For any $t \geq 1$, we have*

$$K^{-1}t^{-\nu} \leq \mathbb{P}\Big(V > t \Big) \leq Kt^{-\nu}, \tag{2.2}$$

where ν is defined in (1.1).

Proof of Lemma 1. We have

$$\begin{aligned}\mathbb{P}\Big(V > t \Big) &= \mathbb{P}\Big(\inf_{s \geq t} X(s) < \inf_{0 \leq u \leq t} X(u) \Big) \\ &= \mathbb{E}\Big[\mathbb{P}\Big(\inf_{s \geq t} X(s) < \inf_{0 \leq u \leq t} X(u) \,\Big|\, X(u);\, 0 \leq u \leq t \Big) \Big].\end{aligned}$$

Given the value of $X(t)$, the post-t process $\{ X(s+t);\, s \geq 0 \}$ is a d-dimensional Bessel process starting from $X(t)$, independent of $\{ X(u);\, 0 \leq u \leq t \}$. Thus by scaling and (2.1), we obtain

$$\mathbb{P}\Big(\inf_{s \geq t} X(s) < \inf_{0 \leq u \leq t} X(u) \,\Big|\, X(u);\, 0 \leq u \leq t \Big) = \Big(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u) \Big)^{2\nu}.$$

Consequently,

$$\mathbb{P}\Big(V > t \Big) = \mathbb{E}\Big[\Big(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u) \Big)^{2\nu} \Big]. \tag{2.3}$$

Since $\inf_{0 \leq u \leq t} X(u) \leq 1$, we have

$$\mathbb{E}\Big[\Big(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u) \Big)^{2\nu} \Big] \leq \mathbb{E}\Big(X^{-2\nu}(t) \Big).$$

Applying a diffusion comparison theorem ([R-Y] Theorem IX.3.7) to square Bessel processes, it is seen that $X(t)$ is stochastically bigger than $R(t)$ (which is intuitively obvious, of course). Thus by scaling, this implies

$$\mathbb{P}\Big(V > t \Big) \leq \mathbb{E}\Big(R^{-2\nu}(t) \Big) = t^{-\nu}\mathbb{E}\Big(R^{-2\nu}(1) \Big),$$

which yields the upper bound in Lemma 1, since $\mathbb{E}\big(R^{-2\nu}(1)\big) < \infty$. To show the lower bound, observe that by (2.3), for any $\lambda > 0$,

$$\begin{aligned}\mathbb{P}\Big(V > t \Big) &\geq \mathbb{E}\Big[\Big(\frac{1}{X(t)} \inf_{0 \leq u \leq t} X(u) \Big)^{2\nu} \mathbb{1}_{\{\inf_{u \geq 0} X(u) > 1/2;\, X(t) < \lambda\sqrt{t}\}} \Big] \\ &\geq (2\lambda)^{-2\nu} t^{-\nu} \mathbb{P}\Big(\inf_{u \geq 0} X(u) > 1/2;\, X(t) < \lambda\sqrt{t} \Big) \\ &\geq (2\lambda)^{-2\nu} t^{-\nu} \Big(\mathbb{P}\big(\inf_{u \geq 0} X(u) > 1/2\big) - \mathbb{P}\big(X(t) > \lambda\sqrt{t}\big) \Big).\end{aligned}$$

Since $\mathbb{P}\big(X(t) > \lambda\sqrt{t}\big) \le \mathbb{P}\big(R(t) > \lambda\sqrt{t}-1\big) = \mathbb{P}\big(X(1) > \lambda - 1/\sqrt{t}\big)$, we can choose λ so large that this probability is smaller than $\frac{1}{2}\mathbb{P}\big(\inf_{u\ge 0} X(u) > 1/2\big)$. The lower bound in Lemma 1 is proved. $\square$

Lemma 1 will be used to obtain accurate estimates of the law of some functionals of ξ. Define for $r > 0$

$$\sigma(r) = \inf\{t > 0 : R(t) = r\},$$

the first hitting time of R at level r, which is (almost surely) finite. Since $R(0) = 0$, the scaling property immediately yields that for any given $r > 0$, $\sigma(r)$ has the same law as $r^2\sigma(1)$. For notational convenience, we write in the sequel

$$\sigma \equiv \sigma(1);$$
$$\xi_\sigma \equiv \xi(\sigma(1)).$$

The random variables σ and ξ_σ play an important rôle in our proof of Theorem 1. Here we give a résumé of their basic properties. The equivalence for the lower tail of σ is known. Recall that ([G-S]) $\lim_{s\to 0} s^\nu e^{1/(2s)}\mathbb{P}(\sigma < s) = 2^{1-\nu}/\Gamma(1+\nu)$. Therefore,

$$K^{-1}s^{-\nu}\exp\big(-\frac{1}{2s}\big) \le \mathbb{P}\Big(\sigma < s\Big) \le Ks^{-\nu}\exp\big(-\frac{1}{2s}\big), \quad 0 < s \le 1. \tag{2.4}$$

The exact upper tail of σ, which involves Bessel functions and their positive zeros, was evaluated respectively by Ciesielski & Taylor [C-T] for integer dimensions d, and by Kent [Ke] and Ismail & Kelker [I-K] for any $d > 0$. Their result implies the following useful estimate for $x \ge 1$:

$$\mathbb{P}\Big(\sigma > x\Big) \le \exp\big(-\frac{x}{K}\big). \tag{2.5}$$

For the variable ξ_σ, it follows from the strong Markov property of R that $\{R(\sigma + t);\, t \ge 0\}$ is a d-dimensional Bessel process starting from 1 (thus behaving like the process X), independent of σ. Since V is the location of the infimum of X over $(0,\infty)$, this yields:

$$\xi_\sigma - \sigma \ \text{ is independent of } \sigma; \tag{2.6}$$

$$\xi_\sigma - \sigma \ \stackrel{(d)}{=}\ V; \tag{2.7}$$

("$\stackrel{(d)}{=}$" denoting identity in distribution). Our next preliminary result is on the joint tail of ξ_σ and σ.

Lemma 2. *Let $x \geq 2$ and $y \geq 1$. Then*

$$\mathbb{P}\Big(\frac{\xi_\sigma}{\sigma} > x\Big) \leq K x^{-\nu}, \tag{2.8}$$

$$\mathbb{P}\Big(\frac{\xi_\sigma}{\sigma} > x;\, y > \sigma > 1\Big) \geq K^{-1} x^{-\nu} - e^{-y/K}. \tag{2.9}$$

Proof of Lemma 2. According to our notation, V is independent of R (thus of σ). We have, by (2.6) and (2.7),

$$\begin{aligned}\mathbb{P}\Big(\frac{\xi_\sigma}{\sigma} > x\Big) &= \mathbb{P}\Big(V > (x-1)\sigma\Big)\\ &= \mathbb{P}\Big(V > (x-1)\sigma;\, \sigma \geq \frac{1}{x-1}\Big) + \mathbb{P}\Big(V > (x-1)\sigma;\, \sigma < \frac{1}{x-1}\Big).\end{aligned}$$

Using (2.2) and (2.4), the above expression is

$$\begin{aligned}&\leq K_1 (x-1)^{-\nu} \mathbb{E}\Big(\sigma^{-\nu} \mathbb{1}_{\{\sigma \geq 1/(x-1)\}}\Big) + \mathbb{P}\Big(\sigma < \frac{1}{x-1}\Big)\\ &\leq K_1 (x-1)^{-\nu} \mathbb{E}(\sigma^{-\nu}) + K_1 (x-1)^{-\nu} \exp(-\frac{x-1}{2})\\ &\leq K x^{-\nu},\end{aligned}$$

the last inequality due to the fact that $\mathbb{E}(\sigma^{-\nu}) < \infty$ (this is easily seen from (2.4)). Therefore (2.8) is proved. To show (2.9), observe that by (2.6), (2.7), (2.2) and (2.5), we have

$$\begin{aligned}\mathbb{P}\Big(\frac{\xi_\sigma}{\sigma} > x;\, y > \sigma > 1\Big) &\geq \mathbb{P}\Big(\frac{\xi_\sigma}{\sigma} > x;\, \sigma > 1\Big) - \mathbb{P}\Big(\sigma \geq y\Big)\\ &= \mathbb{P}\Big(V > (x-1)\sigma;\, \sigma > 1\Big) - \mathbb{P}\Big(\sigma \geq y\Big)\\ &\geq K_1 (x-1)^{-\nu} \mathbb{E}\Big(\sigma^{-\nu} \mathbb{1}_{\{\sigma > 1\}}\Big) - e^{-y/K}\\ &\geq K^{-1} x^{-\nu} - e^{-y/K}.\end{aligned}$$

Lemma 2 is proved. $\square$

Lemma 3. *For any $x \geq 2$, we have*

$$\mathbb{P}\Big(\xi(1) > x\Big) \leq K x^{-\nu}.$$

Proof of Lemma 3. Conditioning on $R(1) = x$, $\xi(1) - 1$ has the same distribution as $x^2 V$ (this is easily seen from the Markov and scaling properties of R). Thus by Lemma 1,

$$\begin{aligned}
\mathbb{P}\Big(\xi(1) > x\Big) &= \mathbb{P}\Big(R^2(1)V > x - 1\Big) \\
&\le \mathbb{P}\Big(R(1) > \sqrt{x-1}\Big) + \mathbb{P}\Big(R^2(1)V > x-1;\ R(1) \le \sqrt{x-1}\Big) \\
&\le \mathbb{P}\Big(\sup_{0\le t\le 1} R(t) > \sqrt{x-1}\Big) \\
&\qquad\qquad + K_1(x-1)^{-\nu}\mathbb{E}\Big(R^{-2\nu}(1)\mathbb{1}_{\{R(1)\le\sqrt{x-1}\}}\Big).
\end{aligned}$$

Since $\mathbb{E}\big(R^{-2\nu}(1)\big) < \infty$, the proof of Lemma 3 is reduced to showing the following estimate:

$$\mathbb{P}\Big(\sup_{0\le t\le 1} R(t) > \sqrt{x-1}\Big) \le K_2 x^{-\nu}. \tag{2.10}$$

This is easily verified. Indeed, by scaling, we have, for any $\lambda > 0$,

$$\mathbb{P}\Big(\sup_{0\le t\le 1} R(t) > \lambda\Big) = \mathbb{P}\Big(\sup_{0\le t\le 1/\lambda^2} R(t) > 1\Big) = \mathbb{P}\Big(\sigma < \frac{1}{\lambda^2}\Big).$$

Taking $\lambda = \sqrt{x-1}$ and using (2.4), we obtain

$$\begin{aligned}
\mathbb{P}\Big(\sup_{0\le t\le 1} R(t) > \sqrt{x-1}\Big) &= \mathbb{P}\Big(\sigma < \frac{1}{x-1}\Big) \\
&\le K(x-1)^{\nu}\exp\Big(-\frac{x-1}{2}\Big) \\
&\le K_2 x^{-\nu},
\end{aligned}$$

which yields (2.10). □

Proof of Theorem 1. We begin with the convergent part. Let $f > 0$ be non-decreasing such that $\int^{\infty} dt/t\, f^{\nu}(t) < \infty$. Thus f increases to infinity. Choose a large initial value n_0 and define $t_n = e^n$ for $n \ge n_0$. By scaling and Lemma 3, we have

$$\begin{aligned}
\mathbb{P}\Big(\xi(t_{n+1})) > t_n f(t_n)\Big) &= \mathbb{P}\Big(\xi(1) > \frac{t_n}{t_{n+1}} f(t_n)\Big) \\
&= \mathbb{P}\Big(\xi(1) > \frac{1}{e} f(t_n)\Big) \\
&\le K f^{-\nu}(t_n).
\end{aligned}$$

Since

$$\sum_{n=n_0+1}^{\infty} f^{-\nu}(t_n) = \sum_{n=n_0+1}^{\infty} \int_{t_{n-1}}^{t_n} \frac{dt}{t_n - t_{n-1}} f^{-\nu}(t_n) \le \frac{e}{e-1} \sum_{n=n_0+1}^{\infty} \int_{t_{n-1}}^{t_n} \frac{dt}{t} f^{-\nu}(t)$$
$$= \frac{e}{e-1} \int_{t_{n_0}}^{\infty} \frac{dt}{t} f^{-\nu}(t) < \infty,$$

the Borel–Cantelli lemma tells us that

$$\limsup_{n\to\infty} \frac{\xi(t_{n+1})}{t_n f(t_n)} \le 1 \quad \text{a.s.}$$

Since replacing f by a multiple of f does not change the test, an argument by monotonicity readily yields $\limsup_{t\to\infty} \xi(t)/t\, f(t) = 0$ almost surely. To verify the divergent part of Theorem 1, pick an f such that $\int^{\infty} dt/t\, f^{\nu}(t)$ diverges. Obviously we only have to treat the case that $f(\infty) = \infty$. Choose a large k_0, and define $t_k = e^k$ as before (it will be seen that t_k is rather a space variable than a time variable, but the notation should not cause any trouble). We shall consider a sequence of *random* times in order to avoid dependence difficulty. Let

$$E_k = \Big\{ \xi(\sigma(t_k)) > \sigma(t_k) f(t_k^3);\ t_k^3 > \sigma(t_k) > t_k^2 \Big\},$$

for $k \ge k_0$. By scaling and (2.9), we have

$$\mathbb{P}(E_k) = \mathbb{P}\Big(\frac{\xi_\sigma}{\sigma} > f(t_k^3);\ t_k > \sigma > 1 \Big) \ge K^{-1} f^{-\nu}(t_k^3) - \exp(-t_k/K).$$

Accordingly,

$$f^{-\nu}(t_k^3) \le K\mathbb{P}(E_k) + K e^{-t_k/K}. \tag{2.11}$$

Since $\sum_{k=k_0}^{\infty} f^{-\nu}(t_k^3) = \infty$ and $\sum_k e^{-t_k/K} < \infty$, the above estimate clearly implies

$$\sum_k \mathbb{P}(E_k) = \infty. \tag{2.12}$$

To apply the Borel-Cantelli lemma, we need to check that the measurable events E_k are almost independent. Let $k_0 \le k < \ell$. Denote by $\xi(s,t)$ the time when R reaches its minimum over (s,t) (thus $\xi(t) = \xi(t,\infty)$ according to our notation). Write

$$\mathbb{P}(E_k E_\ell) = \mathbb{P}\Big(E_k;\ E_\ell;\ \xi(\sigma(t_k)) < \sigma(t_\ell) \Big) + \mathbb{P}\Big(E_k;\ E_\ell;\ \xi(\sigma(t_k)) \ge \sigma(t_\ell) \Big)$$
$$\equiv \Delta_1 + \Delta_2,$$

with obvious notation. Then

$$\begin{aligned}\Delta_1 &\le \mathbb{P}\Big(\, \xi(\sigma(t_k),\sigma(t_\ell)) > \sigma(t_k)f(t_k^3);\ \sigma(t_k) > t_k^2;\\ &\qquad\qquad \xi(\sigma(t_\ell)) - \sigma(t_\ell) > \sigma(t_\ell)\big(f(t_\ell^3)-1\big);\ \sigma(t_\ell) > t_\ell^2 \Big)\\ &\le \mathbb{P}\Big(\, \xi(\sigma(t_k),\sigma(t_\ell)) - \sigma(t_k) > t_k^2\big(f(t_k^3)-1\big);\\ &\qquad\qquad \xi(\sigma(t_\ell)) - \sigma(t_\ell) > t_\ell^2\big(f(t_\ell^3)-1\big)\Big).\end{aligned}$$

Using the strong Markov property of R, it is seen that $\xi(\sigma(t_k),\sigma(t_\ell))$ is independent of $\xi(\sigma(t_\ell)) - \sigma(t_\ell)$. Thus by scaling and (2.7),

$$\begin{aligned}\Delta_1 &\le \mathbb{P}\Big(\, \xi(\sigma(t_k),\sigma(t_\ell)) - \sigma(t_k) > t_k^2\big(f(t_k^3)-1\big)\Big)\,\mathbb{P}\Big(\, \xi_\sigma - \sigma > f(t_\ell^3) - 1\Big)\\ &\le \mathbb{P}\Big(\, \xi(\sigma(t_k)) - \sigma(t_k) > t_k^2\big(f(t_k^3)-1\big)\Big)\,\mathbb{P}\Big(V > f(t_\ell^3)-1\Big)\\ &= \mathbb{P}\Big(V > f(t_k^3)-1\Big)\,\mathbb{P}\Big(V > f(t_\ell^3)-1\Big)\\ &\le K_1 f^{-\nu}(t_k^3) f^{-\nu}(t_\ell^3), \qquad (2.13)\end{aligned}$$

the last inequality following from Lemma 1. Now let us evaluate Δ_2. Clearly we have

$$\begin{aligned}\Delta_2 &\le \mathbb{P}\Big(\, \xi(\sigma(t_k)) > \sigma(t_\ell)f(t_\ell^3);\ \sigma(t_\ell) > t_\ell^2\Big)\\ &\le \mathbb{P}\Big(\, \xi(\sigma(t_k)) - \sigma(t_k) > \sigma(t_\ell)\big(f(t_\ell^3)-1\big);\ \sigma(t_\ell) > t_\ell^2\Big)\\ &\le \mathbb{P}\Big(\, \xi(\sigma(t_k)) - \sigma(t_k) > t_\ell^2\big(f(t_\ell^3)-1\big)\Big),\end{aligned}$$

which, by scaling and (2.7) and (2.2), implies

$$\begin{aligned}\Delta_2 &\le \mathbb{P}\Big(\, \xi_\sigma - \sigma > \big(\frac{t_\ell}{t_k}\big)^2\big(f(t_\ell^3)-1\big)\Big) \le K_1\big(\frac{t_k}{t_\ell}\big)^{2\nu} f^{-\nu}(t_\ell^3)\\ &\le K_1 f^{-\nu}(t_k^3) e^{-2\nu(\ell-k)}. \qquad (2.14)\end{aligned}$$

Since $\mathbb{P}(E_k E_\ell) = \Delta_1 + \Delta_2$, combining (2.13), (2.14) and (2.11) gives

$$\begin{aligned}\mathbb{P}(E_k E_\ell) &\le K_2(\mathbb{P}(E_k) + e^{-t_k/K})(\mathbb{P}(E_\ell) + e^{-t_\ell/K})\\ &\qquad + K_2(\mathbb{P}(E_k) + e^{-t_k/K})e^{-2\nu(\ell-k)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum\sum_{k_0 \le k < \ell \le n} \mathbb{P}(E_k E_\ell) &\le K_2\Big(\sum_{k=k_0}^{n} (\mathbb{P}(E_k) + e^{-t_k/K})\Big)^2\\ &\qquad + \frac{K_2}{1-e^{-2\nu}} \sum_{k=k_0}^{n} (\mathbb{P}(E_k) + e^{-t_k/K}).\end{aligned}$$

Since $\sum_k \mathbb{P}(E_k) = \infty$ and $\sum_k e^{-t_k/K} < \infty$, this yields

$$\limsup_{n\to\infty} \sum_{k=k_0}^{n} \sum_{\ell=k_0}^{n} \mathbb{P}(E_k E_\ell) \Big/ \Big(\sum_{k=k_0}^{n} \mathbb{P}(E_k) \Big)^2 \le K_1 < \infty.$$

According to a well-known version of the Borel–Cantelli lemma (cf. [K-S]), this implies $\mathbb{P}(E_k, \text{ i.o.}) \ge 1/K_1$. In particular, we have

$$\mathbb{P}\Big(\limsup_{t\to\infty} \frac{\xi(t)}{t\, f(t)} \ge 1 \Big) \ge \frac{1}{K_1}.$$

Using Bessel time inversion (which tells that $\{ tR(1/t);\, t > 0 \}$ is again a Bessel process of dimension d) and Blumenthal's 0–1 law, this probability equals 1. Since replacing f by a multiple of f does not change the test, we have established the divergent part of Theorem 1. $\square$

3. Some related problems.

3.1. Theorem 1 is concerned with the location of the future infimum of R. A natural question is to study also the location of the past supremum of a Bessel process R of dimension $d > 0$. Let

$$\eta(t) = \sup\Big\{ u \le t :\, R(u) = \sup_{0\le s\le t} R(s) \Big\}.$$

Thus $\eta(t)$ is the location of the maximum of R over $[0, t]$. Of course there are infinitely many large t's such that $\eta(t) = t$. What about the liminf behaviour of $\eta(t)$? In case $d = 1$ (thus R is a reflecting Brownian motion), the answer to this question can be found in Csáki, Földes & Révész [Cs-F-R]:

$$\liminf_{t\to\infty} \frac{(\log\log t)^2}{t} \eta(t) = \frac{\pi^2}{4} \quad \text{a.s.} \tag{3.1}$$

The corresponding problem for arbitrary dimension d remains open. Nonetheless, some heuristic arguments suggest that the following Chung-type law of the iterated logarithm might hold:

Conjecture. *For any $d > 0$, we have*

$$\liminf_{t\to\infty} \frac{(\log\log t)^2}{t} \eta(t) = j_\nu^2 \quad \text{a.s.,}$$

where j_ν denotes the smallest positive zero of the Bessel function J_ν of index $\nu \equiv d/2 - 1$.

If the above Conjecture is true, by taking $d = 1$ we would recover (3.1), since $j_{-1/2} = \pi/2$.

3.2. There has been several recent papers devoted to the so-called Bessel gap, i.e. the difference between the past supremum and future infimum of R ($d > 2$). See for example Khoshnevisan [Kh]. It also seems interesting to investigate the difference between the locations of the past supremum and future infimum of R, i.e. we propose to study the process $t \mapsto \xi(t) - \eta(t)$. Since $\eta(t) \le t$, it is seen that $\xi(t) - \eta(t)$ have the same upper functions as $\xi(t)$, i.e. Theorem 1 holds for $\xi(t) - \eta(t)$ in the place of $\xi(t)$.

What about the liminf behaviour of $\xi(t) - \eta(t)$? Obviously for any $t > 0$, $\xi(t) - \eta(t)$ is (strictly) positive. A little more thinking convinces that with probability one,

$$\liminf_{t\to\infty}\Big(\xi(t) - \eta(t) \Big) = 0.$$

It would therefore be natural to look for a liminf iterated logarithm law for $t \mapsto \xi(t) - \eta(t)$. This problem is raised by Omer Adelman (personal communication).

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