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0. Introduction

Let B denote a one-dimensional Brownian motion started from 0 and L its local time process at level 0. For fixed $\mu > 0$, the perturbed reflecting Brownian motion X is defined for all $t \geq 0$ by

$$X_t = |B_t| - \mu L_t.$$

It has aroused some interest in the last few years (see Le Gall-Yor [7], Yor [13], chapters 8 and 9, Carmona-Petit-Yor [2], Perman [8]). We are going to make a few remarks concerning this process and give short elementary proofs of some known results, such as the generalized Ray-Knight Theorems for X. Let us just stress that none of the results derived here is new, and that our modest aim is to shed a new light on them, which we hope can improve our understanding of these identities.

We now recall a few relevant facts: For all $a \in R$, $T_a = \inf\{t \geq 0;\ X_t = a\}$ will denote the hitting time of a by X. Except when $\mu = 1$, X is not Markovian; however, for $a > 0$, T_{-a} is the hitting time of a/μ by L and hence a stopping time for B. The strong Markov property then yields that the processes $(X_t, t \geq 0)$ and $(a + X_{t+T_{-a}}, t \geq 0)$ have the same law. We will refer to this property as the 'strong Markov property' for X.
Note also that for $\mu = 1$, Lévy's identity (that is: if $S_t = \sup_{s<t} B_s$, then the processes $(S, S - B)$ and $(L, |B|)$ have the same law) shows that X is in fact a Brownian motion.

1. A hitting time property

In [11], we used the following result: For all $a > 0$, $b > 0$,

$$P(T_{-a} < T_b) = \left(\frac{b}{a+b}\right)^{1/\mu}. \qquad (1)$$

This is a generalization of the classical hitting time property for Brownian motion (which is in fact (1) for $\mu = 1$):

$$P(\sigma_{-a} < \sigma_b) = \frac{b}{a+b}, \tag{2}$$

where $\sigma_x = \inf\{t > 0,\ B_t = x\}$.

In [11], we derived (1) from the explicit law of L_{T_1} derived by Carmona-Petit-Yor [2] (corollary 3.4.1 there) (one has $P(T_{-a} < T_b) = P(L_{T_b} > a/\mu)$). As briefly pointed out in [2], the law of L_{T_1} (and therefore (1)) is in fact also a direct consequence of the explicit solution to Skorokhod's problem by Azéma and Yor [1] (see also exercise (5.9) chapter VI in Revuz-Yor [10]) in a very special case: One just has to compute the right-hand side of (5.9) in [10] for an affine function γ and then use Lévy's identity.

We now give an alternative elementary short proof of (1): First, for all $x \geq 1$ we put

$$g(x) = P(T_{1-x} < T_1).$$

For $x > 1$ and $y > 1$, one has immediately $T_{1-x} < T_{1-xy}$. The 'strong Markov property' at time T_{1-x} and the scaling property imply that

$$g(xy) = P(T_{1-x} < T_1)P(T_{x-xy} < T_x) = g(x)g(y).$$

Moreover, g is continuous decreasing on $[1, \infty)$ and $g(1) = 1$. Hence, for some fixed $c = c(\mu)$,

$$g(x) = x^{-c}. \tag{3}$$

It now remains to show that $c = 1/\mu$: We look at the asymptotic behaviour of

$$f(x) = P(T_{-1} > T_x) = 1 - g(1 + 1/x)$$

as $x \to \infty$. (3) implies that $f(x) = 1 - (1 + 1/x)^{-c} \sim c/x$ as $x \to \infty$. On the other hand, Lévy's identity implies that

$$P(\sigma_{-(x+1/\mu)} < \sigma_{1/\mu}) \leq f(x) \leq P(\sigma_{-x} < \sigma_{1/\mu}),$$

and consequently (using (2)), $f(x) \sim 1/(\mu x)$ as $x \to \infty$, and (1) follows.

2. The generalized second Ray-Knight Theorem as a consequence of (1)

In [2] (see also Yor [13], chapter 9), Carmona-Petit-Yor have derived a generalized second Ray-Knight Theorem for the local times of X (Theorem 3.3 in [2], Theorem 9.1 in [13]; we refer to Yor [13], chapter 3 or Revuz-Yor [10], Chapter XI for the Ray-Knight Theorems for Brownian motion). They then derive the law of L_{T_1} (which

implies (1)) as a consequence of this Theorem. We now briefly point out how this generalized second Ray-Knight Theorem for X can in fact be derived 'backwards', as a consequence of (1), using a general result of Lamperti [5] on semi-stable Markov processes we first recall.
Suppose $(Y_t, t \geq 0)$ is a non-deterministic continuous Markov process in $[0,\infty)$, started from $x \in [0,\infty)$ under the probability measure P_x. Suppose furthermore that Y is semi-stable of index 1 (in the sense of [5]), that is $(c^{-1}Y_{ct}, t \geq 0)$ under P_x and $(Y_t, t \geq 0)$ under $P_{x/c}$ have the same law for all $c > 0$. Then, Theorem 5.1 in Lamperti [5] implies that Y is a multiple of a squared Bessel process (of index $\delta \in R$), which is either absorbed or reflected at 0. This result is the key to our approach.

We now put down some notation and state the generalized second Ray-Knight Theorem. Let ℓ_t^a denote the local time of X at level a and time t, and let τ denote the right-continuous inverse process of ℓ^0. Then:

Theorem (Carmona-Petit-Yor). *The processes* $(\ell_{\tau_1}^a, a \geq 0)$ *and* $(\ell_{\tau_1}^{-a}, a \geq 0)$ *are independent and:*
(i) $(\ell_{\tau_1}^a, a \geq 0)$ *is a squared Bessel process of dimension 0 started from 1 and absorbed at 0.*
(ii) $(\ell_{\tau_1}^{-a}, a \geq 0)$ *is a squared Bessel process of dimension* $2 - 2/\mu$ *started from 0 and absorbed at 0.*

Let $A^+(t) = \int_0^t 1_{\{X_s > 0\}} ds$ and $A^-(t) = \int_0^t 1_{\{X_s < 0\}} ds$. Let also σ^+ (respectively σ^-) denote the right-continuous inverse of A^+ (resp. A^-). We put for all $u \geq 0$, $X_u^+ = X_{\sigma_u^+}$ and $X_u^- = X_{\sigma_u^-}$. In other words and loosely speaking: X^+ (resp. X^-) is obtained by glueing the positive (resp. negative) excursions of X together. Then:

Lemma *The two-processes* X^+ *and* X^- *are independent. Moreover* X^+ *is a reflected Brownian motion.*

There are various possible proofs of this lemma. Yor ([13], Chapter 8) indicates a proof based upon Knight's Theorem on orthogonal martingales. Mihael Perman suggested an excursion-theoretical approach The last part of this note provides yet another possible justification.

This lemma shows immediately that $(\ell_{\tau_1}^a, a \geq 0)$ and $(\ell_{\tau_1}^{-a}, a \geq 0)$ are independent; (i) then follows from the second Ray-Knight Theorem for Brownian motion (it actually also follows from (ii) with $\mu = 1$). It remains to show (ii).
The 'Markov property' for X and the lemma show that $(\ell_{\tau_1}^{-a}, a \geq 0)$ is a Markov process (one just has to apply the Lemma to $(a + X_{T_{-a}+t}, t \geq 0)$). As X is a continuous semi-martingale, Theorem (1.7) in Chapter VI of Revuz-Yor [10] yields that $(\ell_{\tau_1}^{-a}, a \geq 0)$ is continuous. The scaling property for B (which is also the scaling property for X) implies that $(\ell_{\tau_1}^{-a}, a \geq 0)$ is a semi-stable Markov process of index 1

in the sense of Lamperti [5]. Hence, Lamperti's result mentionned at the beginning of this section shows that $(\ell_{\tau_1}^{-a}, a \geq 0)$ is a multiple of a squared Bessel process Y. Let δ denote its dimension and $y = Y_0$. Y is absorbed at 0 since otherwise, $\ell_{\tau_1}^{-a}$ is not identically 0 for all sufficiently large a. It remains to identify δ and y, which can be done using section 1: As ℓ^0 increases,

$$P(T_{-a} < T_b) = P(\ell^0_{T_{-a}} < \ell^0_{T_b}).$$

But $\ell^0_{T_b}$ depends only on X^+ whereas $\ell^0_{T_{-a}}$ depends only on X^-; hence, these two random variables are independent. It is well-known that $\ell^0_{T_b}$ is an exponential random variable of parameter $1/2b$ (see e.g. Proposition (4.6), Chapter 6 in Revuz-Yor [10]). Consequently, if ξ denotes an exponential random variable of parameter $\lambda = 1/(2b)$, if ρ denotes the hitting time of 0 by Y and Z_γ a Gamma-random variable of index $\gamma > 0$ (that is with density $z^{\gamma-1}e^{-z}/\Gamma(\gamma)$ on R_+),

$$\begin{aligned} E(e^{-\lambda/\rho}) &= P(1/\rho < \xi) = P(\inf(X_s, s \leq \tau_1) < -\frac{1}{\xi}) \\ &= P(\ell^0_{T_{-1}} < \ell^0_{T_b}) = \left(\frac{b}{1+b}\right)^{1/\mu} = (1+2\lambda)^{-1/\mu} \end{aligned} \tag{4}$$

which is the Laplace transform of $(2Z_{1/\mu})$. Hence, ρ has the same law as $1/(2Z_{1/\mu})$. On the other hand, if $\rho(\alpha, x)$ is the hitting time of 0 by a squared Bessel process of dimension $2 - 2\alpha$ started from $x > 0$, then it is well-known that $\rho(\alpha, x)$ has the same law as $x^2/(2Z_\alpha)$ (one can for instance compare the Laplace transforms, using the results of Kent [4], and equation (15) section 6.22 in Watson [12]; alternatively, one can note by time-reversal that $\rho(\alpha, x)$ is the last passage time at x by a Bessel process of index $2 + 2\alpha$ started from 0, and use the results of Getoor [3], see also Yor [14]). Hence, $y = 1$ and $\delta = 2 - 2/\mu$, which completes the proof of the theorem.

3. The generalized first Ray-Knight Theorem

We now briefly point out how the same approach also yields the generalization of the first Ray-Knight Theorem for perturbed reflecting Brownian motion derived by Le Gall-Yor [6] (see also Yor [13], Section 3.3). However, in this case, the original proofs are shortish anyway.

Theorem (Le Gall-Yor). *$(\ell_{T_{-1}}^{-1+a}, 0 \leq a \leq 1)$ is a squared Bessel process of dimension $2/\mu$ started from 0 and reflected at 0.*

By time-reversal (since $(B_t, t \leq T_{-1})$ and $(B_{T_{-1}-t}, t \leq T_{-1})$ have the same law), one can consider the process $\tilde{X}_u = |B_u| + \mu L_u$ and its local times taken at infinite time: $\tilde{\ell}^a = \ell^a_\infty(\tilde{X})$, for $a \geq 0$, and remark that $(\ell_{T_{-1}}^{-1+a}, 0 \leq a \leq 1)$ and $(\tilde{\ell}^a, 0 \leq a \leq 1)$ have the same law. As previously, $\tilde{\ell}$ is a continuous Markov process, which is self-similar of index 1 because of the scaling property of $\tilde{X}$. $\tilde{\ell}$ is henceforth (using again

Theorem 5.1 in Lamperti [5]) a multiple of a squared Bessel process β of dimension δ: $\tilde{\ell} = \alpha\beta$, with $\alpha > 0$. This time, β has to be reflected at 0, since almost surely, for all rational $a > 0$, $\tilde{\ell}^a \neq 0$. We now identify δ and α using (4). On the one hand, one has for all $\lambda > 0$ (see e.g. Revuz-Yor [10], line before Corollary (1.4) in Chapter XI):

$$E(e^{-\lambda\alpha\beta_1}) = (1 + 2\lambda\alpha)^{-\delta/2}.$$

On the other hand, (4) implies that (using the same notations ξ, λ as in (4)),

$$E(e^{-\lambda\alpha\beta_1}) = P(\xi > \alpha\beta_1) = P(\ell^0_{T_{1/(2\lambda)}} > \ell^0_{T_{-1}}) = P(T_{1/(2\lambda)} > T_{-1}) = (1 + 2\lambda)^{-1/\mu}$$

and the Theorem follows.

4. The discrete approach

We now mention an approximation of X by a random walk, which converges towards perturbed reflecting Brownian motion as the simple random walk does towards Brownian motion. We define $(S_n, n \geq 0)$ as follows: We fix $\mu > 0$ and we put $q = (1 + \mu)^{-1} \in (0, 1)$.

Let $I_n = \min\{S_0, S_1, \ldots, S_n\}$ and $S_0 = 0$. By induction, for all $n \geq 0$, if $S_0, \ldots, S_n$ are defined, then the law of S_{n+1} is the following:

$$P(S_{n+1} = S_n + 1) = P(S_{n+1} = S_n - 1) = 1/2 \text{ if } S_n \neq I_n$$

and

$$P(S_{n+1} = S_n + 1) = q, \ P(S_{n+1} = S_n - 1) = 1 - q \text{ if } S_n = I_n.$$

Using for instance Lévy's identity and Proposition 2 page 137-138 in Révész [4], one can show that the processes

$$(n^{-1/2} S_{[nt]}, t \in [0, 1])$$

converge weakly towards $(X_t, t \in [0, 1])$ as $n \to \infty$, where $[x]$ denotes the integer part of x. With little extra work, this approach provides another possible proof of the Lemma, since the independence of the positive and negative parts of the random walk $(S_n, n \geq 0)$ is trivial. Equation (1) can also be deduced, since (if $N_p = \inf\{n \geq 0, S_n = p\}$), $P(N_{-1} < N_p)$ and consequently $P(N_{-p'} < N_p)$ and $P(N_{-[ap]} < N_{[bp]})$ can be very easily explicitely computed, when $p > 0$, $p' > 0$, $a > 0$, $b > 0$: Indeed, it is a good undergraduate exercise to see that

$$P(N_{-1} < N_p) = (1 - q) \sum_{k \geq 0} \left(\frac{q(p-1)}{p} \right)^k = (1 + 1/(\mu p))^{-1},$$

and consequently as $P(N_{-p'} < N_p) = P(N_{-1} < N_p) P(N_{-1} < N_{p+1}) \ldots P(N_{-1} < N_{p+p'-1})$,

$$\log P(N_{-[ap]} < N_{[bp]}) = - \sum_{k=[bp]}^{k=[ap]+[bp]-1} \log(1 + 1/(\mu k)) \sim \frac{1}{\mu} \log \left(\frac{b}{a+b} \right)$$

as $p \to \infty$, which yields (1).

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Note added in proof. I would like to mention the two recent preprints by Burgess Davis [15] and Darryl Nester [16], which are very closely related with Section 4 of this note.

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