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C-semigroups on Banach spaces and functional inequalities.

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Abstract. We introduce the notion of C-semigroup on a Banach space. This notion is intimately relevent to classical Dirichlet forms on Banach spaces. We shall prove a sufficient condition for a semigroup on R^d to be a C-semigroup. Then, we prove that C-semigroups satisfy various functional inequalities such as Poincaré inequality, logarithmical Sobolev inequality and Stein-Meyer-Bakry inequalities (Riesz transform).

Key words. C-semigroup, classical Dirichlet form, well-admissible measure on Banach space, symmetric Markov process, Poincaré inequality, logarithmical Sobolev inequality, Stein-Meyer-Bakry inequalities (Riesz transform), stochastic flow.

AMS classification. 31C25, 44A15, 47B99, 60J35, 60J60.

Démontrer l'inégalité de Meyer à partir de la formule

$$\frac{1}{2}\tilde{\nabla}f(x) = E^{\infty}[\int_0^{\tau} \tilde{\nabla}Cf(X_s,B_s)\, e^s\, dB_s\, e^{-\tau} \,|\, X_{\tau} = x].$$

Introduction.

We consider a separable real Banach space B. We assume that B is <u>riggid</u>, i.e. there is a separable real Hilbert space H such that $B^* \subset H \subset B$ densely and continuously. We choose an orthonormal basis $(k_i)_{i\geq 1}$ of H, consisting of elements of B^*. We denote by $\langle \bullet,\bullet \rangle$ the scalar product in H and by $\|\bullet\|$ the associated norm on H.

Let μ be a positive measure on B, charging every open set in B. We suppose that μ is σ-finite on the cylindrical sets. Let $FC_f^2(B)$ denote the family of real cylindrical two times continuously differentiable bounded functions f on B such that $\mu(\{f \neq 0\}) < \infty$. For $u \in FC_f^2(B)$, we define

$$\frac{\partial u}{\partial k_i}(x) = \lim_{t\downarrow 0} \frac{1}{t}(u(x + tk_i) - u(x)), \ i = 1, 2, ...$$

We in addition assume that the measure μ is <u>well-admissible</u>, i.e., for any $i \geq 1$, there exists a function $\eta_i \in L^2(B,\mu)$ such that

$$\int \frac{\partial u}{\partial k_i}(x)\,\mu(dx) = -\int u(x)\,\eta_i(x)\,\mu(dx),\ \forall\, u \in FC_f^2(B).$$

For a well-admissible measure μ, we can define a differential operator in the following way. For a function $f \in L^2(B,\mu)$, we shall say that f is differentiable (in directions k_i), if there exists $g \in L^2(B,H,\mu)$ such that, for any $i \geq 1$, for any $u \in FC_f^2(B)$,

$$\int \left[\frac{\partial u}{\partial k_i}(x) + u(x)\,\eta_i(x)\right] f(x)\,\mu(dx) = -\int u(x)\,\langle g(x), k_i\rangle\,\mu(dx).$$

The function g, which is uniquely determined by f (cf. Song [29]), will be denoted by $\tilde{\nabla} f$. Clearly, $\tilde{\nabla} u$ coincides with $\nabla u = (\frac{\partial u}{\partial k_i}, i \geq 1)$ if $u \in FC_f^2(B)$.

The gradient operator $\tilde{\nabla}$ being defined, we can now introduce the notion of C-semigroup. In the following definition, $\Lambda(H)$ denotes the space of bounded linear transformations in H equipped with the usual operator norm $\|\bullet\|$ (there will be no confusion with the norm on H).

Definition 1. Let (Q_t) be a semigroup of symmetric Markov operators on $L^2(B,\mu)$. We shall say that (Q_t) is a C-semigroup (with respect to the measure μ), if

i. for any $f \in L^2(B,\mu)$, for any $t > 0$, $Q_t f$ is differentiable;

ii. there exists a B-valued Hunt process X such that Q_t is the transition semigroup of X;

iii. there exists a $\Lambda(H)$-valued càdlàg process C_t (which will be called a C-process), adapted with respect to the natural filtration of X, with bounded variation, such that $\tilde{\nabla} Q_t f(x) = E_x[\tilde{\nabla} f(X_t)\, C_t]$, μ-a.s. x, for any differentiable function $f \in L^2(B,\mu)$, for any $t > 0$;

iv. $\alpha(Q_\bullet) = \text{esssup}\ \sup_{t>0} \frac{1}{t} \log \|C_t\| < \infty$.

v. C_t^{-1} exists and has bounded variation.

vi. for any $T > 0$, the identity $C_{T-s} \circ \iota_T = (C_s')^{-1}\, C_T'$ holds, where C_s' denotes the adjoint operator of C_s, and ι_T denotes the time inversion operator of the Hunt process X.

vi. for any $s \geq 0$, $t \geq 0$, $C_{t+s} = (C_s \circ \theta_t) C_t$, where θ_t, $t \geq 0$, denotes the translation operator associated with X.

Relatively to a C-semigroup (Q_t), we introduce the operators $\tilde{Q}_t$, $t \geq 0$, on the space $L^2(B,H,\mu)$: $\tilde{Q}_t(F)(x) = E_x[F\, C_t]$, $F \in L^2(B,H,\mu)$. We have the following lemma:

Lemma 2. The operators $(\tilde{Q}_t)$ are bounded, symmetric and form a semigroup.

This lemma will be proved in §5. We shall call the semigroup $(\tilde{Q}_t)$ the tengent semigroup of (Q_t).

A priori, C-semigroup property can concern any semigroup on $L^2(B,\mu)$. But, the far intimately relevent semigroup is the semigroup associated with the Dirichlet form defined as follows: We define the space $W(\mu)$ to be the family of differentiable functions f in $L^2(B,\mu)$. We introduce the form:

$$\mathcal{E}_\mu(f,h) = \int \langle \tilde{\nabla} f, \tilde{\nabla} h\rangle(x)\, \mu(dx), \quad f, h \in W(\mu).$$

It is known that the form $(\mathcal{E}_\mu, W(\mu))$ is a Dirichlet form on $L^2(B,\mu)$ (see Albeverio-Röckner [1]). This Dirichlet form will be called the classical Dirichlet form associated with the (well-admissible) measure μ.

Occasionally, we also need the next definition. As it will be seen below, this notion concerns especially the C-semigroups associated with a classical Dirichlet form.

Definition 3. Let γ be a measurable mapping from B into $\Lambda(H)$. Let (Q_t, X_t, C_t) be a C-semigroup with its associated Hunt process X and C-process (C_t). We shall say that the C-process is logarithmically differentiable with log-derivative γ, if $\int_0^t \|\gamma(X_s)\|\, ds < \infty$ almost surely, for any $t > 0$, and C_t satisfies the stochastic integral equation

$$C_t = I + \int_0^t \gamma(X_s)\, C_s\, ds, \ \forall\, t > 0,$$

where I denotes the identity transform in H.

We shall see that the constant $\alpha(Q_\bullet)$ in Definition 1.iv is a functional of the log-derivative of C_t, if it exists.

The introduction of the notion of C-semigroup has been stimulated by our experiences on the studies of classical Dirichlet spaces on Banach spaces. We have noticed that, for a symmetric Markovian semigroup Q_t on $L^2(B,\mu)$, many problems will have very simple solutions, if we can say about $\tilde{\nabla}Q_t$. The notion of C-semigroup synthesizes what about $\tilde{\nabla}Q_t$ would be useful.

Examples of interventions of C-semigroups are numerous. The Brownian semigroups, the Ornstein-Uhlenbeck semigroups, the Bessel semigroups (cf. Song [31]), the symmetric convolution semigroups, etc., are C-semigroups. Also, the transition semigroup of a stochastic differentiable flow on R^d of the form

$$dX_t = d\beta_t + \nabla b(X_t)dt,$$

where b is a $C_b^2(R^d)$-function, gives rise to a C-semigroup. Notice that, in this last example, a C-process is simply given by $C_t = (\partial_j X^i(t),\ 1 \le i,j \le d)$, which is obviously logarithmically differentiable with log-derivative $(\partial_{ij} b)$. We shall see later (Theorem 1.1, Part I) that this will remain true for a larger class of functions b on R^d.

The notion of C-semigroup already had been introduced in [31], where we proved that, for a Markovian semigroup Q_t on R^1 to be the transition semigroup of a two parameter continuous symmetric Markov process in the sense of Hirsch-Song [13], [14], Q_t must be a C-semigroup with $\alpha(Q_\bullet) \le 0$. More recently in [30], we show that C-semigroup property can be used to prove the Markovian uniqueness for Dirichlet operators on Banach spaces. In the present paper we shall show that C-semigroups satisfy various functional inequalities.

The paper is organised in two parts. In the first part, we shall give a sufficient condition for a semigroup on R^d to be a C-semigroup.

In the second part, we shall study the Poincaré inequality, the logarithmical Sobolev inequality, the Stein-Meyer-Bakry inequalities, for a C-semigroup associated with a classical Dirichlet form.

Let us say two words on the hypothesis of our paper. First, we have limited ourselves to consider only Banach spaces. But, our knowledge on Bakry's paper [3], in which Bakry has already used the tengent semigroup $\tilde{Q}_t$, convinces us of the possibility to apply our method to the studies of diffusions on Riemmanian manifolds. Secondly, the notion of C-semigroup is introduced by use of the gradient operator $\tilde{\nabla}$. This deprives us of considering that semigroups which possess an "<u>opérateur carré du champs</u>" $\Gamma(f,f)$ other than $\langle\tilde{\nabla}f,\tilde{\nabla}f\rangle$.

Part I. Existence of C-semigroups on R^d.

§1. Hypothesis.

In this section, we describe the semigroups which will be proved to be C-semigroups. Our description uses the notion of Dirichlet form, for which we refer to the book of Fukushima [9]. For the special case of Dirichlet forms on Banach spaces, we also can refer to Ma-Röckner [19], Song [29], [28], and the references therein.

We work on R^d equipped with the Lebesgue measure dx. We consider the classical Sobolev space $H^{1,2}(R^d)$ and the Dirichlet form

$$\mathcal{E}(u,v) = \int \sum_{1\le i\le d} \partial_i u(x)\, \partial_i v(x)\, dx = \int \langle \nabla u(x), \nabla v(x) \rangle\, dx,\ u, v \in H^{1,2}(R^d),$$

where $\nabla u(x) = (\partial_1 u(x),\dots,\partial_d u(x))$, and ∂_i is in the distribution sense. Let P_t be the semigroup associated with $\mathcal{E}$. This semigroup has a nice representation: $P_t f(x) = E[f(x + \sqrt{2}\,\beta_t)]$, where β denotes the standard d-dimensional Brownian motion started from zero.

Let Ω denote $C(R_+,R^d)$. The points in Ω will be denoted by ω, while the coordinate process will be denoted by $(\omega_t, t \ge 0)$. We shall denote the law of $x + \sqrt{2}\beta$ on Ω by P_x. If ξ denotes an R^d-valued random variable whose law is the Lebesgue measure, independent of β, the law of $\xi + \sqrt{2}\beta$ on Ω will be denoted by P.

We choose now a function b in $L^2(R^d,dx)$ and put $b_n = P_{1/n}b$ for any integer $n \ge 1$. We assume the following hypotheses on the function b.

Hy1: The fonction b is an $\mathcal{E}$-quasi-continuous function in the space $H^{1,2}$ and there are constants $C > 0$, $\upsilon > 0$, $2 > \rho > 0$ such that $|b(x)| \le C \exp\{\upsilon |x|^\rho\}$.

Under this hypothesis, the functions b_n belong to $C^\infty(R^d)$.

Hy2: $e^b \in L^2(R^d)$.

Let us show then that $e^{b_n} \in L^2(R^d)$ for any $n \ge 1$. Indeed, for any $t > 0$, we have

$$P_t[e^{2b}] \ge e^{2P_t b}.$$

This implies

$$\int e^{2b_n(x)}\,dx \le \int P_{1/n}[e^{2b}](x)\,dx = \int e^{2b(x)}\,dx < \infty.$$

Hy3: The functions e^b and e^{b_n} belong to $H^{1,2}(R^d)$.

Hy4: For any $1 \le i, j \le d$, $\partial_{ij}b$ are signed measures whose positive and negative parts are of finite energy with respect to $\mathcal{E}$.

Hy5: There exists a constant $0 < C < \infty$ such that, for any $n \ge 1$, for dx-a.s. $x \in R^d$, the matrix $\partial^2 b_n(x) - CI_d$ is definite negative.

The measures $\partial_{ij}b$ are decomposed in positive parts and negative parts: $\partial_{ij}b = \partial_{ij}b^+ - \partial_{ij}b^-$. We denote $\partial_{ij}b^+ + \partial_{ij}b^-$ by $\left|\partial_{ij}b\right|$. By definition, $\left|\partial_{ij}b\right|$ is a finite $\mathcal{E}$-energy measure. Let $B_t(\omega) = (B_{ij}(\omega)(t))$ be the matrix-valued process on Ω such that each $B_{ij}(\omega)(t)$ is, under P, the additive functional associated with the measures $\partial_{ij}b$. We shall denote the matrix $(\partial_{ij}b)$ by $\partial^2 b$ and $(\partial_{ij}b_n)$ by $\partial^2 b_n$. Remark that, when $\partial_{ij}b$ are functions, $B_{ij}(\omega)(dt)$ is just $\partial_{ij}b(X_t(\omega))\,dt$.

By hypothesis Hy2 and its consequence, we can define the bounded measures $\mu(dx) = e^{2b(x)}dx$ and $\mu_n(dx) = e^{2b_n(x)}dx$. With respect to the measure μ, we have an integration by parts formula:

$$\int \partial_i u(x)\,\mu(dx) = -\int u(x)\,2\partial_i b(x)\,\mu(dx),\quad u \in C_c^1(R^d).$$

Notice that by Hy3, $\partial_i b \in L^2(R^d,\mu)$. Therefore, the measure μ is well-admissible and the operator $\tilde{\nabla}$ introduced in Introduction is well define on the space $L^2(R^d,\mu)$. We then consider the classical Dirichlet form $(\mathcal{E}_\mu, W(\mu))$ associated with μ.

Let Q_t denote the Markov semigroup on $L^2(R^d,\mu)$ associated with $\mathcal{E}_\mu$ (cf. Fukushima [9]). The main object of Part I is to prove the following theorem:

Theorem 1. Under the hypotheses Hyl to Hy5, (Q_t) is a C-semigroup. Moreover, if $\partial_{ij}b$, $1 \le i, j \le d$, are functions on R^d, a C-process can be associated with (Q_t), which is logarithmically differentiable with logarithmic derivative $\partial^2 b = (\partial_{ij}b)$.

§2. Estimate on the function b.

Recall that there is a conservative μ-symmetric diffusion process X in R^d whose Dirichlet form coincides with $\mathcal{E}_\mu$ (see Takeda [34]). The process X satisfies the stochastic equation:

$$X_t = X_0 + \sqrt{2}\,\beta_t + \int_0^t 2\nabla b(X_s)\, ds.$$

The law of X on Ω will be denoted by Q_x when $X_0 = x$, and by Q when X_0 has the law of μ. We know how to describe the semigroup Q_t of X: According to Takeda [34], there is a multiplicative functional N defined on Ω such that

$$N_t(\omega) = \exp\{\int_0^t \langle 2\nabla b(\omega_s), d\omega_s\rangle - 4\int_0^t \langle \nabla b, \nabla b\rangle(\omega_s)\, ds\},$$

for P_x-a.s. $\omega \in \Omega$, for any $x \in R^d$. Then, $Q_t f(x) = P_x[N_t(\omega)f(\omega_t)]$, or more generally, $Q_x[F(\omega^t)] = P_x[N_t F(\omega^t)]$, where $\omega^t = (\omega_s,\ 0 \le s \le t)$. It is known that N is a P_x-martingale for any $x \in R^d$.

Notice that the above results hold again when the function b is replaced by the function b_n. We shall denote the corresponding objects by $(X^n, \mu_n, Q^n_x, Q^n, Q^n_t, N^n_t)$.

Lemma 1. We have the following three convergences in $L^2(P)$, the convergences of the processes being with respect to the uniform norm over any compact set in R_+:

$$b_n(\omega_\bullet) \to b(\omega_\bullet);\quad \int_0^\bullet \nabla b_n(\omega_s)\, ds \to \int_0^\bullet \nabla b(\omega_s)\, ds;\quad \int_0^\bullet \partial^2 b_n(\omega_s)\, ds \to B_\bullet(\omega).$$

Proof. These are consequences of results of Hirsch-Song [13], [14]. $\square$

Let Λ denote the family of stopping times τ such that

$$\text{P-esssup}_\omega \{\sup_n \int_0^{\tau(\omega)} \|\partial^2 b_n\|(\omega_s)\, ds + \sup_n \sup_{0\le s\le \tau(\omega)} \left|b_n(\omega_s)\right|\} < \infty.$$

Lemma 2. For any $\tau \in \Lambda$, P-esssup$_\omega$ sup$_n$ $N^n_\tau(\omega)$ is finite.

Proof. It is enough to notice that, under P,

$$\int_0^t \langle \nabla b_n(\omega_s), d\omega_s \rangle = b_n(\omega_t) - b_n(\omega_0) - \int_0^t \Delta b_n(\omega_s)\, ds. \square$$

Lemma 3. There is an increasing sequence (τ_n) in Λ such that $\lim_n \tau_n(\omega) = \infty$, P_x-a.s. ω, for dx-a.s. $x \in R^d$.

Proof. Set

$$S_t(\omega) = \sup_n \int_0^t \|\partial^2 b_n\|(\omega_s)\, ds + \sup_n \sup_{0 \le s \le t} \left| b_n(\omega_s) \right| .$$

As a consequence of Lemma 1, S_t is finite and continuous P_x-almost surely, for dx-a.s. $x \in R^d$. To prove the lemma, it is enough to set $\tau_n(\omega) = \inf\{t;\ S_t(\omega) \ge n\}$. $\square$

§3. C-Process.

For any $\omega \in \Omega$, let $C(\omega)$ and $C^n(\omega)$ denote respectively the following two d×d matrix-valued processes on Ω determined by the equations:

$$C_t(\omega) = I_d + 2\int_0^t dB_s(\omega)\, C_s(\omega), \text{ P-a.s. } \omega, \text{ and}$$

$$C^n_t(\omega) = I_d + 2\int_0^t \partial^2 b_n(\omega_s)\, C^n_s(\omega)\, ds, \text{ P-a.s. } \omega.$$

It can be shown that the solutions of these equations exist and are unique. Moreover, we have:

Lemma 1. Let A_t be the unique solution of the equation: $A_t = I_d - 2\int_0^t A_s\, dB_s$, $t \ge 0$.

Then, for any $t \ge 0$, $A_t = C_t^{-1}$ the inverse of C_t. Similar result holds also for C^n_t.

Proof. It is because $d(A_tC_t) = 0$. $\square$

Lemma 2. $C^n(\omega) \to C(\omega)$ **uniformly on each compact intervals for P-a.s.** ω.

Proof. Consider the difference between C and C^n:

$$C_t(\omega) - C_t^n(\omega) = 2\int_0^t dB_s(\omega)\, C_s(\omega) - 2\int_0^t \partial^2 b_n(\omega_s)\, C_s^n(\omega)\, ds$$

$$= 2\int_0^t (dB_s(\omega) - \partial^2 b_n(\omega_s)ds)\, C_s(\omega) + 2\int_0^t \partial^2 b_n(\omega_s)\, (C_s(\omega) - C_s^n(\omega))\, ds.$$

Consider the two integrals in the last term. The second one is overestimated by

$$2\, \sup_n \int_0^t \|\partial^2 b_n\|(\omega_s)\, ds \quad \sup_{0\le s\le t} \left| C_s(\omega) - C_s^n(\omega) \right|,$$

while the first one converges uniformly to zero in any compact interval for P-a.s. ω, which is the consequence of Lemma §2.1. Now, to finish the proof of the lemma, it is enough to apply Wendroff inequality (cf. Mao [20], p.24-32). $\square$

Lemma 3. For any integer $p \ge 1$, set $\phi(p,s) = k2^{-p}$, where k is the unique integer such that $s \in [k2^{-p}, (k+1)2^{-p}[$. For a fixed integer $n \ge 1$, let $C_t(n,p)(\omega)$ be solution of the equation

$$C_t(n,p)(\omega) = I_d + 2\int_0^t \partial^2 b_n(\omega_{\phi(p,s)})\, C_s(n,p)(\omega)\, ds.$$

Then, $C^n(\omega)$ is the limit of $C_t(n,p)(\omega)$ when p tends to the infinity.

Proof. This can be proved by the usual Gronwall inequality method. $\square$

Lemma 4. Let M be a d×d real symmetric matrix. Let $\lambda_1(M) \ge \lambda_2(M) \ge \ldots \ge \lambda_d(M)$ be the eigenvalues of M. Let D be the diagonal matrix corresponding to (λ_i) and let U be an orthonormal matrix such that $UMU^* = D$. Let $\exp\{tD\}$ denote the diagonal matrix corresponding to $(\exp\{t\lambda_i\})$. Let v(t) be the solution of the equation:

$$v(t) = I_d + \int_0^t M\, v(s)\, ds, \quad t \geq 0.$$

Then, $v(t) = U^* \exp\{tD\} U$.

Proof. Let $w(t) = Uv(t)U^*$. Then, $w(t)$ satisfies the equation:

$$w(t) = I_d + \int_0^t UMU^*\, w(s)\, ds = I_d + \int_0^t D\, w(s)\, ds, \quad t \geq 0.$$

Clearly, $w(t) = \exp\{tD\}$ is the unique solution of the equation. The lemma is proved. □

Corollary 5. $\|v(t)\| \leq \exp\{t\lambda_1(M)\}$.

Let $\rho = \sup_n \sup_x \lambda_1(2\partial^2 b_n(x))$. Remark that, by Hy5, $\rho < \infty$. We have the following lemma.

Lemma 6. We have $\|C_t\| \leq e^{\rho t}$, P-almost surely. Consequently, $\alpha(Q_\bullet) \leq \rho$.

Proof. Fix $n \geq 1$ and $p \geq 1$. Consider the process $C_t(n,p)$ introduced in Lemma 3. Set $A(t) = C_t(n,p)$. Let $M_k = \partial^2 b_n(\omega_{k2^{-p}})$, $k \geq 1$. Let $v_k(t)$ be the solution associated with M_k defined in Lemma 4. We check easily that $A(t + k2^{-p})\, A^{-1}(k2^{-p})$ and $v_k(t)$ for $t \in [0,2^{-p}[$ satisfy the same equation. By uniqueness, we conclude, if $h = \phi(p,t)2^p$ and $t_i = i2^{-p}$,

$$A(t) = A(t)A^{-1}(t_h)\, A(t_h)A^{-1}(t_{h-1}) \ldots A^{-1}(t_2)A(t_1)A^{-1}(t_1)A(t_0)$$

$$= v_h(t - t_h)\, v_{h-1}(t_h - t_{h-1}) \ldots v_1(t_2 - t_1)v_0(t_1).$$

Now, by Corollary 5 applied to each of v_i, we see $\|C_t(n,p)(\omega)\| \leq e^{\rho t}$, for any $t \geq 0$. Now, it is enough to apply Lemma 3 and Lemma 2 to complete the proof. □

Lemma 7. Let $T > 0$. Let ι_T denote the inversion operator : $\iota_T(\omega)_t = \omega_{T-t}$. Then, $C_{T-t} \circ \iota_T = (C_t')^{-1}\, C_T'$, $0 \leq t \leq T$, P-almost surely, where ' denotes the transposition of matrix.

Proof. We have P-almost surely

$$C_{T-t}\circ\iota_T = I_d + 2\int_0^{T-t} B\circ\iota_T(ds)\, C_s\circ\iota_T$$

$$= I_d + 2\int_0^{T} B\circ\iota_T(ds)\, C_s\circ\iota_T - 2\int_{T-t}^{T} B\circ\iota_T(ds)\, C_s\circ\iota_T$$

$$= C_T\circ\iota_T - 2\int_{T-t}^{T} B\circ\iota_T(ds)\, C_s\circ\iota_T = C_T\circ\iota_T - 2\int_0^{t} B(du)\, C_{T-u}\circ\iota_T.$$

From this formula we see that $C_{T-t}\circ\iota_T\,(C_T\circ\iota_T)^{-1}$ and $(C_t')^{-1}$ satisfy the same equation. By uniqueness, we conclude $C_{T-t}\circ\iota_T\,(C_T\circ\iota_T)^{-1} = (C_t')^{-1}$, $0 \le t \le T$. In particular, when $t = T$, we have $(C_T\circ\iota_T)^{-1} = (C_T')^{-1}$. This proves the lemma. □

Lemma 8. Let θ_t, $t \ge 0$, denote the translation operator on Ω: $\theta_t(\omega)_s = \omega_{t+s}$. Then, we have the relation $C_{t+s} = (C_s\circ\theta_t)C_t$, $s \ge 0$, P-almost surely.

Proof. It is enough to notice that, under the measure P, $C_{t+s}\,C_t^{-1}$ and $C_s\circ\theta_t$ satisfy both the same equation: $A_s = I_d + \int_0^s B\circ\theta_t(du)\, A_u$, $s \ge 0$. □

§4. Derivative of the process X.

Lemma 1. For any $n \ge 1$, for $x \in R^d$, let $X^n(x,\omega)$ be the unique solution (under P) of the equation:

$$dX_t^n(x) = \sqrt{2}\, d\beta_t + 2\nabla b_n(X_t^n)\, dt, \qquad X_0^n = x.$$

Then, $\partial_i(X_t^n)_j(x,\omega) = (C_t^n)_{ij}(\omega)$, P-a.s. ω.

Proof. This can be proved using the results in Ikeda-Watanabe [15], using the localisation, and finally using the fact that X^n is conservative. □

Lemma 2. We have $\tilde{\nabla} Q_t^n u(x) = Q_x^n[\tilde{\nabla} u(\omega_t) C_t^n(\omega)]$, $u \in W(\mu)$, where the vector $\tilde{\nabla} u(x)$ in H is represented horizontally.

Proof. It is enough to prove it for $u \in C_c^1(R^d)$. But, by Lemma §3.6, we have

$$Q_x^n[\,||\nabla u(\omega_t)C_t^n(\omega)||^2\,] \leq e^{2\alpha t}\, Q_x^n[\,||\nabla u(\omega_t)||^2\,] < \infty.$$

So, we can apply Fubini's lemma to finish the proof. □

Lemma 3. For any $\tau \in \Lambda$, for $t \geq 0$, for any $u, v \in C_c^1(R^d, H)$,

$$\lim_n Q^n[\,\langle v(\omega_0), \nabla u(\omega_t)C_t^n(\omega)\rangle\,;\,\tau(\omega) > t] = Q[\,\langle v(\omega_0), \nabla u(\omega_t)C_t(\omega)\rangle\,;\,\tau(\omega) > t].$$

Proof. Indeed,

$$\lim_n Q^n[\,\langle v(\omega_0), \nabla u(\omega_t)C_t^n(\omega)\rangle\,;\,\tau(\omega) > t]$$

$$= \lim_n P[\,e^{b_n(\omega_0)}\, N_t^n(\omega)\,\langle v(\omega_0), \nabla u(\omega_t)C_t^n(\omega)\rangle\,;\,\tau(\omega) > t]$$

$$= P[\,e^{b(\omega_0)}\, N_t(\omega)\,\langle v(\omega_0), \nabla u(\omega_t)C_t(\omega)\rangle\,;\,\tau(\omega) > t]$$

$$= Q[\,\langle v(\omega_0), \nabla u(\omega_t)C_t(\omega)\rangle\,;\,\tau(\omega) > t].$$

Lemma 4. $\mathrm{Lim}_n\, Q^n[\tau(\omega) > t] = Q[\tau(\omega) > t]$.

Proof. Indeed

$$\mathrm{Lim}_n\, Q^n[\tau(\omega) > t] = \mathrm{Lim}_n\, P[N_t^n(\omega); \tau(\omega) > t] = P[N_t(\omega); \tau > t] = Q[\tau(\omega) > t].$$ □

Corollary 5. For any $\varepsilon > 0$, for any $t > 0$, there is a $\tau \in \Lambda$ such that

$$\mathrm{limsup}_n\, Q^n[\tau(\omega) \leq t] < \varepsilon.$$

Proof. This is because that, by Lemma §2.3, there is an increasing sequence $(\tau_k) \subset \Lambda$ which tends to infinity P_x-almost surely, for dx-a.s. $x \in R^d$. Since Q_x is locally absolutely continuous with respect to P_x, we have therefore $\lim_k Q[\tau_k \leq t] = 0$. Now, the corollary follows from Lemma 4. □

Theorem 6. $\tilde{\nabla} Q_t u(x) = Q_x[\tilde{\nabla} u(\omega_t)C_t(\omega)]$, $u \in W(\mu)$.

Proof. It is enough to prove it for $u \in C_c^1(R^d)$. Let $1 \le i \le d$. Set $\partial_i{*}v = \partial_i v + v\partial_i b$, $v \in C_c^1(R^d)$. Let $\varepsilon > 0$. Let $\tau \in \Lambda$ such that $\limsup_n Q^n[\tau \le t] < \varepsilon$. We have, for $v \in C_c^1(R^d)$,

$$\int \partial_i{*}v(x)\, Q_x[u(\omega_t)]\, \mu(dx)$$

$$= \int \partial_i{*}v(x)\, Q_x[u(\omega_t);\, \tau(\omega) > t]\, \mu(dx) + \int \partial_i{*}v(x)\, Q_x[u(\omega_t);\, \tau(\omega) \le t]\, \mu(dx)$$

$$= \lim_n \int \partial_i{*}v(x)\, Q_x^n[u(\omega_t);\, \tau(\omega) > t]\, \mu_n(dx) + O(\varepsilon)$$

$$= \lim_n \int \partial_i{*}v(x)\, Q_x^n[u(\omega_t)]\, \mu_n(dx) + 2\, O(\varepsilon)$$

$$= -\lim_n \int v(x)\, (Q_x^n[\nabla u(\omega_t)\, C_t^n(\omega)])_i\, \mu_n(dx) + 2\, O(\varepsilon)$$

$$= -\lim_n \int v(x)\, (Q_x^n[\nabla u(\omega_t)\, C_t^n(\omega);\, \tau(\omega) > t])_i\, \mu_n(dx) + 3\, O(\varepsilon)$$

$$= -\int v(x)\, (Q_x[\nabla u(\omega_t)\, C_t(\omega);\, \tau(\omega) > t])_i\, \mu(dx) + 3\, O(\varepsilon)$$

$$= -\int v(x)\, (Q_x[\nabla u(\omega_t)\, C_t(\omega)])_i\, \mu(dx) + 4\, O(\varepsilon).$$

Since ε is arbitrary, we have proved

$$\int \partial_i{*}v(x)\, Q_x[u(\omega_t)]\, \mu(dx) = -\int v(x)\, (Q_x[\nabla u(\omega_t)\, C_t(\omega)])_i\, \mu(dx).$$

This is equivalent to $\tilde{\nabla} Q_t u(x) = Q_x[\nabla u(\omega_t) C_t(\omega)]$, $u \in C_c^1(R^d)$. $\square$

Now, we can claim that Theorem §1.1 is proved.

§5. Tengent semigroup.

Let us prove Lemma 2 in Introduction.

Lemma 1. The operators (Θ_t) is a semigroup.

Proof. By the relation $C_{t+s} = (C_s \circ \theta_t) C_t$, we have

$$\Theta_{t+s}[F](x) = Q_x[F(\omega_{t+s}) C_{t+s}] = E_x[\, F(\omega_s) \circ \theta_t\, (C_s \circ \theta_t)\, C_t\,]$$

$$= Q_x[\, Q_{X_t}(F(\omega_s) C_s)\, C_t] = \Theta_t \Theta_s[F](x).\ \square$$

Lemma 2. Θ_t is μ-symmetric.

Proof. Notice that the law Q is invariant under the operator ι_T, for any $T > 0$. By the relation $C_T \circ \iota_T = C_T'$, we obtain that, for any $F, G \in L^2(R^d, H, \mu)$,

$$\int \langle G, \Theta_T(F)\rangle \, d\mu = \int \langle G(x), Q_x[F(\omega_T)\, C_T]\rangle \, \mu(dx)$$

$$= Q[\, \langle G(\omega_0), F(\omega_T) C_T\rangle \,] = Q[\, \langle G(\omega_0 \circ \iota_T), F(\omega_T) \circ \iota_T \;\; C_T \circ \iota_T \rangle \,]$$

$$= Q[\, \langle G(\omega_T), F(\omega_0) C_T'\rangle \,] = Q[\, \langle G(\omega_T) C_T, F(\omega_0)\rangle \,] = \int \langle \Theta_T(G), F\rangle \, d\mu. \; \square$$

Lemma 3. $\langle \Theta_t F(x), \Theta_t F(x)\rangle^{1/2} \le e^{\alpha t} \, Q_t[\langle F, F\rangle^{1/2}](x)$, with $\alpha = \alpha(Q_\bullet)$.

Proof. Indeed, for any $G \in H$, we have

$$\langle G, \Theta_t F(x)\rangle = \langle G, Q_x[F(\omega_t) C_t]\rangle \le Q_x[\, \langle G, G\rangle^{1/2} \langle F(\omega_t) C_t, F(\omega_t) C_t\rangle^{1/2} \,]$$

$$\le e^{\alpha t} \, Q_x[\, \langle G, G\rangle^{1/2} \langle F(\omega_t), F(\omega_t)\rangle^{1/2} \,] = \langle G, G\rangle^{1/2} e^{\alpha t} \, Q_t[\langle F, F\rangle^{1/2}](x). \; \square$$

Let us now give a description of the generator of the tengent semigroup Θ_t.

Lemma 4. For any bounded continuous $F, G \in L^2(R^d, H, \mu)$, for any $t \ge 0$,

$$\int \langle G(x), Q_x[\int_0^t F(\omega_s)\, B(ds)\, C_s]\rangle \, \mu(dx) = \int_0^t ds \int \langle \Theta_s G(x), F(x)\, \partial^2 b(dx)\rangle.$$

Proof. It is enough to look at the limit state of the expression:

$$\int \langle G(x), Q_x^n[\int_0^t F(\omega_s)\, \partial^2 b_n(\omega_s)\, C_s^n \, ds]\rangle \, \mu_n(dx). \; \square$$

Proposition 5. For any $u \in C_c^3(R^d)$, $G \in L^2(R^d, H, \mu)$ continuous and bounded, we have

$$\frac{\partial}{\partial t} \int \langle G(x), \Theta_t[\nabla u](x)\rangle \, \mu(dx) = \frac{\partial}{\partial t} \int \langle G(x), Q_x[\nabla u(\omega_t) C_t]\rangle \, \mu(dx)$$

$$= \int \langle \Theta_t G(x), \nabla u(x)\partial b(dx)\rangle + \int \langle \Theta_t G(x), L\nabla u(x)\rangle\, \mu(dx),$$

where L denotes the infinitesimal generator of Q_t.

Proof. Apply Ito's formula and use the preceding lemma. $\square$

Part II. Functional inequalities.

In this part we consider a well-admissible probability measure μ on a separable real riggid Banach space B. We consider the classical Dirichlet space $(\mathcal{E}_\mu, W(\mu))$ on $L^2(B,\mu)$ and its associated semigroup P_t (we use P_t instead of Q_t to denote the semigroup, the latter will be used to denote the Cauchy semigroup associated with P_t). We assume that there is a diffusion process X in B whose transition semigroup coincides with P_t.

Under the assumption:

$$P_t \text{ is a C-semigroup,}$$

we shall study three types of functional inequalities: Poincaré's inequality, logarithmical Sobolev inequality, and Stein-Meyer-Bakry inequalities.

§1. Poincaré's inequality.

Let us begin with Poincaré's inequality and logarithmical Sobolev inequality. We shall see that a C-semigroup behaves like the Ornstein-Uhlenbeck semigroups. We refer to Bakry-Emery [2], Davies [6], Rothaus [27], etc., on this subject. Recall the number $\alpha = \alpha(Q_\bullet)$ introduced in Definition 1,iv.

Theorem 1. Suppose $\alpha < 0$. Then, for any $t > 0$,

$$\int u^2(x)\, \mu(dx) - \int (P_t u)^2(x)\, \mu(dx) \le \frac{1}{-\alpha}\, \mathcal{E}_\mu(u, u),\ u \in W(\mu).$$

Proof. Let us denote the points in $C(R_+,B)$ by ω and the law of X started from μ on $C(R_+,B)$ by P. Notice that the following inequality holds:

$$\langle \tilde{\nabla} u(\omega_t)C_t, \tilde{\nabla} u(\omega_t)C_t\rangle \le e^{2\alpha t}\langle \tilde{\nabla} u(\omega_t), \tilde{\nabla} u(\omega_t)\rangle,\ \forall\, u \in W(\mu),\ \text{P-a.s..}$$

If we denote by L the infinitesimal generator of P_t on $L^2(B,\mu)$, we have:

$$\int u^2(x)\,\mu(dx) - \int (P_t u)^2(x)\,\mu(dx) = -2\int_0^t \int P_s u(x)\, LP_s u(x)\,\mu(dx)\,ds$$

$$= 2\int_0^t ds \int \langle \tilde{\nabla} P_s u(x), \tilde{\nabla} P_s u(x)\rangle\,\mu(dx)$$

$$= 2\int_0^t ds \int \langle P_x[\tilde{\nabla} u(\omega_s) C_s], P_x[\tilde{\nabla} u(\omega_s) C_s]\rangle\,\mu(dx)$$

$$\leq 2\int_0^t ds \int P_x[\langle \tilde{\nabla} u(\omega_s) C_s, \tilde{\nabla} u(\omega_s) C_s\rangle]\,\mu(dx)$$

$$\leq 2\int_0^t ds \int P_x[e^{2\alpha s}\langle \tilde{\nabla} u(\omega_s), \tilde{\nabla} u(\omega_s)\rangle]\,\mu(dx) \leq \frac{1}{-\alpha}\,\mathcal{E}_\mu(u,u). \square$$

Corollary 2. (Spectral gap) Suppose $\alpha < 0$. Let $(E_\lambda, \lambda \geq 0)$ denote the spectral family of the self-adjoint operator L. Let $f \in E_{]0,-\alpha[}[L^2(B,\mu)]$. Then, $f = 0$.

Proof. By Theorem 1, for such a function f, we have

$$\int_{0+}^{(-\alpha)-} \lambda\, d(E_\lambda f,f) = \mathcal{E}_\mu(f,f) \geq (-\alpha)\left(\int_{0+}^{(-\alpha)-} d(E_\lambda f,f) - \int_{0+}^{(-\alpha)-} e^{-2t\lambda} d(E_\lambda f,f)\right)$$

for any $t \geq 0$. Let t tend to infinity, we obtain

$$\int_{0+}^{(-\alpha)-} \lambda\, d(E_\lambda f,f) \geq (-\alpha) \int_{0+}^{(-\alpha)-} d(E_\lambda f,f).$$

But, this can hold only if $f = 0$. $\square$

§2. Logarithmical Sobolev inequality.

Theorem 1. Suppose $\alpha < 0$. Then, for any $t > 0$, for any function $f \in W(\mu)$ such that $f \geq \varepsilon$ for some constant $\varepsilon > 0$, we have

$$\int f(x) \log f(x)\mu(dx) - \int P_t f(x) \log P_t f(x)\mu(dx) \leq \frac{1}{-\alpha} \int \langle \tilde{\nabla} f(x), \tilde{\nabla} f(x)\rangle \frac{1}{f(x)} \mu(dx).$$

Proof. First of all we remark that P_t is μ-invariant. So, we have

$$\int f(x) \log f(x)\, \mu(dx) - \int P_t f(x) \log P_t f(x)\, \mu(dx)$$

$$= - \int_0^t ds \left[\int LP_s f(x) \log P_s f(x)\, \mu(dx) + \int LP_s f(x)\, \mu(dx) \right]$$

$$= - \int_0^t ds \int LP_s f(x) \log P_s f(x)\, \mu(dx) = \int_0^t ds \int \langle \tilde{\nabla} P_s f(x), \tilde{\nabla} P_s f(x)\rangle \frac{1}{P_s f(x)} \mu(dx).$$

We can overestimate:

$$\langle \tilde{\nabla} P_s f(x), \tilde{\nabla} P_s f(x)\rangle^{1/2} = \langle P_x[\tilde{\nabla} f(\omega_s) C_s],\, P_x[\tilde{\nabla} f(\omega_s) C_s]\rangle^{1/2}$$

$$\leq P_x[\langle \tilde{\nabla} f(\omega_s) C_s,\, \tilde{\nabla} f(\omega_s) C_s\rangle^{1/2}] \leq e^{\alpha s} P_x[\langle \tilde{\nabla} f(\omega_s),\, \tilde{\nabla} f(\omega_s)\rangle^{1/2}]$$

$$\leq e^{\alpha s} P_x[\langle \tilde{\nabla} f(\omega_s),\, \tilde{\nabla} f(\omega_s)\rangle \frac{1}{f(\omega_s)}]^{1/2} P_x[f(\omega_s)]^{1/2}.$$

So, we obtain

$$\int f(x) \log f(x)\, \mu(dx) - \int P_t f(x) \log P_t f(x)\, \mu(dx)$$

$$\leq \int_0^t ds \int e^{\alpha s} P_x[\langle \tilde{\nabla} f(\omega_s),\, \tilde{\nabla} f(\omega_s)\rangle \frac{1}{f(\omega_s)}]\, P_x[f(\omega_s)] \frac{1}{P_s f(x)} \mu(dx)$$

$$= \int_0^t ds \int e^{\alpha s} P_x[\langle \tilde{\nabla} f(\omega_s),\, \tilde{\nabla} f(\omega_s)\rangle \frac{1}{f(\omega_s)}]\, \mu(dx)$$

$$\leq \frac{1}{-\alpha} \int \langle \tilde{\nabla} f(x), \tilde{\nabla} f(x) \rangle \frac{1}{f(x)} \mu(dx). \square$$

§3. Stein-Meyer-Bakry inequalities.

Let $\lambda \geq 0$. We define the Cauchy semigroup associated with $e^{-\lambda t}P_t$:

$$Q_t^\lambda = \int_0^\infty m(t,s)\, e^{-\lambda s}\, P_s\, ds, \text{ where } m(t,s) = \frac{t}{2\sqrt{\pi}}\, s^{-3/2} \exp\{-\frac{t^2}{4s}\}.$$

Let C^λ be the infinitesimal generator of Q_t^λ. The domain of C^λ is denoted by $D(C^\lambda)$. When $\lambda = 0$, we denote $C^0 = C$. Let $\tilde{P}_t$ be the tangent semigroup associated with P_t. The Cauchy semigroup associated with $\tilde{P}_t$ and its generator are denoted by respectively $\tilde{Q}_t^\lambda$ and $\tilde{C}^\lambda$.

The Stein-Meyer-Bakry inequalities state the mutual overestimates between the operators $\tilde{\nabla}$ and C^λ for a suitable $\lambda \geq 0$. There are already many studies on Stein-Meyer-Bakry inequalities. We can refer to Stein [32], Meyer [21], [22], [23], [24], Bakry [3], [4], Feyel [8], Pisier [25], Wu [37], Gundy [12], Varopoulos [35], Lohoué [18], Dellacherie-Maisonneuve-Meyer [7], Gundy-Silverstein [10], Gundy-Varopoulos [11], Banuelos [5], Strichartz [33], etc. These studies concern various type of semigroups on different spaces. The motivation for us to write again about Stein-Meyer-Bakry inequality comes from the desire of understanding the papers of Bakry [3] and [4], in the former of which, Bakry had already used implicitely the C-semigroup notion (see also Dellacherie-Maisonneuve-Meyer [7]). We have remarked that, for C-semigroups, explicit formulas exist relating the gradient $\tilde{\nabla}$ with the Cauchy operator C^λ. Using these formulas, we shall prove hereafter that C-semigroups satisfy Stein-Meyer-Bakry inequalities.

When $f(x,t)$ is a function of two variables (x,t), the operators $\tilde{\nabla}$, P_s, Q_s^λ, etc, will operate on the variable x. The resulted functions will be denoted by $\tilde{\nabla} f(x,t)$, $P_s f(x,t)$, $Q_s^\lambda f(x,t)$, etc. We shall use also the operator D which is the differential with respect to the variable t of the function $f(x,t)$. When $f(x)$ is a function in $L^2(B,\mu)$ (resp. in $L^2(B,H,\mu)$), we shall write $f(x,t)$ for the function $(x,t) \to Q_t^\lambda f(x)$ (resp. $(x,t) \to \tilde{Q}_t^\lambda f(x)$) defined on $B \times R_+$ (we shall not write the parameter λ), and we shall call it the <u>extension</u> of f onto $B \times R_+$. For p > 1, we denote the various L^p-norms below by $\|\bullet\|_p$ as well as by N_p.

In what follows, we shall calculate many times integrals and derivatives. We shall not each time prove that they are meaningful, because the technique to do so is routine.

For any $f \in W(\mu)$, we denote $\partial_i f = \langle \tilde{\nabla} f, k_i \rangle$, $i \geq 1$, where (k_i) is the basis of H introduced in Introduction.

As in Meyer [21], we introduce the process (X_t, B_t), where X_t is the B-valued μ-symmetric diffusion associated with P_t, and B_t is a Brownian motion started from a point $a > 0$ such that $\langle B_t, B_t \rangle = 2t$. We set $\tau = \inf\{t; B_t = 0\}$. We denote the law of $(X_t, B_t)_{t \geq 0}$ by E^a.

The corner stone of Stein-Meyer-Bakry inequalities are martingale inequalities. We shall use constantly three of them.

Lemma 1. (Doob's inequality) For any $p > 1$, for any non negative submartingale S,

$$N_p[\sup_{s \geq 0} S_s^p] \leq \frac{p}{p-1} N_p[S_\infty^p].$$

Lemma 2. (vector Burkholder-Davis-Gundy inequality) For any $p > 1$, there is constants c_p and C_p such that, for any sequence of continuous real martingales $(M_i(t); i \geq 1)$,

$$c_p N_p[(\sum_{i \geq 1} M_i^2(\infty))^{1/2}] \leq N_p[(\sum_{i \geq 1} \langle M_i \rangle(\infty))^{1/2}] \leq C_p N_p[(\sum_{i \geq 1} M_i^2(\infty))^{1/2}],$$

here obviously $\langle M_i \rangle$ denotes the increasing process associated with M_i.

Lemma 3. (Lenglart-Lépingle-Pratelli [17], Théorème 3.2) For any $p > 0$, there is a constant c_p such that, for any continuous submartingale $Z = M + A$, where M is a continuous local martingale and A is a continuous increasing process started from zero,

$$E[A_\infty^p] \leq c_p E[\sup_{s \geq 0} |Z_s|^p].$$

There are numerous forms of Stein-Meyer-Bakry inequalities. Let us begin with a general result which holds for any classical Dirichlet form.

Theorem 4. (Without C-semigroup assumption) Let $p > 1$. There are constants c_p such that, for any $f \in W(\mu)$, for any $a > 0$, we have

$$N_p[\langle \tilde{\nabla} f, \tilde{\nabla} f \rangle^{1/2}] \leq c_p N_p[\langle Q_a \tilde{\nabla} f, Q_a \tilde{\nabla} f \rangle^{1/2}] + c_p N_p[Cf],$$

$$N_p[Cf] \leq c_p N_p[Q_a Cf] + c_p N_p[\langle \tilde{\nabla} f, \tilde{\nabla} f \rangle^{1/2}].$$

Proof. Recall the following two inequalities: there are constants C such that, for any vector valued function $h = (h_i)$, with $h_i \in L^p(B,\mu)$, for any $a > 0$, we have firstly

$$N_p[(\int_0^\tau \sum_i (Dh_i(X_s,B_s))^2\, ds)^{1/2}] \le C\, N_p[(\sum_i h_i^2)^{1/2}]$$

$$\le C\, N_p[(\sum_i (Q_a h_i)^2)^{1/2}] + C\, N_p[(\int_0^\tau \sum_i Dh_i(X_s,B_s)^2\, ds)^{1/2}];$$

and secondly

$$N_p[(\int_0^\tau \sum_i \langle \tilde{\nabla} h_i, \tilde{\nabla} h_i\rangle(X_s,B_s)\, ds)^{1/2}] \le C\, N_p[(\sum_i h_i^2)^{1/2}]$$

$$\le C\, N_p[(\sum_i (Q_a h_i)^2)^{1/2}] + C\, N_p[(\int_0^\tau \sum_i \langle \tilde{\nabla} h_i, \tilde{\nabla} h_i\rangle(X_s,B_s)\, ds)^{1/2}].$$

These inequalities have been proved in Bakry [4], when h is scalar valued. But, it is easy to generalize them to the case of vector valued functions, thanks to the corresponding Burkholder-Davis-Gundy inequality.

Based on that remark, the truth of the theorem results immediately from the identity $\tilde{\nabla} Cf(x,t) = D\tilde{\nabla} f(x,t)$, valid for any $f \in D(L)$, the domain of the generator L. For example, to prove the left side inequality of the theorem, applying the first inequality to $\tilde{\nabla} f = (\partial_i f,\ i \ge 1)$, we write

$$N_p[\langle \tilde{\nabla} f, \tilde{\nabla} f\rangle^{1/2}]$$

$$\le C\, N_p[\langle Q_a \tilde{\nabla} f, Q_a \tilde{\nabla} f\rangle^{1/2}] + C\, N_p[(\int_0^\tau \langle D\tilde{\nabla} f, D\tilde{\nabla} f\rangle(X_s,B_s)\, ds)^{1/2}]$$

$$\le C\, N_p[\langle Q_a \tilde{\nabla} f, Q_a \tilde{\nabla} f\rangle^{1/2}] + C\, N_p[(\int_0^\tau \langle \tilde{\nabla} Cf, \tilde{\nabla} Cf\rangle(X_s,B_s)\, ds)^{1/2}]$$

$$\le C\, N_p[\langle Q_a \tilde{\nabla} f, Q_a \tilde{\nabla} f\rangle^{1/2}] + C\, N_p[Cf],$$

where the last step is obtained by applying the second inequalities to the function Cf. $\square$

Remark. The simplicity of the proof of the inequalities is due to the simple form of the "opérateur carré du champ" associated with the classical Dirichlet form.

Remark. If $\alpha < 0$, there is a gap in the spectrum of the generator L (cf. Corollary §1.2). We can hence take limits in the inequalities of Theorem 4, when a tends to infinity. Then, under some boundedness condition and ergodicity, the term $N_p[Q_a Cf]$ tends to zero, while the term $N_p[\langle Q_a\tilde{\nabla}f, Q_a\tilde{\nabla}f\rangle^{1/2}]$ tends to zero if $\mu(B) = \infty$, tends to $\langle E\nabla f, E\nabla f\rangle$ if $\mu(B) < \infty$. In particular cases such as the Ornstein-Ulenbeck semigroup, $\langle E\nabla f, E\nabla f\rangle$ can be easily controled by $\|Cf\|^2$. This provides a proof of Meyer inequality (cf. Meyer [24]).

How to cancel the terms $N_p[Q_a Cf]$ and $N_p[\langle Q_a\tilde{\nabla}f, Q_a\tilde{\nabla}f\rangle^{1/2}]$ from the right hand sides of the inequalities in Theorem 4, if α is not necessarily negative? To answer the question, the following formula, in which we recognize the intervention of the C-process, gives a good starting point.

Lemma 5. Let $\lambda > \alpha\vee 0$. Then, we have the formula:

$$\frac{1}{2}\tilde{\nabla}f(x) = E^{\infty}\Big[\int_0^{\tau} e^{\lambda s}\,\tilde{\nabla}C^{\lambda}f(X_s,B_s)\,(C_s')^{-1}\,dB_s\,C_{\tau}'\,e^{-\lambda\tau}\,\Big|\,X_{\tau}=x\Big],\ f\in D(L),$$

where $E^{\infty} = \lim_{a\uparrow\infty} E^a$.

Proof. Let g be a function in $L^2(B,\mu)$. We consider its extension g onto $B\times R_+$. Recall (cf. Bakry [3]) that, because $\lambda > 0$, the L^p-norm of $Q_t^{\lambda}g$ decreases exponentially to zero, when t tends to infinity. This will justify the convergences of various integrals which will be coming.

The formula of the lemma is the differentiable form of the following one, which is well known when $\lambda = 0$:

$$E^a\Big[\int_0^{\tau} e^{\lambda s}\,g(X_s,B_s)\,dB_s\,e^{-\lambda\tau}\,\Big|\,X_{\tau}=x\Big]$$

$$= E^a\Big[\int_0^{\tau} e^{\lambda s}\,g(X_s,B_s)\,2\,\partial\log m(B_s,\tau-s)\,ds\,e^{-\lambda\tau}\,\Big|\,X_{\tau}=x\Big]$$

obtained by enlarging the filtration $\sigma(B)_t$ by the variable τ (see Jeulin [16]), where ∂ denotes the derivative with respect to the variable whose place is occupied by B_s;

$$= E^a[\int_0^\tau e^{-\lambda(\tau-s)} P_{\tau-s} g(x,B_s)\, 2\, \partial \log m(B_s, \tau - s)\, ds]$$

by the symmetry of the process X;

$$= E[\int_0^{L_a} e^{-\lambda(L_a-s)} P_{L_a-s} g(x,\sqrt{2}Z_{L_a-s})\, 2\, \partial \log m(\sqrt{2}Z_{L_a-s}, L_a - s)\, ds]$$

by the "retournement du temps" (cf. Revuz-Yor [26]), where Z is a 3-dimensional Bessel process started from zero and $L_a = \sup\{t;\, Z_t = a\}$;

$$= E[\int_0^{L_a} e^{-\lambda u} P_u g(x,\sqrt{2}Z_u)\, 2\, \partial \log m(\sqrt{2}Z_u, u)\, du]$$

$$\rightarrow \int_0^\infty du \int_0^\infty dy\, y\, 2\, m(\sqrt{2}y, u)\, e^{-\lambda u} P_u g(x,\sqrt{2}y)\, 2\, \partial \log m(\sqrt{2}y,u)$$

when a tends to the infinity; (for the potential of Z_t, see Revuz-Yor [26])

$$= 4 \int_0^\infty y\, dy \int_0^\infty du\, e^{-\lambda u} P_u g(x,\sqrt{2}y)\, \partial m(\sqrt{2}y,u)$$

$$= 4 \int_0^\infty y\, dy \int_0^\infty du\, e^{-\lambda u} \partial(P_u g(x,\bullet)\, m(\bullet,u))_{\bullet=\sqrt{2}y}$$

$$- 4 \int_0^\infty y\, dy \int_0^\infty du\, e^{-\lambda u} \partial(P_u g(x,\bullet))_{\bullet=\sqrt{2}y}\, m(\sqrt{2}y,u)$$

$$= 4 \int_0^\infty y\, dy\, \partial(Q^\lambda_{\bullet+\bullet} g(x))_{\bullet=\sqrt{2}y} - 4 \int_0^\infty y\, dy \int_0^\infty du\, e^{-\lambda u} P_u C^\lambda g(x,\sqrt{2}y)\, m(\sqrt{2}y,u)$$

$$= 4\int_0^\infty y\, dy\, 2\, C^\lambda Q^\lambda_{2\sqrt{2y}} g(x) - 4\int_0^\infty y\, dy\, Q^\lambda_{2\sqrt{2y}} C^\lambda g(x)$$

$$= 4\int_0^\infty y\, dy\, Q^\lambda_{2\sqrt{2y}} C^\lambda g(x) = \frac{1}{2}(C^\lambda)^{-1} g(x).$$

Replacing g by $C^\lambda f$, we obtain:

$$\frac{1}{2} f(x) = E^\infty[\int_0^\tau e^{\lambda s}\, C^\lambda f(X_s,B_s)\, dB_s\, e^{-\lambda\tau} \,\Big|\, X_\tau = x],\ f \in D(C^\lambda).$$

Now, to prove the lemma, it is enough to take the gradient $\tilde\nabla$ on both sides of this formula when $f \in D(L)$. On the left hand side, we obtain $\frac{1}{2}\,\tilde\nabla f$. To compute the right hand side, we first employ the above technique of enlargement of filtration, next, we use the C-semigroup property, then, we invert the time, finally, we obtain the formula of the lemma. $\square$

Before studying the consequence of the formula in Lemma 5 on a general C-semigroup, let us first try it with the Ornstein-Uhlenbeck process on B (so X is now O-U). In such case, $C_s = e^{-s}$ and $\alpha = -1$. Taking the limit in the formula when λ decreases to zero. We obtain:

$$\frac{1}{2}\,\tilde\nabla f(x) = E^\infty[\int_0^\tau \tilde\nabla C f(X_s,B_s)\, e^s\, dB_s\, e^{-\tau} \,\Big|\, X_\tau = x].$$

(In fact, this formula can be proved directly and very easily. Chronologically, this formula was the germ of that in Lemma 5.) Set $N_t = \int_0^t \tilde\nabla C f(X_s,B_s)\, dB_s$. By integration by parts, we can write:

$$\frac{1}{2}\,\tilde\nabla f(x) = E^\infty[N_\tau - \int_0^\tau N_s\, e^{s-\tau}\, ds \,\Big|\, X_\tau = x].$$

It yields immediately:

$$||\tilde{\nabla}f||_p \le 2\,||N_\tau||_p + 2\,||\,\sup_{s\le\tau}|N_s|\,||_p.$$

To finish the estimate, we first apply the Burkholder-Davis-Gundy inequality to the vector martingale N, then apply the inequalities mentioned in the beginning of the proof of Theorem 4. We conclude $||\tilde{\nabla}f||_p \le c_p\,||Cf||_p$. By duality, we conclude also the inverse inequality: $||Cf||_p \le c_p\,||\tilde{\nabla}f||_p$. The formula given by Lemma 5 provides thus a second proof of Meyer's inequality.

Give up the Ornstein-Uhlenbeck semigroup and consider again our C-semigroup P_t. We notice that the above technique remains applicable, if the C-process has a bounded log-derivative and $\alpha < 0$. We can therefore claim our second form of Stein-Meyer-Bakry inequalities:

Theorem 6. Assume that the C-process has a bounded log-derivative γ and $\alpha < 0$. Then, the following inequality holds:

$$||\tilde{\nabla}f||_p \le (2\,c_p + 2\,\frac{1}{-\alpha}\,c_p\,||\gamma||_\infty)\,||Cf||_p,\ \forall\, f \in W(\mu),$$

where c_p is a martingale inequality constant. An inverse inequality also holds by duality.

Nevertheless, Theorem 6 is not the optimal form of the inequalities, while all power of martingale inequalities has not been exhausted yet. Let us start up off again with the following representation of the norm $||\tilde{\nabla}f||_p$:

$$||\tilde{\nabla}f||_p = 2\,\sup_h E^\infty[\langle \int_0^\tau e^{\lambda s}\,\tilde{\nabla}C^\lambda f(X_s,B_s)\,(C_s')^{-1}\,dB_s\,C_\tau'\,e^{-\lambda\tau},\ h(X_\tau)\rangle\,],$$

where the supremum is taken over the family of h in $L^q(R^d,H,\mu)$ such that $||h||_q = 1$, where q is the conjugate number of p. Let us set $h(x,y) = \tilde{Q}_y^\lambda h(x)$ for such a function h. The function h is related with the martingale: for $t < \tau$,

$$M_t = E[e^{-\lambda\tau}\,h(X_\tau)C_\tau\,|F_t] = e^{-\lambda t}\,\tilde{Q}^\lambda_{B_t}h(X_t)\,C_t = e^{-\lambda t}\,h(X_t,B_t)\,C_t.$$

The martingale M_t, for $t < \tau$, has another expression:

$$M_t = \int_0^t e^{-\lambda s}\,\tilde{C}^\lambda\,h(X_s,B_s)\,C_s\,dB_s + \int_0^t e^{-\lambda s}\sum_i \partial_i h(X_s,B_s)\,C_s\,d\sqrt{2}\beta_t^i,$$

where β^i are independent brownian motions. Using the function h(x,t), we have the following estimation:

$$E^{\infty}[\langle \int_0^{\tau} e^{\lambda s}\, \tilde{\nabla} C^{\lambda} f(X_s,B_s)\,(C_s')^{-1}\, dB_s\, C_{\tau}'\, e^{-\lambda\tau},\, h(X_{\tau})\rangle\,]$$

$$= E^{\infty}[\langle \int_0^{\tau} e^{\lambda s}\, \tilde{\nabla} C^{\lambda} f(X_s,B_s)\,(C_s')^{-1}\, dB_s,\, e^{-\lambda\tau}\, h(X_{\tau})C_{\tau}\rangle\,]$$

$$= E^{\infty}[\int_0^{\tau} \langle e^{\lambda s}\, \tilde{\nabla} C^{\lambda} f(X_s,B_s)\,(C_s')^{-1},\, e^{-\lambda s}\, \tilde{C}^{\lambda}\, h(X_s,B_s)\, C_s\rangle\, 2\, ds\,]$$

$$= 2\, E^{\infty}[\int_0^{\tau} \langle \tilde{\nabla} C^{\lambda} f(X_s,B_s),\, \tilde{C}^{\lambda}\, h(X_s,B_s)\rangle\, ds\,]$$

$$\leq 2\, \|(\int_0^{\tau} \langle \tilde{\nabla} C^{\lambda} f, \tilde{\nabla} C^{\lambda} f\rangle(X_s,B_s)\, ds\,)^{1/2}\|_p\ \|(\int_0^{\tau} \langle \tilde{C}^{\lambda}\, h, \tilde{C}^{\lambda}\, h\rangle(X_s,B_s)\, ds)^{1/2}\|_q.$$

Let us estimate separately the above two norms. We need the following notation. Let λ be a real number. We shall write $\lambda dt \geq -C_t\, dC_t^{-1}$, if, for any H-valued process g_t, we have

$$\int_0^{\infty} (\,\lambda\langle g_t, g_t\rangle\, dt + \langle g_t, g_t C_t\, dC_t^{-1}\rangle\,) \geq 0.$$

Lemma 7. If $\lambda dt \geq -C_t\, dC_t^{-1}$, we have $\|C_t\|^2 \leq e^{2\lambda t}$.

Proof. Let $\varepsilon > 0$. Set $f(t) = (\langle gC_t^{-1}, gC_t^{-1}\rangle + \varepsilon)^{-1}$. Set $g_t = f(t)\, g\, C_t^{-1} 1_{[0,a]}(t)$ in the above definition, where $a > 0$, $g \in H$. We have

$$-2\int_0^a \lambda\, f(t)\, \langle gC_t^{-1}, gC_t^{-1}\rangle\, dt \leq 2\int_0^a f(t)\, \langle gC_t^{-1}, gdC_t^{-1}\rangle$$

$$= \log[\langle gC_a^{-1}, gC_a^{-1}\rangle + \varepsilon] - \log[\langle g,g\rangle + \varepsilon].$$

Taking the limit in the above inequality when ε tends to zero, and replacing g by hC_a, we obatin: $-2\lambda a \le \log\langle h,h\rangle - \log\langle hC_a, hC_a\rangle$. This proves the lemma. □

Remark. This lemma provides another proof of Lemma §3.6, Part I.

Lemma 8. Set $h_t = h(X_t,B_t)$. There is constant c_q such that, for any $\lambda > 0$ such that $\lambda dt \ge -C_t\, dC_t^{-1}$, we have:

$$\|(\int_0^\tau \langle \tilde{C}^\lambda h_s, \tilde{C}^\lambda h_s\rangle ds)^{1/2}\|_q \le c_q \|\langle h(X_\tau),h(X_\tau)\rangle^{1/2}\|_q \le c_q \|h\|_q.$$

Proof. Indeed, by Itô's formula, we have:

$$\begin{aligned} d\langle h_t,h_t\rangle = {} & 2\langle h_t, \tilde{C}^\lambda h_t\rangle dB_t + 2\sum_i \langle h_t, \partial_i h_t\rangle d\sqrt{2}\beta_t^i \\ & + 2\lambda \langle h_t, h_t\rangle dt + 2\langle h_t, h_t C_t\, dC_t^{-1}\rangle \\ & + 2\sum_j \sum_i (\partial_i h_t^j)^2 dt + 2\langle Dh_t, Dh_t\rangle dt. \end{aligned}$$

Remark $Dh_t = \tilde{C}^\lambda h(X_t,B_t)$. Since $\lambda dt \ge -C_t\, dC_t^{-1}$, this formula implies that $\langle h_t,h_t\rangle$ is a submartingale. Applying Lemma 3 to the couple $\langle h_t,h_t\rangle$ and $\int \langle Dh_t, Dh_t\rangle dt$, we obtain:

$$\|(\int_0^\tau \langle \tilde{C}^\lambda h_s, \tilde{C}^\lambda h_s\rangle ds)^{1/2}\|_q \le C\|\sup_{s\le\tau}\langle h_s,h_s\rangle^{1/2}\|_q.$$

In order to replace $\sup_{s\le\tau}\langle h_s,h_s\rangle^{1/2}$ by $\langle h_\tau,h_\tau\rangle^{1/2}$, we shall prove that $\langle h_t,h_t\rangle^{1/2}$ also is a submartingale. Look at the bounded variation part of the semi-martingale $\langle h_t,h_t\rangle^{1/2}$. It is given by:

$$\begin{aligned} & \frac{1}{2}\langle h_t,h_t\rangle^{-1/2} (2\lambda \langle h_t, h_t\rangle dt + 2\langle h_t, h_t C_t\, dC_t^{-1}\rangle) \\ & \quad + \frac{1}{2}\langle h_t,h_t\rangle^{-1/2}\, 2\sum_j \sum_i (\partial_i h_t^j)^2 dt + \frac{1}{2}\langle h_t,h_t\rangle^{-1/2}\, 2\langle Dh_t, Dh_t\rangle dt \\ & \quad\quad - \frac{1}{8}\langle h_t,h_t\rangle^{-3/2}\, 8\, [\langle h_t, \tilde{C}^\lambda h_t\rangle^2 dt + \sum_i \langle h_t, \partial_i h_t\rangle^2 dt]. \end{aligned}$$

The Schwarz inequality yields that this is bigger than

$$\langle h_t,h_t\rangle^{-1/2}\,(\lambda\,\langle h_t, h_t\rangle\, dt + 2\,\langle h_t, h_t C_t\, dC_t^{-1}\rangle)$$

which is non negative. So, $\langle h_t,h_t\rangle^{1/2}$ is a submartingale. Now, by Doob's inequality, we can write:

$$\| \sup_{s\le\tau} \langle h_s,h_s\rangle^{1/2} \|_q \le C\, \| \langle h_\tau,h_\tau\rangle^{1/2} \|_q,$$

which is what we wanted. $\square$

Lemma 9. There is a constant c_p such that:

$$\|(\int_0^\tau \langle \tilde\nabla C^\lambda f, \tilde\nabla C^\lambda f\rangle(X_s,B_s)\, ds\,)^{1/2} \|_p \le c_p \, \| C^\lambda f \|_p, \quad f \in D(L).$$

Proof. Notice that $C^\lambda f(X_t,B_t) = e^{\lambda t}\, E^a[C^\lambda f(X_\tau)\, e^{-\lambda\tau} \,|\, X_t,B_t]$, $t < \tau$. We can now make the same arguments as that used in the proof of Lemma 6, to the submartingale $\left| C^\lambda f(X_t,B_t) \right|$. $\square$

Corollary 10. Set $\alpha' = \inf\{\lambda > 0;\ \lambda dt \ge -\, C_t\, dC_t^{-1}\}$. Then, there is constant c_p such that, for any $f \in W(\mu)$, $\|\tilde\nabla f\|_p \le c_p\, \|C^{\alpha'} f\|_p$.

Proof. We need only to consider $f \in D(L)$. Let $\lambda > 0$ such that $\lambda dt \ge -\, C_t\, dC_t^{-1}$. According to Lemma 7, the formula in Lemma 5 is valid. We can therefore write:

$$\|\tilde\nabla f\|_p = 2 \sup_h E^\infty[\langle \int_0^\tau e^{\lambda s}\, \tilde\nabla C^\lambda f(X_s,B_s)\, (C_s')^{-1}\, dB_s\, C_\tau'\, e^{-\lambda\tau},\ h(X_\tau)\rangle\,]$$

$$\le 4 \sup_h \|(\int_0^\tau \langle \tilde\nabla C^\lambda f, \tilde\nabla C^\lambda f\rangle(X_s,B_s)\, ds\,)^{1/2} \|_p \, \|(\int_0^\tau \langle \tilde C^\lambda h_s,\ \tilde C^\lambda h_s\rangle\, ds)^{1/2} \|_q$$

$$\le C\, \|C^\lambda f\|_p.$$

But the constant in the above inequality is the same for all $\lambda > \alpha'$. The lemma is proved by taking the limit when λ decreases to α'. $\square$

Theorem 11. There is constants $c_p > 0$ such that, for any $f \in W(\mu)$, we have

$$\frac{1}{c_p} \|\tilde{\nabla} f\|_p \leq \|C^{\alpha'} f\|_p \leq (c_p \|\tilde{\nabla} f\|_p + \alpha' \|f\|_p).$$

Proof. The left hand side inequality is proved in the preceding corollary. The right hand side inequality can be proved by duality. $\square$

Remark. We can substitute $\|Cf\|_p + \|f\|_p$ for $\|C^{\alpha'} f\|_p$ (see Bakry [3]).

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