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ON THE EXISTENCE OF DISINTEGRATIONS

by
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ABSTRACT. Whether, with respect to every partition of the unit square, S, consisting of Borel subsets of S, Lebesgue measure on S admits a countably additive disintegration, is undecidable with the usual axioms for set theory. Also reported herein: There are Borel partitions of S with respect to which, Lebesgue measure admits no proper, integrable disintegrations, not even one that is finitely additive.

Section 1. Introduction and Summary.

One is often concerned with the conditional probability of an event, B, given (the occurrence of) an event h, $P(B/h)$, where B ranges over a collection $\mathcal{B}$ of events and h ranges over a collection, π, of exhaustive and pairwise incompatible events. The condition,

$$(1) \qquad P(h/h) = 1 \text{ for all } h \text{ in } \pi,$$

is sometimes, as in the theory of regular conditional distributions, not required. Presumably, the forfeiture of (1), an intuitively necessary condition, has been made in order to accommodate certain requirements of measurability and countable additivity.

In the present paper, which joins earlier ones in the study of the existence of proper disintegrations, Condition (1) is required.

If (1) holds, and if the integral, or expectation, of $P(B/h)$ with respect to a probability measure, Q, on π, has the unconditional probability, $P(B)$, for its value, for all B in a collection $\mathcal{B}$, then P on $\mathcal{B}$ has a *proper disintegration*, a notion more formally defined below. Since the present paper concerns no disintegrations other than those that are proper, "disintegration" will mean "proper disintegration".

Principal interest herein is the case in which P is countably additive and nonatomic, and defined on the sigma-field, $\mathcal{B}$, of all Borel subsets of some uncountable Borel subset, S, of a complete, separable metric space. Since all such

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are Borel isomorphic (see [P, 1967, Theorem 2.12]), it may be better to fix S to be the usual coin-tossing space, the unit square, or the unit interval, and P to be the usual fair-coin distribution, or Lebesgue measure.

If every element of a partition of S is a Borel subset of S, the partition itself is, in this paper, called *Borel*. A disintegration, D, is *conventional* if the partition π is Borel, and if these two additional usual conditions obtain:

(2) The marginal of P on π can serve as the integrating measure, Q;

(3) D is countably additive.

(Some formal definitions are postponed until the next section.)

It is known that there are Borel π for which Lebesgue measure possesses no proper conventional disintegrations. The set of atoms of the tail sigma-field for ordinary coin-tossing measure provides such an example; see [DH, 1983].

Therefore, interest arises in the existence of disintegrations if Condition (2) or (3) (or both) is not required, and the purpose of this paper is to report these two findings:

(4a) There is a Borel partition of the square, S, with respect to which Lebesgue measure possesses no proper disintegration that satisfies (2), not even one that is finitely additive.

(4b) The question whether, for every Borel partition of the square, S, Lebesgue measure possesses a disintegration that satisfies (3), is not decidable with the usual axioms for set theory.

The question whether, when neither (2) nor (3) is required, there exists, for each Borel π, a disintegration of Lebesgue measure, we do not see how to settle.

Introduce the notation, $\mathfrak{P}$, for the set of P that are countably additive and nonatomic. Of course, since all such P are Borel isomorphic, a fact corresponding to (4a) holds for each such P. As it turns out, however, this stronger fact obtains:

(4c) There is a Borel partition of S with respect to which no P in $\mathfrak{P}$, possesses an integrable disintegration.

Some references containing material related to the present study are: [De, 1930, 1972, 1974], [BR, 1963], [Bo, 1969, p. 39, Proposition 13], [BD, 1975, Theorem 2], [D, 1977], [SV, 1979, Theorem 1.1.8], [DH, 1983] and [MR, 1988].

Section 2. Definitions and Notations.

π-Measurability. Always, π designates a partition of S, sometimes Borel, and functions are real-valued, defined on S, and usually bounded. Let π^* designate the set of functions whose restriction to each member h of π is constant, and call such functions *π-measurable*. Plainly, each π-measurable function can be identified with a unique function whose domain is π, and vice versa. The useful convention of identifying a set with its indicator, that is, with that function that is 1 on the set and 0 off the set, is borrowed from de Finetti, and is used herein. Plainly, the π-measurable sets then constitute a sigma-field, indeed a complete Boolean algebra. Equally plainly, a function, f, is π-measurable if, and only if, the inverse image under f of every set is π-measurable. Calling an element of π a *π-fiber*, it is evident that a set is π-measurable if, and only if, it is a union of π-fibers.

Proper. A mapping, κ, of a collection of functions into π^* is proper, at a π-fiber, h, if the value of κf at h depends only on the values of f on h, that is, if f and f' are members of the collection that agree on h, then κf and $\kappa f'$ agree at h. If κ is proper at each h in π, then κ is *proper*.

π-Kernels. A π-proper, κ, defined on a linear space, F, of functions, that includes the constant functions, c, is a *π-kernel* if κ is linear and order-preserving, and normalized by the condition $\kappa(c) = c$ for constants c.

For later reference, it is noted here that if an f in F is π-measurable, then κf equals f. Moreover, even if not in F, if f is π-measurable, it is natural to define κf to be f. There is some convenience in enlarging the scope of κ to the linear space, $F + \pi^*$, of all $f + g$ for f in F and g in π^*, by

$$\begin{aligned} \kappa(f+g) &= \kappa f + \kappa g. \\ &= \kappa f + g. \end{aligned} \tag{5}$$

As is easily verified, this is a valid definition, and the enlarged κ, too, is a π-kernel. Thus enlarged, κ is idempotent, that is, $\kappa(\kappa)$ is κ, or more fully, $\kappa(\kappa(f))$ is $\kappa(f)$.

Expectations. Each π-kernel determines a family of conditional probabilities, or expectations, one for each π-fiber, h, where the expectation corresponding to h is supported by h. If not otherwise stated, an expectation is not required to be countably additive, so an expectation here is a linear functional, Q, defined on a linear space of bounded functions, that satisfies for each f, Qf is at most the least upper bound of f. Here, the expectation corresponding to h has as its domain the linear space, Fh, of fh for f in F, where fh agrees with f on h and is 0 off h, and the expectation assigns to fh the value of κf at h. Because κ is proper, this is indeed a well-defined expectation.

Following de Finetti again, [De, 1972, p. 117], probability measures and their corresponding expectations are designated by the same letter, herein usually by P or Q.

Disintegration. Consider the following two conditions that a pair κ and P may satisfy:

(6a) For all f, if κf is everywhere nonnegative then, for all positive numbers, ε, the P probability that f is less than $-\varepsilon$ is 0.

(6b) There is an expectation, Q, defined on the range of κ such that

$$Pf = Q(\kappa f), \text{ all } f \text{ in } F; \tag{7a}$$

or, more briefly,

(7b) $$P = Q\kappa.$$

Proofs, when straightforward, as for the following lemma, are often omitted.

Lemma 1. *(6a) implies (6b).*

If a π-kernel, κ, satisfies (6b), then P has a ***π-disintegration***, and the pair $[\kappa, Q]$ constitutes a disintegration of P, or, more fully, a ***π-disintegration*** of P. Plainly, for any κ and P, there is at most one Q defined on the range of κ such that $[\kappa, Q]$ is a disintegration of P. So, when Q does exist, it is appropriate, too, to say that κ is a disintegration of P.

Integrable Disintegrations. If (6b) holds and κf is P-integrable, then P and Q agree on κf, for then:

(8) $$P(\kappa f) = Q[\kappa(\kappa(f))] = Q(\kappa f).$$

If, for all f in F, κf is P-integrable, κ is ***P-integrable***. In this case, Q and P agree on the range of κ, and, letting P designate also its restriction to that range, $[\kappa, P]$ is a disintegration of P. Such a disintegration is ***integrable***. Recapitulating, if κ is a P-integrable π-kernel, then

(7*a) $$Pf = P(\kappa f), \text{ all } f \text{ in } F;$$

or, more briefly,

(7*b) $$P = P\kappa.$$

Say that a partition π is ***P-integrable*** if, for some π-kernel, κ, (7*b) holds.

Proposition 0. *For P to have a π-disintegration, it is necessary and sufficient that some extension of P possess an integrable π-disintegration.*

Proof. Suppose that P has a π-disintegration, that is, suppose that (6b) holds. Then, if f and f' are in F, and if $\kappa f = f'$, then $Pf = Pf'$, as is evident by this calculation.

$$Pf = Q(\kappa f) = Q[\kappa(\kappa f)] = P(\kappa f) = Pf'.$$

As is now easily verified, if $\tilde{P}$ assigns to each bounded function of the form $f' + \kappa f$ the value $Pf' + Pf$, then $\tilde{P}$ is a well-defined expectation that extends P. And if $\tilde{\kappa}$ assigns to $f' + \kappa f$ the function $\kappa f' + \kappa f$, then $\tilde{\kappa}$ is a well-defined π-kernel that extends κ. It is then easily verified that for $g = f' + \kappa f$, $\tilde{P}g = \tilde{P}(\tilde{\kappa}g)$, that is, $\tilde{P}$ possesses an integrable disintegration. The converse is immediate from this easily verified fact.

Fact. *If $\tilde{P}$ is an extension of P, and $\tilde{P}$ has a π-disintegration, then so does P.*

Proof. Express $\tilde{P}$ as $\tilde{Q}\tilde{\kappa}$. Then let κ be the restriction of $\tilde{\kappa}$ to the domain of P, and let Q be the restriction of $\tilde{Q}$ to the range of κ. It is then a triviality to verify that $P = Q\kappa$.

Section 3. Universally Nonintegrable Partitions.

Theorem 1. *There exist Borel partitions with respect to which no nonatomic, countably additive probability, P, possesses an integrable disintegration.*

The proof of Theorem 1 requires some preliminaries.

Call π *connective* if, for any two disjoint uncountable Borel sets, there is a π-fiber that intersects each of them. Call π *binary* if each of its fibers has at most two elements. A significant step towards the proof of Theorem 1 is this Proposition.

Proposition 1. *There exist binary π that are connective.*

The proof of Proposition 1 is of a type that has frequently been used, at least since Felix Bernstein, [B, 1907].

Proof of Proposition 1. Index the pairs p of uncountable Borel sets by the ordinals less than the minimal ordinal whose cardinality is the continuum, and let α and β designate such ordinals. Let D be an enumerably infinite subset of S. Suppose that for some β, and for each $\alpha < \beta$, there is an h_α consisting of two elements, that satisfies: h_α is a subset of the union of the pair, p_α; h_α intersects each element of p_α, and, for the $\alpha < \beta$, the h_α are disjoint, and also disjoint from D. As is well-known, and as follows from [P, 1967, Theorem 2.8], each uncountable Borel set has the continuum as its cardinality, the process can continue, and the h_α become defined for all α less than the continuum. The subset of S not covered by the h_α, say, V is infinite, for it includes D. Partition V arbitrarily into sets, each of which has two elements, and let π consist of this partition together with the h_α. Plainly, such a π satisfies the lemma. □

[As is evident, the proof of Proposition 1 made use of the axiom of choice. The question arises whether it may be possible to find a proof that does not rely on that axiom. We believe that the answer is negative, and that that can be seen to follow from certain work of Solovay [S, 1970]. Also, from what Dellacherie has kindly told us, a negative answer perhaps follows from certain work of Sierpinski published in Fundamenta, but we do not know of a precise reference.]

In the interests of brevity of exposition introduce a definition. If no nonatomic, countably additive, probability, P, on S, possesses an integrable π-disintegration, call π universally nonintegrable. So, Theorem 1 asserts the existence of universally nonintegrable partitions.

Proposition 2. *Binary connective π are universally nonintegrable.*

Plainly, binary π are Borel, so Theorem 1 follows immediately from Propositions 1 and 2.

Preliminary to the proof of Proposition 2, call π *ubiquitous* if every uncountable Borel set includes a π-fiber. Of course, Proposition 2 follows from these two facts:

Proposition 2.1. *Binary connective π are ubiquitous.*

Proposition 2.2. *If π is both connective and ubiquitous, then it is universally nonintegrable.*

Proof of Proposition 2.1. Let B be an uncountable Borel set. Let U and V be disjoint, uncountable Borel subsets of B, which of course exist. Since π is connective, there is a π-fiber, h, that intersects U and V. Since every fiber has at most two elements, this h has two elements, and is a subset of the union of U and V, and hence of B. □

The next goal, a proof of Proposition 2.2, requires some preparation. First, two definitions: A subset of S is *thin*, if each of its Borel subsets is countable. A real-valued function, g, defined on S is *unwavering* if, there is a constant, c, such that, for each countably additive, nonatomic, P, for which g is integrable, g assumes the value c with P-probability 1.

Lemma 2. *For a set to be thin it is necessary and sufficient that it have measure zero for every countably additive, nonatomic P for which it is measurable.*

Proof. Each countably additive probability, P, on a complete separable metric space, is tight, see [P, 1967, Theorem 3.2]. Consequently, each P-measurable set A of positive P probability includes a compact and, therefore, Borel, K, of positive probability. Since P is nonatomic, K is uncountable. So K, and hence A, is not thin. Summarizing, the condition is necessary. To establish sufficiency, suppose that A is not thin, that is, that it includes an uncountable Borel B. As is well known, see, for example, [P, 1967, Theorem 2.8], such a B includes a subset, K, homeomorphic to the usual coin-tossing space. Therefore, some countably additive probability, P, is supported by K, and hence by B and by A. □

Lemma 3. *Each of the following conditions on π implies its successors.*

(a) *π is connective.*

(b) *For every π-measurable set, either it, or its complement, is thin.*

(c) *Suppose that V is a π-measurable set. Then, either V is thin and (therefore) a P-null set for all P in $\mathfrak{P}$ for which it is measurable, or its complement is.*

(d) *Every real-valued, π-measurable function is unwavering.*

Proof. The arguments that (a) implies (b) and that (b) implies (c) are straightforward and omitted. So suppose (c), and let g be real-valued and π-measurable. It may be supposed that, for some P in $\mathfrak{P}$, g is P-integrable, for otherwise, g is clearly unwavering. Fix a positive number ε, and partition the real line into half-closed intervals, I, of length ε. The inverse image of I under g, say V, is both π-measurable and P-measurable. By (c), either it is thin and $PV = 0$, or its complement is thin and $PV = 1$. Since the I are disjoint, so are the V. Therefore $PV = 1$ for at most one V. Since P is countably additive, PV cannot be 0 for all V. So the complement of V is thin for precisely one V, and, for that V, $PV = 1$. Consider the I corresponding to that V and label it $I(\varepsilon)$. It is now routine to let ε be of the form $1/k$ where k is a power of 2, and to verify that the corresponding sequence of I have as their intersection a single number c. So g is unwavering, and (d) holds. □

Remark 1. Were it useful for the main purposes of this paper, Lemma 3 could have been stated in a stronger form. For each of the four conditions are equivalent to the others. To verify this, it suffices to verify that if (a) does not hold, neither does (d). So, suppose that (a) does not hold. Then there exist disjoint, uncountable, Borel sets B and C with the property that each π-fiber that intersects B is disjoint from C. So the complement, N, of the smallest π-measurable set, M, that includes B, includes C. There is a P in $\mathfrak{P}$ for which PB and PC are $1/2$. For such a P, M and N are P-measurable, and the π-measurable function, g, that is 1 on M and -1 off M is P-measurable. Plainly, g is not unwavering. So, (d) does not hold.

Lemma 4. *If π is connective, then, for all π-kernels, κ, and bounded, Borel f, κf is unwavering.*

Proof. Apply Lemma 3.

Turn now to the property of being ubiquitous.

Let the *π-interior* of a set, C, be the largest π-measurable set included in it, and designate it by πiC. It is obvious that $C \backslash \pi iC$ includes no π fibers.

Lemma 5. *Suppose π is ubiquitous, C is a Borel set, and P is in $\mathfrak{P}$. Then, if the π-interior of C is a P-null set, so is C.*

Proof. Express C as the union of two sets, its π-interior and the remainder of C, say, D. Since, both C and its π-interior are P-measurable, so is D. Since D includes no π-fibers and π is ubiquitous, D includes no uncountable Borel set. Therefore, D is a P-null set. Since C is the union of two P-null sets, it, too, is P-null. □

Lemma 6. *Suppose π is connective and ubiquitous. Suppose, too, κ is a π-kernel, and f is a bounded, real-valued, Borel function. Then, there is a constant c such that, for all P in $\mathfrak{P}$ for which κf is P-integrable, $f = c$ with P-probability 1.*

Proof. By Lemma 4, there is a c such that, for all described P, $\kappa f = c$ on a set of P-probability 1. Let ε be a positive real number, and let C be the event that f is at least $c + \varepsilon$. Then, on any π-fiber included in C, and, therefore, on the π-interior of C, κf is at least $c + \varepsilon$. So, the π-interior of C is a P-null set. Then, by Lemma 5, C, too, is a P-null event. Likewise, so is the event that f is at most $c - \varepsilon$. Plainly, since P is countably additive, $f = c$, with P-probability 1. □

Since Proposition 2.2 is an immediate consequence of Lemma 6, Proposition 2 has been proven. Therefore, the proof of Theorem 1 is complete.

This special consequence of Theorem 1 is recorded here.

Corollary 1. *There is a Borel partition of the unit interval with respect to which Lebesgue measure possesses no integrable disintegration, not even one that is finitely additive.*

Section 4. Countably Additive Disintegrations.

There is more than one notion for an expectation, E, on a linear space, F, to be countably additive; see [DH, 1984]. Herein, F includes only Borel functionss, and attention is restricted to the usual, and strongest notion in which E is the restriction to F of the usual L_1-space of some countably additive probability measure on $\mathcal{B}$. A kernel, κ, is *countably additive* if, for each h, the corresponding expectation is countably additive. A disintegration is *countably additive* if both κ and Q are countably additive.

Notice that, as defined, countably additive disintegrations need not be conventional. For a disintegration of P to be *conventional*, in addition to being countably additive, it is required to be P-integrable.

The purpose of this section is to prove:

Theorem 2. *The assertion that, with respect to every Borel partition, π, of the unit interval, Lebesgue measure possesses a countably additive disintegration, is undecidable with the usual axioms for set theory.*

The usual set of axioms of set theory, designated by ZFC, are the Zermelo-Fraenkel axioms, together with the axiom of choice.

Of course, the undecidability of an assertion is equivalent to the consistency of it, as well as of its negation, with ZFC. So Theorem 1 is equivalent to the conjunction of two propositions, the first of which is:

Proposition 3. *It is consistent with the usual axioms of set theory that Lebesgue measure on the unit interval possesses a countably additive π-disintegration for every Borel π.*

Recall two notions: If K is a collection of disjoint non-empty sets, then a *K-selection* is a set included in their union that has a single point in common with each of the members of K. In particular, a π-selection is a subset, V, of S that contains one, and only one, point of each π-fiber. And the *π-saturation* of a set, C, is the smallest π-measurable set that includes C.

Lemma 7. *Each of the following conditions on P and π implies its successors.*

(i) *If a set of π-fibers has cardinality less than the continuum, then each Borel subset of its union is a P-null set.*

(ii) *There is a π-selection, V, of outer P-probability 1.*

(iii) *There exists a countably additive, π-disintegration of P.*

Proof. (i) $\rightarrow$ (ii). Assign to each ordinal α of cardinality less than the continuum a Borel set, B_α, of positive P-probability so that all Borel sets of positive P-probability are listed. Fix α, and suppose that for each $\beta < \alpha$, there is assigned a point x_β in B_β, and let h_β be the π-fiber containing x_β. Plainly, the set of these fibers for $\beta < \alpha$ has cardinality less than the continuum. So, by (i), B_α contains a point, x_α, not in any fiber, h_β, for $\beta < \alpha$. Let V be the union of the set of these x_α with any selection from the set of fibers complementary to the set of h_α. Plainly, V satisfies (ii).

(ii) $\rightarrow$ (iii). Let κ be the π-kernel that assigns to the Borel set, B, the indicator of the π-saturation of VB, the intersection of V with B. Equivalently, κ associates to the π-fiber, h, the one-point dirac delta measure at the singleton Vh. Plainly, κ satisfies (6a) for V has outer measure 1. Hence, by Lemma 1, (6b) holds. Verify that the set of π-saturations of VB, B Borel, is a sigma-field, say $\mathcal{V}$, and each κf is measurable with respect to that sigma-field. There remains only to verify that Q is countably additive on $\mathcal{V}$. For this purpose, verify that $\mathcal{V}$ is isomorphic to the sigma-field, $\mathcal{W}$, of subsets of V of the form VB (which holds for any selection V). To conclude that Q is countably additive, one need only observe that $Q(\kappa(B)) = PB = P^*(VB)$ (where P^* denotes P-outer measure), for V has outer measure 1. For this shows that Q is isomorphic to the restriction of the outer measure P^* to $\mathcal{W}$, which, as is well-known, is countably additive.

Proof of Proposition 3. As is easily verified, it suffices to consider π's each of whose elements is a Borel set of Lebesgue measure zero. The continuum hypothesis then implies (i) of Lemma 7. For the union of countably many sets of measure zero has measure zero. Therefore, (i) is consistent with the usual axioms of set theory, and, in view of Lemma 7, so is (iii). □

Remark 2. Proposition 3 has wider validity than asserted. For the argument works for all countably additive P. Furthermore, all restrictions on the nature of π can be removed, at the expense of some complication of the proof.

Remark 3. The role played by the continuum hypothesis in the proof of Proposition 3 could have been played by a weaker axiom, an axiom known as Martin's axiom. For this axiom, too, is strong enough to imply (i) of Lemma 7. One formulation of Martin's axiom is: If a compact Hausdorff space, H, admits no disjoint collection of nonempty open sets, other than countable collections, then the union of fewer than a continuum of closed subsets of H, each of which has no interior, has no interior. (See [K, 1983, Theorem 3.4, p. 65].)

Proposition 4. *It is consistent with the usual axioms of set theory that there be a Borel partition of the unit interval with respect to which Lebesgue measure has no countably additive disintegration.*

Several lemmas are needed.

Lemma 8. *For any π and any countably additive π-kernel, κ, the sigma-field generated by the functions κB for B in $\mathcal{B}$, say $\mathcal{M}$ ($\mathcal{M}$ for marginal sigma-field) is countably generated. Moreover, π is the set of atoms of $\mathcal{M}$.*

Proof. By the monotone class argument, the image under κ of a countable boolean algebra that generates $\mathcal{B}$ generates $\mathcal{M}$, so $\mathcal{M}$ is countably generated. To see that π is the set of atoms of $\mathcal{M}$, notice first that π is a subset of $\mathcal{M}$, for $\kappa h = h$ for all h in π. Each π-fiber is an $\mathcal{M}$-atom, for none of its proper subsets is a member of $\mathcal{M}$. There are no other $\mathcal{M}$-atoms, for the union of these $\mathcal{M}$-atoms is the entire space, S. □

Recall certain terminology and facts: An *atom* of a sigma-field, $\mathcal{U}$, is a minimal element of $\mathcal{U}$; an atom of a probability, P, on $\mathcal{U}$, is a set of positive measure that has no subsets of smaller positive measure. Let α be the sum of the measures of the atoms. If α is 1, the measure is atomic, and, otherwise, P is a unique convex combination of an atomic and nonatomic probability. The next lemma is well-known.

Lemma 9. *Suppose that A is an atom for a countably additive probability measure, P, defined on a sigma-field, $\mathcal{U}$. Then, if $\mathcal{U}$ is countably generated, A is represented by an atom of $\mathcal{U}$.*

For the convenience of the reader, a less well-known fact in the literature [GP, 1984, Theorem 9.2] is reformulated as the next lemma, and a proof is provided. A definition facilitates the formulation.

A cardinal number is *small* if every set of reals of that cardinality has Lebesgue measure 0, or, as is equivalent, no set of positive outer Lebesgue measure is of that cardinality.

Lemma 10. *Every countably additive, finite measure, Q, defined on a sigma-field of subsets of a set, H, of small cardinality, is atomic. Consequently, only the Q that vanishes identically has no atoms.*

Proof. What must be seen is that the nonatomic part of Q vanishes. If it did not, then, by renorming that part, and changing notation, it may be assumed that Q itself is a nonatomic probability measure. Then there exist a sequence of independent events of probability 1/2. The indicators of these events provide a sequence of independent zero-one valued functions, which determine a mapping, φ, of the probability space, H, into fair coin-tossing space. Let C be any Borel subset of coin-tossing space that covers the range of φ. Plainly, its inverse image under φ, being H, certainly has measure 1. Since φ is measure-preserving, C has probability 1 (for the fair coin-tossing measure). So the range of φ has outer probability 1. Therefore, the range is not of small cardinality, so H certainly is not of small cardinality. This contradicts the hypothesis. □

Lemma 11. *Suppose that the cardinality of a partition, π, is small, and P is countably additive, with or without atoms, and has a countably additive π-disintegration. Then the sum of the P-probabilities of the π-fibers is 1.*

Proof. For any κ, $\kappa h = h$. So $Qh = Q\kappa h = Ph$, and it suffices to verify that the sum of the Qh is 1. By Lemmas 8 and 9, this sum is the same as the sum of QA over all Q-atoms A. Since Q is, in effect, defined on a sigma-field of subsets of π, and π is of small cardinality by hypothesis, Lemma 10 applies to complete the proof.

Lemma 11 has a corollary.

Corollary 2. *If π is a partition of small cardinality all of whose fibers are of P-probability 0, then P possesses no countably additive π-disintegrations.*

Proof Proposition 4. In the Cohen model for the negation of the continuum hypothesis, there is a set π, of small cardinality, whose members are disjoint Borel null subsets of the unit interval, I, that covers I, as is proven in [K, 1984]. Therefore, the existence of such π is indeed consistent with the usual axioms for set theory. This, together with Corollary 2, completes the proof.

Since Theorem 2 is nothing other than the conjunction of Propositions 3 and 4, its proof is complete.

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