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On quantum extensions of the Azéma martingale semigroup

by A.M. Chebotarev and F. Fagnola

1. Introduction

In this note we study some quantum extensions of classical Markovian semigroups related to the Azéma martingales with parameter β ($\beta \neq 0$, $\beta \neq -1$) (see [1], [5], [6]). The formal infinitesimal generator given by

$$(\mathcal{L}_0 f)(x) = (\beta x)^{-2} \left(f(cx) - f(x) - \beta x f'(x)\right)$$

on bounded smooth functions f can be written formally as follows (see [9])

$$\mathcal{L}(m_f) = G m_f + L^* m_f L + m_f G^*$$

where m_f denotes the multiplication operator by f, the operator G is the infinitesimal generator of a strongly continuous contraction semigroup on $L^2(\mathbb{R};\mathbb{C})$ (see Section 2) and L is related to G by the formal condition $G + G^* + L^*L = 0$. The associated minimal quantum dynamical semigroup, can be easily constructed, for example as in [2], [3], [4], [8]. We show that this semigroup is conservative if $\beta < \beta_*$ and it is not if $\beta > \beta_*$ where β_* is the unique solution of the equation

$$\exp(\beta) + \beta + 1 = 0.$$

Therefore it is a natural conjecture that the minimal quantum dynamical semigroup is a ultraweakly continuous extension to $\mathcal{B}(h)$ of the Azéma martingale semigroup when $\beta < \beta_*$. However we can not prove this fact because the characterisation of the classical infinitesimal generator is not known. The above quantum dynamical semigroup is not such an extension when $\beta > \beta_*$ because the corresponding classical Markovian semigroup is identity preserving.

We were not able to study the critical case $\beta = \beta_*$ although it seems reasonable that conservativity holds also in this case. In fact, as shown by Emery in [5], the Azéma martingale with parameter β starting from $x \neq 0$ can hit 0 in finite time only if $\beta > \beta_*$. The operators G and L we consider are singular at the point 0, hence, in this case, boundary conditions on G at 0 should appear to describe the behaviour of the process.

The cases when $\beta < \beta_*$ and $\beta > \beta_*$ are studied in Section 3 by checking a necessary and sufficient condition obtained in [2]. In Section 5 we apply a sufficient condition for conservativity obtained in [3]. This condition yields the previous result when $\beta \leq -1.5$; since $\beta_* = -1.278...$, it is quite "close" to the necessary and sufficient one.

2. Notation and preliminary results

Let β be a fixed real number with $\beta \neq -1$, $\beta \neq 0$ and let $c = \beta+1$. Let $h = L^2(\mathbb{R};\mathbb{C})$ and denote by $\mathcal{B}(h)$ the *-algebra of all bounded operators on h. Let us consider the strongly continuous contraction semigroup P on h defined by

$$(P(t)u)(x) = \begin{cases} \left(1-\frac{2t}{\beta x^2}\right)^{(1-\beta)/4\beta} u\left(x\left(1-\frac{2t}{\beta x^2}\right)^{1/2}\right) & \text{if } 1-\frac{2t}{\beta x^2} > 0 \\ 0 & \text{if } 1-\frac{2t}{\beta x^2} \leq 0 \end{cases}$$

The dual semigroup P^* can be easily computed

$$(P^*(t)u)(x) = \begin{cases} \left(1+\frac{2t}{\beta x^2}\right)^{-(1+\beta)/4\beta} u\left(x\left(1+\frac{2t}{\beta x^2}\right)^{1/2}\right) & \text{if } 1+\frac{2t}{\beta x^2} > 0 \\ 0 & \text{if } 1+\frac{2t}{\beta x^2} \leq 0 \end{cases}$$

Let D_0 be the linear manifold of infinitely differentiable functions with compact support vanishing in a neighbourhood of 0. The infinitesimal generators G and G^* of the semigroups P and P^* satisfy

$$Gu(x) = -\frac{1}{\beta x}u'(x) + \frac{\beta-1}{2\beta^2x^2}u(x), \qquad G^*u(x) = \frac{1}{\beta x}u'(x) - \frac{\beta+1}{2\beta^2x^2}u(x)$$

for all $u \in D_0$. In fact P has been obtained integrating the first order partial differential equation

$$\frac{\partial w(t,x)}{\partial t} = -\frac{1}{\beta x}\frac{\partial w(t,x)}{\partial x} + \frac{\beta-1}{2\beta^2x^2}w(t,x)$$

by the characteristics method. Consider the operator M on h defined by

$$D(M) = \left\{ u \in h \mid x^{-1}u(x) \in h \right\}, \qquad Mu(x) = (\beta x)^{-1}u(x).$$

and let S be the unitary operator on h

$$Su(x) = |c|^{-1/2}u(c^{-1}x).$$

The form $\mathcal{L}$ on h given by

$$\langle v, \mathcal{L}(f)u\rangle = \langle G^*v, fu\rangle + \langle SMv, fSMu\rangle + \langle v, fG^*u\rangle$$

for all $u, v \in D_0$, transforms $f \in D_0$ into the multiplication operator on h by

$$(\mathcal{L}f)(x) = (\beta x)^{-2}\left(f(cx) - f(x) - \beta x f'(x)\right).$$

Thus the restriction of $\mathcal{L}$ to D_0 coincides with the restriction to D_0 of the infinitesimal generator of an Azéma-Emery martingale with parameter β (see [5]).

The domain D_0, however, may be too small for the operators G and G^*. In fact it can be shown that D_0 is a core for G if and only if $\beta \geq -1/2$ and is a core for G^* if and only if $\beta \leq 1/2$. The domains of G and G^* however can be described as the range of the resolvent operators $R(1;G) = (1-G)^{-1}$ and $R(1;G^*) = (1-G^*)^{-1}$.

Proposition 2.1. *For all $u \in h$ define a function u_0 by*

$$u_0(x) = \begin{cases} \int_x^{\mathrm{sgn}(x)\infty} \exp(-\beta y^2/2)|y|^{-(\beta+1)/2\beta} y u(y) dy & \text{if } \beta > 0 \\ -\int_0^x \exp(-\beta y^2/2)|y|^{-(\beta+1)/2\beta} y u(y) dy & \text{if } \beta < 0 \end{cases}$$

Then the operator $R(1; G^*)$ is given by

$$\left(R(1; G^*)u\right)(x) = \beta |x|^{(\beta+1)/2\beta} \exp(\beta x^2/2) u_0(x). \tag{2.1}$$

Proof. For all $u, v \in h$ we have

$$\langle v, R(1; G^*)u\rangle = \int_{\mathbb{R}} v(x)dx \int_0^\infty \mathrm{e}^{-t}\left(1 + \frac{2t}{\beta x^2}\right)^{-(\beta+1)/4\beta} u\left(x\sqrt{1 + \frac{2t}{\beta x^2}}\right) dt.$$

By the change of variables $y = x\sqrt{1 + 2t/\beta x^2}$ in the integral with respect to t we obtain the representation formula (2.1). □

Proposition 2.2. *The domain of the operator M contains the domain of the operator G^*. Moreover the necessary condition for $\mathcal{T}$ to be conservative,*

$$\langle G^* v, u\rangle + \langle SMv, SMu\rangle + \langle v, G^* u\rangle = 0 \tag{2.2}$$

for all vectors u, v in the domain of G^*, is fulfilled.

Proof. Clearly, to establish the inclusion $D(G^*) \subset D(M)$, it suffices to show that, for all $u \in h$, the integral

$$\int_{\mathbb{R}} x^{-2} \left|\left(R(1; G^*)u\right)(x)\right|^2 dx \tag{2.3}$$

is convergent. To this end, let us first fix $r \in (0, 1)$ and denote by $I(r)$ the set $(-r^{-1}, -r) \cup (r, r^{-1})$. Integrating by parts we have

$$\begin{aligned} \int_{I(r)} (\beta x)^{-2} \left|\left(R(1; G^*)u\right)(x)\right|^2 dx &= \int_{I(r)} (\beta x)^{-1}|x|^{1/\beta} \cdot \beta \mathrm{sgn}(x) \exp(\beta x^2)|u_0(x)|^2 dx \\ &\doteq \beta r^{-1/\beta} \exp(\beta r^{-2}) \left(|u_0(r^{-1})|^2 - |u_0(-r^{-1})|^2\right) \\ &+ \beta r^{1/\beta} \exp(\beta r^2) \left(|u_0(-r)|^2 - |u_0(r)|^2\right) \\ &- 2\int_{I(r)} \beta^2 |x|^{1+1/\beta} \exp(\beta x^2)|u_0(x)|^2 dx \\ &+ 2\Re e \int_{I(r)} \beta |x|^{(1+\beta)/2\beta} \exp(\beta x^2/2) \bar{u}(x) u_0(x) dx \end{aligned}$$

The first two terms vanish as r tends to zero. In fact consider, for example, the case $\beta > 0$, then, using the Schwartz inequality, we can write the estimate

$$r^{1/\beta} \exp(\beta r^2)|u_0(r)|^2 \le \|u\|^2 \exp(\beta r^2) r^{1/\beta} \int_r^\infty \exp(-\beta y^2) y^{1-1/\beta} dy. \tag{2.4}$$

Clearly, when $\beta > 1/2$, the right-hand side integral is bounded, therefore (2.4) vanishes as r tends to 0. On the other hand, when $\beta \in (0, 1/2]$, by the De L'Hôpital rule, we have

$$\lim_{r\to 0+} r^{1/\beta} \int_r^\infty \exp(-\beta y^2) y^{1-1/\beta} dy = \lim_{r\to 0+} \beta r^{1+1/\beta} \exp(\beta r^2) r^{1-1/\beta} = 0.$$

In a similar way one can compute the other limits and show that the first two terms vanish. The third and fourth term clearly converge to

$$-2\left\|R(1;G^*)u\right\|^2, \qquad 2\Re e\,\langle u, R(1;G^*)u\rangle$$

respectively. Therefore the integral (2.3) is convergent. Moreover, by the identity $R(1;G^*) - I = G^* R(1;G^*)$, we have

$$\begin{aligned}\left\|MR(1;G^*)u\right\|^2 &= -2\left\|R(1;G^*)u\right\|^2 + 2\Re e\,\langle u, R(1;G^*)u\rangle \\ &= -2\Re e\,\langle R(1;G^*)u, G^* R(1;G^*)u\rangle\end{aligned}$$

Therefore we proved also the identity (2.2), with $v = u$, because the operator S is unitary. The proof for arbitrary v, u is the same. □

Having found a Lindblad form for the infinitesimal generator of the classical process we investigate whether the corresponding minimal quantum dynamical semigroup (abbreviated to *m.q.d.s.* in the sequel) on $\mathcal{B}(h)$ is identity preserving i.e. conservative. Recall that the m.q.d.s. $\mathcal{T}$ is the ultraweakly continuous semigroup on $\mathcal{B}(h)$ defined as follows (see [2], [3], [4], [8]). For all positive element X of $\mathcal{B}(h)$, let us consider the increasing sequence

$$\begin{aligned}\left\langle u, \mathcal{T}_t^{(0)}(X)u\right\rangle &= \langle P^*(t)u, XP^*(t)u\rangle \\ \left\langle u, \mathcal{T}_t^{(n+1)}(X)u\right\rangle &= \langle P^*(t)u, XP^*(t)u\rangle \\ &\quad + \int_0^t \left\langle SMP^*(t-s)u, \mathcal{T}_s^{(n)}(X)SMP^*(t-s)u\right\rangle ds.\end{aligned}$$

The bounded operator $\mathcal{T}_t(X)$ is given by

$$\mathcal{T}_t(X) = \sup_{n\geq 0} \mathcal{T}_t^{(n)}(X).$$

Proposition 2.3. *The abelian subalgebra $L^\infty(\mathbb{R};\mathbb{C})$ of $\mathcal{B}(h)$ is invariant for the m.q.d.s. $\mathcal{T}$.*

Proof. In fact, for every $X \in L^\infty(\mathbb{R};\mathbb{C})$, a straightforward computation shows that

$$\mathcal{T}_t^{(n)}(X) \in L^\infty(\mathbb{R};\mathbb{R})$$

for all $t \geq 0$ and all integer $n \geq 0$. □

Let β_* be the unique solution of the equation

$$\exp(\beta) + \beta + 1 = 0.$$

It is easy to check the inequality

$$-1.2785 < \beta_* < -1.2784.$$

In the following sections we shall prove the

Theorem 2.4. *The m.q.d.s. is conservative if $\beta < \beta_*$ and is not conservative if $\beta > \beta_*$.*

We recall the necessary and sufficient condition for the m.q.d.s. to be conservative obtained in [2]. Let $\mathcal{Q} : \mathcal{B}(h) \to \mathcal{B}(h)$ be the normal and monotone map defined by

$$\langle v, \mathcal{Q}(X)u\rangle = \int_0^\infty \mathrm{e}^{-t} \langle SMP^*(t)v, XSMP^*(t)u\rangle \, dt.$$

Theorem 2.5. *Let G^*, M and S be the above defined operators. The following conditions are equivalent:*

i) the semigroup $\mathcal{T}$ is conservative,

ii) the sequence $(\mathcal{Q}^n(I))_{n\geq 1}$ converges strongly to 0,

iii) the equation $\mathcal{L}(X) = X$ has no nonzero positive solution in $\mathcal{B}(h)$.

We refer to [7] Th. 3.3, Prop. 3.5, 3.6 for the proof. The technical condition (B) used there can always be assumed without loss of generality as shown in [3] Lemma 2.4.

3. The case $\beta < \beta_*$

We shall check the condition *ii)* of Theorem 2.5. As a first step we establish a useful formula.

Lemma 3.1. *Let f be a positive bounded measurable function on $\mathbb{R}$. Identify f with the corresponding multiplication operator. The operator $\mathcal{Q}(f)$ agrees with the multiplication operator by the positive measurable function*

$$(\mathcal{Q}(f))(y) = (-2\beta)^{-1} \int_0^\infty ds \exp\left(\beta y^2 s/2\right) (1+s)^{-1+1/(2\beta)} f\left(cy(1+s)^{1/2}\right) \quad (3.1)$$

Proof. Let u be a smooth function with compact support contained in $\mathbb{R} - \{0\}$. Denote by $q(\cdot,\cdot)$ and $p(\cdot,\cdot)$ the functions

$$q(t,y) = \left(1 - 2t/(\beta y^2)\right)^{1/2}, \qquad p(t,x) = \begin{cases} (1 + 2t/(\beta x^2))^{1/2} & \text{if } x^2 > -2t/\beta \\ 0 & \text{if } x^2 \leq -2t/\beta. \end{cases}$$

A straightforward computation yields

$$\begin{aligned} \langle u, \mathcal{Q}(f)u\rangle &= \int_0^\infty \mathrm{e}^{-t} \langle MP^*(t)u, (S^* f S)MP^*(t)u\rangle \, dt \\ &= \int_0^\infty \mathrm{e}^{-t} dt \int_{\mathbb{R}} dx \frac{1}{\beta^2 x^2} p(t,x)^{-1-1/\beta} f(cx) \left|u(xp(t,x))\right|^2. \end{aligned}$$

By the change of variables $xp(t,x) = y$, the right-hand side can be written in the form

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-t} dt \int_{\mathbb{R}} dy |u(y)|^2 (\beta y)^{-2} \left(q(t,y)\right)^{-2+1/\beta} f(cyq(t,y)) \\ &= \int_{\mathbb{R}} dy |u(y)|^2 (\beta y)^{-2} \int_0^\infty dt \, \mathrm{e}^{-t} \left(q(t,y)\right)^{-2+1/\beta} f(cyq(t,y)). \end{aligned}$$

Changing the variable t to $s = -2t/(\beta y^2)$ we obtain the formula (3.1). □

Let $\mathcal{F}$ be the cone of positive measurable function on $\mathbb{R}$ bounded on open subsets of $\mathbb{R}$ not containing 0. In view of Lemma 3.1 we can extend the map $\mathcal{Q}$ to $\mathcal{F}$ defining $\mathcal{Q}(f)$ as the unique positive selfadjoint operator such that

$$\langle u, \mathcal{Q}(f)u\rangle = \sup_{n\geq 1} \langle u, \mathcal{Q}(f \wedge n)u\rangle$$

for all $u \in D_0$. Clearly the operator $\mathcal{Q}(f)$ agrees with the moltiplication operator by the function $(\mathcal{Q}(f))$ given by (3.1). Moreover we have the following

Lemma 3.2. *Let f, g be two elements of $\mathcal{F}$ satisfying the inequality $f \leq g$. Then the operators $\mathcal{Q}(f)$, $\mathcal{Q}(g)$ satisfy the inequality*

$$\mathcal{Q}(f) \leq \mathcal{Q}(g).$$

Let q_η denote the function

$$q_\eta : \mathbb{R} - \{0\} \to \mathbb{R}, \qquad q_\eta(x) = |x|^{-\eta}$$

for all $\eta > 0$. This function will be often identified with the corresponding positive self-adjoint multiplication operator.

Lemma 3.3. *The operator $\mathcal{Q}(I)$ satisfies the following inequalities*

$$\mathcal{Q}(I) \leq 1, \qquad\qquad \mathcal{Q}(I) \leq (1+\beta)^{-2}(1-2\beta)^{-1}q_2.$$

Proof. Use the formula (3.1), f being the constant function 1. To establish the second inequality it suffices to estimate the first factor in the integrand by the constant 1 and compute the integral. The first inequality is well known; however it can be checked here in the same way. □

Lemma 3.4. *For all $\eta > 0$ the following inequality holds*

$$\mathcal{Q}(q_\eta) \leq \left[(1-\eta\beta)|1+\beta|^\eta\right]^{-1} q_\eta.$$

Proof. In fact, by Lemma 3.1, we have

$$\begin{aligned}(\mathcal{Q}(q_\eta))(y) &= |y|^{-\eta} \cdot |c|^{-\eta}(-2\beta)^{-1} \int_0^\infty ds \, \exp\left(\beta y^2 s/2\right)(1+s)^{-1+1/(2\beta)-\eta/2} \\ &\leq |y|^{-\eta} \cdot |c|^{-\eta}(-2\beta)^{-1} \int_0^\infty ds \, (1+s)^{-1+1/(2\beta)-\eta/2} \\ &= \left((1-\eta\beta)|1+\beta|^\eta\right)^{-1} q_\eta(y).\end{aligned}$$

This proves the Lemma. □

We can now prove the first part of Theorem 2.4.

Proposition 3.5. *Suppose $\beta < \beta_*$. Then the m.q.d.s. $\mathcal{T}$ is conservative.*

Proof. Since β is smaller than β_* we have the inequality

$$\lim_{\eta \to 0+} (1-\eta\beta)^{1/\eta}\,|\beta+1| = \exp(-\beta)|\beta+1| > 1.$$

Therefore we can choose a real number $\eta \in (0,2]$ such that

$$(1-\eta\beta)\,|\beta+1|^{\eta} > 1.$$

From Lemma 3.3 it follows that there exists a positive constant ξ, depending only on β, such that

$$\mathcal{Q}(I) \leq \xi q_{\eta}.$$

Thus, using Lemma 3.4, we can easily establish the following estimate by induction

$$\mathcal{Q}^n(I) \leq \xi\,[(1-\eta\beta)|\beta+1|^{\eta}]^{-n}\,q_{\eta}.$$

Letting n tend to infinity we check the condition ii) of Theorem 2.5 is fulfilled. □

4. The case $\beta > \beta_*$

We shall show that the condition *iii)* of Theorem 2.5 is not satisfied. Fix $\lambda = 1$. We consider first $\mathcal{L}$ as the differential operator $\mathcal{L}_d$ on some function space given by

$$(\mathcal{L}_d(f))\,(x) = (\beta x)^{-2}\,(f(cx) - f(x) - \beta x f'(x)) \tag{4.1}$$

and construct a nonzero positive bounded continuous function f on $\mathbb{R}$ solving the differential equation $f = \mathcal{L}_d(f)$. Then we show that the function f satisfies the condition

$$\langle v, fu\rangle = \langle G^*v, fu\rangle + \langle SMv, fSMu\rangle + \langle v, fG^*u\rangle = \langle v, \mathcal{L}(f)u\rangle \tag{4.2}$$

for all vectors v, u in the domain of G^*.

For every open subset E of $\mathbb{R}$ we denote by $\mathcal{C}^k(E;\mathbb{R})$ (resp. $\mathcal{C}_b^k(E;\mathbb{R})$) the vector space of real-valued continuous (resp. bounded continuous) functions on E with continuous (resp. bounded continuous) derivatives of the first k orders.

Lemma 4.1. *Let g be an element of $\mathcal{C}_b^0(\mathbb{R}-\{0\};\mathbb{R})$ and let f be an element of $\mathcal{C}_b^0(\mathbb{R}-\{0\};\mathbb{R})$. If $\beta > 0$ the following conditions are equivalent:*

a) $f \in \mathcal{C}^1(\mathbb{R}-\{0\};\mathbb{R})$ and, for all $x \in \mathbb{R}-\{0\}$, we have

$$f(x) - (\beta x)^{-2}\,(f(cx) - f(x) - \beta x f'(x)) = g(x), \tag{4.3}$$

b) for all $x \in \mathbb{R}-\{0\}$ we have

$$f(x) = |x|^{-1/\beta}\exp(-\beta x^2/2)\int_0^x |t|^{1/\beta}\exp(\beta t^2/2)\left[\beta t g(t) + (\beta t)^{-1} f(ct)\right] dt. \tag{4.4}$$

If $\beta < 0$ the condition a) is equivalent to the following condition:
c) for all $x \in I\!R - \{0\}$ we have

$$f(x) = -|x|^{-1/\beta} \exp(-\beta x^2/2) \int_x^{\mathrm{sgn}(x)\infty} |t|^{1/\beta} \exp(\beta t^2/2) \left[\beta t g(t) + (\beta t)^{-1} f(ct)\right] dt \tag{4.5}$$

Proof. The differential equation (4.3) can be written in the normal form

$$f'(x) = -\left(\beta x + (\beta x)^{-1}\right) f(x) + \left[\beta x g(x) + (\beta x)^{-1} f(cx)\right].$$

Therefore, integrating this first order differential equation, for all $x, x_0 \in I\!R$ with $0 < x_0 < x$ or $x < x_0 < 0$ we have

$$\begin{aligned} f(x) &= f(x_0)|x_0|^{1/\beta} \exp(\beta x_0^2/2)|x|^{-1/\beta} \exp(-\beta x^2/2) \\ &+ |x|^{-1/\beta} \exp(-\beta x^2/2) \int_{x_0}^{x} |t|^{1/\beta} \exp(\beta t^2/2) \left[\beta t g(t) + (\beta t)^{-1} f(ct)\right] dt. \end{aligned}$$

The function f being bounded, if $\beta > 0$, we can let x_0 go to 0 and obtain (4.4). If $\beta < 0$, we can exchange x and x_0, let x_0 go to $\mathrm{sgn}(x)\infty$ and obtain (4.5). Conversely, differetiating (4.4) and (4.5) we obtain (4.3). □

Let $C_b^0(I\!R - \{0\}; I\!R_+)$ denote the cone of nonnegative functions in $C_b^0(I\!R - \{0\}; I\!R)$. The following proposition gives essentially the construction of the so-called minimal solution to the Feller-Kolmogorov equation of a classical stochastic process.

Proposition 4.2. *There exists a map*

$$\mathcal{R} : C_b^0(I\!R - \{0\}; I\!R_+) \to C_b^0(I\!R - \{0\}; I\!R_+) \cap C^1(I\!R - \{0\}; I\!R)$$

with the following properties:
a) for every $g \in C_b^0(I\!R - \{0\}; I\!R_+)$, the function $\mathcal{R}(g)$ satisfies the equation (4.3),
b) for every $g \in C_b^0(I\!R - \{0\}; I\!R_+)$, we have the inequality

$$\|\mathcal{R}(g)\|_\infty \le \|g\|_\infty,$$

c) for every $g, \tilde{g} \in C_b^0(I\!R - \{0\}; I\!R_+)$, such that $g \le \tilde{g}$ and every function $\tilde{f} \in C_b^0(I\!R - \{0\}; I\!R_+) \cap C^1(I\!R - \{0\}; I\!R)$ satisfying the equation (4.3) with $g = \tilde{g}$ we have the inequality

$$\mathcal{R}(g) \le \tilde{f}.$$

Proof. Consider, for example, the case $\beta > 0$. Let $(f_n)_{n \ge 0}$ be the sequence of elements of $C_b^0(I\!R - \{0\}; I\!R_+)$ defined by

$$\begin{aligned} f_0(x) &= 0, \\ f_{n+1}(x) &= |x|^{-1/\beta} \exp(-\beta x^2/2) \int_0^x |t|^{1/\beta} \exp(\beta t^2/2) \left[\beta t g(t) + (\beta t)^{-1} f_n(ct)\right] dt. \end{aligned}$$

We can easily show by induction the inequality

$$\|f_n\|_\infty \le \|g\|_\infty \, . \tag{4.6}$$

In fact, (4.6) holds when $n = 0$. Suppose it has been established for an integer n. Then, for all $x \in I\!R - \{0\}$, we have the inequalities

$$\begin{aligned} |f_{n+1}(x)| &\le \|g\|_\infty \, |x|^{-1/\beta} \exp(-\beta x^2/2) \int_0^x |t|^{1/\beta} \exp(\beta t^2/2) \left[\beta t + (\beta t)^{-1}\right] dt \\ &= \|g\|_\infty \, |x|^{-1/\beta} \exp(-\beta x^2/2) \int_0^x \frac{d}{dt} \left(|t|^{1/\beta} \exp(\beta t^2/2)\right) dt \\ &\le \|g\|_\infty \, , \end{aligned}$$

which prove the inequality (4.6) for the integer $n+1$. Since g is nonnegative-valued, we can also show by induction that the sequence $(f_n)_{n\ge 0}$ is increasing. Let us consider the function $\mathcal{R}(g)$ defined by

$$\left(\mathcal{R}(g)\right)(x) = \sup_{n\ge 0} f_n(x).$$

Clearly $\mathcal{R}(g)$ is nonnegative and satisfies the condition b). Moreover $\mathcal{R}(g)$ satisfies the equation (4.4). Hence it belongs to $C_b^0\left(I\!R - \{0\}; I\!R\right) \cap C^1\left(I\!R - \{0\}; I\!R\right)$ and satisfies the equation (4.3). Finally let $g, \tilde{g}$ and f be as in c) and let $(f_n)_{n\ge 0}$ be the above defined sequence. Notice that, for all integer n and all $x \in I\!R$ with $x \ne 0$ the difference $\tilde{f}(x) - f_n(x)$ an be written in the from

$$|x|^{-1/\beta} \exp(-\beta x^2/2) \int_0^x |t|^{1/\beta} \exp(\beta t^2/2) \left[(\beta t)^{-1} (\tilde{f} - f_n)(ct) + \beta t \left(\tilde{g}(t) - g(t)\right)\right] dt.$$

Therefore, since $\tilde{f} \ge 0 = f_0$, we can prove by induction the inequality $\tilde{f} \ge f_n$ for every n. By the definition of $\mathcal{R}_\lambda$ the property c) follows. The case when $\beta < 0$ can be dealt with in the same way. □

Lemma 4.3. *For all $\beta > \beta_*$ there exists $\eta \in (0,1]$ such that*

$$|\beta + 1|^\eta - (1 + \eta\beta) = 0. \tag{4.7}$$

Proof. The function

$$\varphi : [0,1] \to I\!R, \qquad \varphi(x) = |\beta + 1|^x - 1 - \beta x$$

has the following properties

$$\varphi(0) = 0, \qquad \varphi(1) \le 0, \qquad \varphi'(0) = \log|\beta + 1| - \beta < 0,$$

because $\beta > \beta_*$. This proves the Lemma. Notice that, in the case $\beta \ge -1$, we can choose $\eta = 1$ since $\varphi(1) = |\beta + 1| - \beta - 1 = 0$. □

Proposition 4.4. *Suppose $\beta > \beta_*$ and consider a number $\eta \in (0,1]$ fulfilling (4.7). Let g be an element of $\mathcal{C}_b^0(I\!R; I\!R_+)$ satisfying the inequality*

$$0 \leq g(x) \leq k|x|^\eta/(1+|x|^\eta)$$

for all $x \in I\!R$, k being a positive constant. Then the function $\mathcal{R}(g)$ satisfies the inequality

$$0 \leq \mathcal{R}(g) \leq k|x|^\eta/(1+|x|^\eta).$$

Proof. Consider the function

$$\tilde{f} : I\!R \to I\!R, \qquad \tilde{f}(x) = k|x|^\eta/(1+|x|^\eta).$$

Clearly (4.7) implies the inequality $|\beta+1|^\eta(1-\eta\beta) < 1$. Then a straightforward computation yields

$$\left(\mathcal{L}_d\left(\tilde{f}\right)\right)(x) = k|x|^\eta \frac{\left(|\beta+1|^\eta - (1+\eta\beta)\right) + \left(|\beta+1|^\eta(1-\eta\beta) - 1\right)|x|^\eta}{(\beta x)^2\left(1+|c|^\eta|x|^\eta\right)\left(1+|x|^\eta\right)^2} \leq 0 \quad (4.8)$$

for all $x \in I\!R - \{0\}$. Let us consider the function $g \in \mathcal{C}_b^0(I\!R - \{0\}, I\!R_+)$ defined by

$$\tilde{g} = \tilde{f} - \mathcal{L}_d\left(\tilde{f}\right).$$

Because of (4.8) and the definition of $\tilde{f}$ we have the inequalities

$$\tilde{g} \geq \tilde{f} \geq g.$$

Applying the Proposition 4.2 c), we complete the proof. □

Proposition 4.5. *Suppose $\beta > \beta_*$. Let $g \in \mathcal{C}_b^0(I\!R; I\!R)$ be a Lipschitz continuous function. Then there exists a function $f \in \mathcal{C}_b^0(I\!R; I\!R) \cap \mathcal{C}^1(I\!R - \{0\}; I\!R)$ such that $f(0) = g(0)$ and*

$$f(x) - \left(\mathcal{L}_d(f)\right)(x) = g(x) \quad (4.9)$$

for all $x \neq 0$.

Proof. Fix $\eta \in (0,1]$ satisfying (4.7) and write g in the form

$$g(0) + (g - g(0))^+ - (g - g(0))^-.$$

Since g is Lipschitz continuous there exists a positive constant k such that

$$0 \leq (g - g(0))^+(x) \leq \frac{k|x|^\eta}{1+|x|^\eta}, \qquad 0 \leq (g - g(0))^-(x) \leq \frac{k|x|^\eta}{1+|x|^\eta}.$$

Consider the function f defined by $f(0) = g(0)$ and

$$f(x) = g(0) + \left(\mathcal{R}\left((g-g(0))^+\right)\right)(x) - \left(\mathcal{R}\left((g-g(0))^-\right)\right)(x)$$

for all $x \neq 0$. This function satisfies the equation (4.9) because of Proposition 4.2 a) and the fact that (4.9) is linear. Moreover, from Proposition 4.4, we have the inequality

$$|f(x) - g(0)| \leq k|x|^\eta/(1+|x|^\eta)$$

which shows that f is continuous at the point 0. □

Proposition 4.6. *Suppose $\beta > \beta_*$. Then there exists a function $f \in C_b^0(I\!R; I\!R) \cap C^0(I\!R - \{0\}; I\!R)$ satisfying the differential equation*

$$f(x) = (\mathcal{L}_d(f))(x) \tag{4.10}$$

for all $x \neq 0$ and the condition $f(0) > 0$.

Proof. Let us consider the function

$$f_1 : I\!R \to I\!R, \qquad f_1(x) = \left(1 + x^2\right)^{-1},$$

and the function $g \in C_b^0(I\!R; I\!R)$ defined as the only continuous extension of the function

$$g(x) = f_1(x) - (\mathcal{L}_d(f_1))(x)$$

for all $x \neq 0$. A straightforward computation yields

$$g(x) = \frac{2 + (c^2 - 2c)x^2 + c^2x^4}{(1 + c^2x^2)(1 + x^2)^2}.$$

By Proposition 4.5 there exists a function $f_2 \in C_b^0(I\!R; I\!R) \cap C^1(I\!R - \{0\}; I\!R)$ satisfying the equation (4.9) and the condition $f_2(0) = g(0) = 2$. The function

$$f = f_2 - f_1$$

satisfies the differential equation (4.10) and the condition $f(0) = 1$. □

The following propositions show that a function f satisfying the conditions of Proposition 4.6 is non-negative everywhere.

Proposition 4.7. *Let f be an element of $C_b^0(I\!R - \{0\}; I\!R) \cap C^1(I\!R - \{0\}; I\!R)$ satisfying the differential equation (4.10). Then*

$$\lim_{|x| \to \infty} f(x) = 0. \tag{4.11}$$

Proof. In the case $\beta > 0$, for all $x \neq 0$, we have the inequality

$$|f(x)| \leq \|f\|_\infty \frac{\left|\int_0^x |t|^{1/\beta} \exp(\beta t^2/2)\, dt\right|}{\beta |x|^{1/\beta} \exp(\beta x^2/2)}.$$

Therefore (4.11) follows computing the limits, for example by the De L'Hôpital rule. The proof in the case $\beta < 0$ is similar. □

Proposition 4.8. *Let f be an element of $C_b^0(I\!R; I\!R) \cap C^1(I\!R - \{0\}; I\!R)$ satisfying the differential equation (4.9) and the condition $f(0) > 0$. Then f is non-negative everywhere.*

Proof. We use the well-known minimum principle. Suppose that there exists $a \in I\!R$ such that $f(a) < 0$. Then, by Proposition 4.7, there exists $b \in I\!R - \{0\}$ such that

$$\min_{x \in I\!R} f(x) = f(b) < 0.$$

The point b does not coincide with 0, hence f is differentiable at that point and $f'(b)$ vanishes. Then we have the contradiction

$$f(b) = (\beta b)^{-2} (f(cb) - f(b)) \geq 0.$$

This shows that f must be non-negative everywhere. □

Proposition 4.9. *Let f be an element of $C_b^0(I\!R;(0,+\infty))\cap C^1(I\!R-\{0\};(0,+\infty))$ satisfying the differential equation (4.10). Then f satisfies also the identity (4.2) for all $v,u\in D(G^*)$.*

Proof. Suppose, for example, $\beta>0$. Let us consider two vectors in the domain of G^* represented in the form $R(1;G^*)v, R(1;G^*)u$ with $v,u\in h$. For all $r\in(0,1)$ denote by I_r the set $(-r^{-1},-r)\cup(r,r^{-1})$. The scalar product $\langle R(1;G^*)v, fR(1;G^*)u\rangle$ can be written as

$$\begin{aligned}
&\lim_{r\to 0}\int_{I_r}\left(R(1;G^*)\bar{v}\right)(x)\,(\beta x)^{-2}\,f(cx)\left(R(1;G^*)u\right)(x)\,dx\\
&-\lim_{r\to 0}\int_{I_r}\left(R(1;G^*)\bar{v}\right)(x)(\beta x)^{-2}f(x)\left(R(1;G^*)u\right)(x)\,dx\\
&-\lim_{r\to 0}\int_{I_r}\left(R(1;G^*)\bar{v}\right)(x)(\beta x)^{-1}f'(x)\left(R(1;G^*)u\right)(x)\,dx.
\end{aligned}$$

Integrating by parts the third integral, we can write the above sum of the second and third term in the form

$$\begin{aligned}
&\langle SMR(1;G^*)v, fSMR(1;G^*)u\rangle\\
&+2\langle R(1;G^*)v, fR(1;G^*)\rangle-\langle v, fR(1;G^*)u\rangle-\langle R(1;G^*)v, fu\rangle\\
&+\beta\lim_{r\to 0}|r|^{1/\beta}\exp(\beta r^2)\bar{v}_0(r)u_0(r)f(r)\\
&-\beta\lim_{r\to 0}|r|^{-1/\beta}\exp(\beta r^{-2})\bar{v}_0(r^{-1})u_0(r^{-1})f(r^{-1})\\
&-\beta\lim_{r\to 0}|r|^{1/\beta}\exp(\beta r^2)\bar{v}_0(-r)u_0(-r)f(-r)\\
&+\beta\lim_{r\to 0}|r|^{-1/\beta}\exp(\beta r^{-2})\bar{v}_0(-r^{-1})u_0(-r^{-1})f(-r^{-1})
\end{aligned}$$

where v_0 and u_0 are defined as in Proposition 2.1. It is easy to see as in the proof of Proposition 2.2 that all the above limits vanish. Then the proof can be completed using the identity $R(1;G^*)-I=G^*R(1;G^*)$. □

Thus we have proved the condition *iii)* of Theorem 2.5 is not fulfilled.

Proposition 4.10. *The m.q.d.s. $\mathcal{T}$ is not conservative if $\beta>\beta_*$.*

5. Applying the sufficient condition of [3]

In this section we will show that the m.q.d.s. $\mathcal{T}$ is conservative if $\beta\le -3/2$ by checking a sufficient condition obtained in [3]. We transform first the form $\mathcal{L}$ defined in Section 2 shifting the spectrum operator G^* by $-1/2$ and considering the operators

$$L_1=I,\qquad L_2=SM.$$

The "shifted" m.q.d.s. generated by the form

$$(v,Xu)\to\langle G^*v,Xu\rangle+\langle L_1v,XL_1u\rangle+\langle L_2v,XL_2u\rangle+\langle v,XG^*u\rangle$$

is conservative if and only if the "unshifted" one is also. Let us recall that, if $\beta < 0$, $R(1;G^*)$ is modified as follows

$$(R(1;G^*)u)(x) = \beta|x|^{(\beta+1)/2\beta}\exp(3\beta x^2/4)u_0(x)$$

where

$$u_0(x) = -\int_0^x \exp(-3\beta y^2/4)|y|^{-(\beta+1)/2\beta} y u(y)dy$$

Consider the selfadjoint operator defined by

$$D(C) = \left\{ u \in h \mid x^{-2}u(x) \in h \right\}, \qquad Cu(x) = \left(1 + (\beta x)^{-2}\right) u(x).$$

The domain of its positive square root $C^{1/2}$ coincides with the domain of the operator M. Therefore, by Proposition 2.2, it is contained in the domain of G^* and, for all $v, u \in D(G^*)$, we have

$$-\langle G^*v, u\rangle - \langle v, G^*u\rangle = \left\langle C^{1/2}v, C^{1/2}u\right\rangle.$$

Moreover C has a bounded inverse which is bounded from below by the identity operator. We shall denote by C_ε (for $\varepsilon > 0$) the bounded operator $(\varepsilon I + C^{-1})^{-1}$.

Let $\mathcal{Q} : \mathcal{B}(h) \to \mathcal{B}(h)$ be the normal and monotone map defined by

$$\langle v, \mathcal{Q}(X)u\rangle = \int_0^\infty \mathrm{e}^{-t}\left(\langle P^*(t)v, XP^*(t)u\rangle + \langle SMP^*(t)v, XSMP^*(t)u\rangle\right)dt.$$

Theorem 5.1. *Suppose that there exists a positive constant b such that:*
i) for all $u \in D(G^*)$ the following inequality holds

$$-2\Re e\left\langle G^*u, C^{-1}u\right\rangle \le b\|u\|^2, \tag{5.1}$$

ii) for all $\varepsilon > 0$ and all u in a dense subset of h contained in the domain of $C^{1/2}$ we have

$$\langle u, \mathcal{Q}(C_\varepsilon)u\rangle \le \left\|C^{1/2}u\right\|^2 + b\|u\|^2. \tag{5.2}$$

Then the m.q.d.s. $\mathcal{T}$ is conservative.

We refer to [3] Th. 4.2 for the proof and check the conditions *i), ii)*.

Lemma 5.2. *Suppose $\beta < 0$. Then the inequality (5.1) holds with $b = 3$.*

Proof. We use vectors in the domain G^* represented in the form $R(1;G^*)$ with $u \in h$. The identity $G^*R(1;G^*) = R(1;G^*) - I$ yields

$$\begin{aligned}&-2\Re e\left\langle G^*R(1;G^*)u, C^{-1}R(1;G^*)u\right\rangle \\ &\quad= -2\Re e\langle R(1;G^*)u, C^{-1}R(1;G^*)u\rangle + 2\Re e\left\langle u, C^{-1}R(1;G^*)u\right\rangle.\end{aligned}$$

The operator C^{-1} is positive, hence it suffices to show that the second term is bounded from above. Let r be an element of $(0,1)$ and denote by I_r the set $(-r^{-1},-r)\cup(r,r^{-1})$. Integrating by parts we have

$$\begin{aligned}
&\langle u, C^{-1}R(1;G^*)u\rangle \\
&= \beta \lim_{r\to 0}\int_{I_r}\left[x|x|^{-\frac{\beta+1}{2\beta}}\mathrm{e}^{-3\beta x^2/4}\bar{u}(x)\right]\frac{\beta^2x^2}{1+\beta^2x^2}\mathrm{sgn}(x)|x|^{\frac{1}{\beta}}\mathrm{e}^{3\beta x^2/2}u_0(x)dx \\
&= \lim_{r\to 0}\Big\{\frac{\beta^3r^2}{1+\beta^2r^2}r^{\frac{1}{\beta}}\exp(3\beta r^2/2)\left(|u_0(r)|^2-|u_0(-r)|^2\right) \\
&\quad+\frac{\beta^3r^{-2}}{1+\beta^2r^{-2}}r^{-\frac{1}{\beta}}\exp(3\beta r^2/2)\left(|u_0(-r^{-1})|^2-|u_0(r^{-1})|^2\right) \\
&\quad+\beta\int_{I_r}\frac{2\beta^2}{(1+\beta^2x^2)^2}|x|^{1+\frac{1}{\beta}}\exp(3\beta r^2/2)|u_0(x)|^2dx \\
&\quad+\beta^2\int_{I_r}\frac{1}{1+\beta^2x^2}|x|^{1+\frac{1}{\beta}}\exp(3\beta r^2/2)|u_0(x)|^2dx \\
&\quad+3\beta^2\int_{I_r}\frac{\beta^2x^2}{1+\beta^2x^2}|x|^{1+\frac{1}{\beta}}\exp(3\beta r^2/2)|u_0(x)|^2dx \\
&\quad-\beta\int_{I_r}\frac{\beta^2x^2}{1+\beta^2x^2}|x|^{\frac{1+\beta}{2\beta}}\exp(3\beta r^2/4)\bar{u}_0(x)u(x)dx\Big\}.
\end{aligned}$$

As in the proof of Proposition 2.2 we can prove that

- the first and second term vanish,
- the third term is negative because $\beta<0$,
- the sum of the fourth and fifth term is bounded from above by $3\,\|R(1;G^*)u\|^2$,
- the sixth term converges to $-\langle R(1;G^*)u, C^{-1}u\rangle$.

This proves the lemma. □

Remark. A similar proof shows that the inequality (5.1) holds also when $\beta>0$ with $b=3(1+\beta)$.

Lemma 5.3. *The condition ii) of Theorem 5.1 holds when $\beta\le -3/2$.*

Proof. Let $\varepsilon>0$ and let u be a smooth function with compact support contained in $\mathbb{R}-\{0\}$. Remark that C_ε coincides with the multiplication operator by the function m_ε given by

$$m_\varepsilon(x)=\left(\varepsilon+\left(1+\beta^{-2}x^{-2}\right)^{-1}\right)^{-1}.$$

Let $q(\cdot,\cdot)$ and $p(\cdot,\cdot)$ be the functions defined in the proof of Lemma 3.1. A straightforward computation yields

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-t}\left(\langle SMP^*(t)u, C_\varepsilon SMP^*(t)u\rangle+\langle P^*(t)u, C_\varepsilon P^*(t)u\rangle\right)dt \\
&=\int_0^\infty \mathrm{e}^{-2t}dt\int_{\mathbb{R}}dx\left(\frac{m_\varepsilon(cx)}{\beta^2x^2}+m_\varepsilon(x)\right)p(t,x)^{-1-\frac{1}{\beta}}\left|u\left(xp(t,x)\right)\right|^2.
\end{aligned}$$

By the change of variables $xp(t,x) = y$, the right-hand side can be written in the form

$$\int_0^\infty e^{-2t}dt \int_{I\!R} dy|u(y)|^2 \left(q(t,y)\right)^{\frac{1}{\beta}} \left(\frac{m_\varepsilon(cyq(t,y))}{\beta^2 y^2 (q(t,y))^2} + m_\varepsilon(yq(t,y)) \right)$$
$$= \int_{I\!R} dy|u(y)|^2 \int_0^\infty dt\, e^{-2t} \left(q(t,y)\right)^{\frac{1}{\beta}} \left(\frac{m_\varepsilon(cyq(t,y))}{\beta^2 y^2 (q(t,y))^2} + m_\varepsilon(yq(t,y)) \right)$$

Changing the variable t to $s = -2t/(\beta y^2)$ and letting ε tend to 0 we can estimate the integral with respect to t by

$$\frac{1}{-2\beta^3 c^2 y^2} \int_0^\infty \exp(\beta y^2 s)\,(1+s)^{-2+1/(2\beta)}\, ds$$
$$+ \frac{2}{-2\beta} \int_0^\infty \exp(\beta y^2 s)\,(1+s)^{-1+1/(2\beta)}\, ds$$
$$- \frac{1}{2} \int_0^\infty \beta y^2 \exp(\beta y^2 s)\,(1+s)^{-1+1/(2\beta)}\, ds.$$

Since β is negative, this sum can be estimated by

$$\frac{1}{-2\beta^3 c^2 y^2} \int_0^\infty (1+s)^{-2+1/(2\beta)}\, ds + \frac{2}{-2\beta} \int_0^\infty (1+s)^{-1+1/(2\beta)}\, ds$$
$$- \frac{1}{2} \int_0^\infty \beta y^2 \exp(\beta y^2 s)\, ds$$
$$= \left((1-2\beta)(1+\beta)^2 \beta^2 y^2\right)^{-1} + 5/2.$$

Therefore, recalling the definition of the operator C given in this section, the condition *ii)* of Theorem 5.1 holds, in the case $\beta < 0$ whenever

$$(1-2\beta)(1+\beta)^2 \geq 1$$

i.e. $\beta \leq -3/2$. □

Theorem 5.4. *The m.q.d.s.* $\mathcal{T}$ *is conservative if* $\beta \leq -3/2$.

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