Séminaire de probabilités (Strasbourg)

Yukuang Chiu

From an example of Lévy's

Séminaire de probabilités (Strasbourg), tome 29 (1995), p. 162-165
<http://www.numdam.org/item?id=SPS_1995__29__162_0>

From an Example of Lévy's

Yukuang Chiu
Department of Mathematics, 0112
University of California, San Diego
La Jolla, CA 92093-0112, USA

The motive of this paper is to prove completely an assertion of P. Lévy [3], who claimed that for each positive integer n, there exists a polynomial F_n of degree n such that the Wiener integral with respect to Brownian motion $\{B(u); 0 \leq u\}$

$$X(t) = \int_0^t F_n\left(\frac{u}{t}\right) dB(u)$$

is again a Brownian motion and

$$\mathbf{B}(X;t) \subsetneq \mathbf{B}(B;t).$$

Here $\mathbf{B}(X;t)$ is the σ–field generated by $\{X(s); s \leq t\}$.

Over the last few years, the non-canonical representation of Brownian motion of this kind related has been of interest to many authors, in particular Th. Jeulin & M. Yor [4,5,6,7] and M. Hitsuda [2].

Let $X_0(t) = B(t), (t \geq 0)$ be a standard Brownian motion. In this paper, our precise purpose is to construct a sequence of Brownian motion $\{X_n(t); t \geq 0\} (n \geq 0)$ such that $X_n(t)$ can be represented as a Wiener integral

$$X_n(t) = \int_0^t F_n\left(\frac{u}{t}\right) dB(u), \quad (n \geq 1).$$

Here $F_n(t)$ is a polynomial of degree n in t. And if $\mathbf{M}(X_n;t)$ is a linear span of $\{X_n(u); 0 \leq u \leq t\}$ in $L^2(\Omega, P)$, then for all $t > 0$,

$$\mathbf{M}(X_{n+1};t) \subsetneq \mathbf{M}(X_n;t), \ (n \geq 0), \tag{1}$$

and further, for all $t > 0$ and $n \geq 1$,

$$\int_0^t F_n\left(\frac{u}{t}\right) u^k du = 0, \ k = 1, 2, \ldots, n. \tag{2}$$

Let $F_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. We know by calculating the covariance that $X_n(t)$ is a Brownian motion if and only if the coefficients of $F_n(t)$ satisfy the following equations

$$\begin{cases} \frac{a_0 a_0}{1} + \frac{a_0 a_1}{2} + \cdots + \frac{a_0 a_n}{n+1} = 1, \\ \frac{a_1 a_0}{2} + \frac{a_1 a_1}{3} + \cdots + \frac{a_1 a_n}{n+2} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{a_n a_0}{n+1} + \frac{a_n a_1}{n+2} + \cdots + \frac{a_n a_n}{2n+1} = 0. \end{cases} \tag{3}$$

In the simplest case

$$\begin{cases} a_n = \frac{2n+1}{n}, a_0 = -\frac{n+1}{n}, \\ a_1 = a_2 = \cdots = a_{n-1} = 0. \end{cases}$$

is a solution of equation (3).

Theorem 1 *If* $P_n(t) = \frac{2n+1}{n}t^n - \frac{n+1}{n}$ *and if* $F_n(t)$ *are defined by the following recursive formula*

$$\begin{cases} F_1(t) = P_1(t) \\ F_n(\frac{u}{t}) = F_{n-1}(\frac{u}{t}) - \int_u^t F_{n-1}(\frac{u}{\tau})\frac{\partial}{\partial\tau}P_n(\frac{\tau}{t})d\tau, \ (n \geq 2) \end{cases},$$

then $F_n(t)$ *satisfies (2), the coefficients of* $F_n(t)$ *are given by*

$$a_k = (-1)^{n+k}\binom{n}{k}\binom{n+1+k}{n}, \ k = 0, 1, \cdots, n$$

and

$$X_n(t) := \int_0^t F_n\left(\frac{u}{t}\right) dB(u), \ (n \geq 1)$$

are Brownian motions satisfying condition (1). Further, $X_n(t)$ *and* $X_{n+1}(t)$ *are related by*

(4) $$X_{n+1}(t) = \int_0^t P_{n+1}\left(\frac{u}{t}\right) dX_n(u), \ (n \geq 0).$$

In order to prove the theorem, we prepare the following lemma.

Lemma 1 *If* $s < n$, *we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{s} = 0.$$

To prove this, we note

$$\begin{aligned} \frac{1}{s!}\left(\frac{d}{dx}\right)^s((1+x)^n x^n) &= \frac{1}{s!}\left(\frac{d}{dx}\right)^s\left(\sum_{k=0}^{n}\binom{n}{k}x^{n+k}\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{s}x^{n+k-s}. \end{aligned}$$

The result follows by letting $x = -1$.

The validity of the coefficients of $F_n(t)$ can be established by mathematical induction. The assertion is trivial for $n = 1$. Suppose the assertion holds for n. Now using lemma 1 and then noting

$$\binom{n}{k}\binom{n+1+k}{n}\left(1 - \frac{2(n+1)+1}{n+1-k}\right) = -\binom{n+1}{k}\binom{n+2+k}{n+1},$$

$$\binom{n}{k}\binom{n+1+k}{n}\frac{2(n+1)+1}{n+1-k} = \frac{2(n+1)+1}{n+1}\binom{n+1}{k}\binom{n+1+k}{n},$$

we see that

$$\begin{aligned} F_{n+1}\left(\frac{u}{t}\right) &= F_n\left(\frac{u}{t}\right) - \int_u^t F_n\left(\frac{u}{\tau}\right)\frac{\partial}{\partial\tau}P_{n+1}\left(\frac{\tau}{t}\right)d\tau \\ &= \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+1+k}{n}\left(1 - \frac{2(n+1)+1}{n+1-k}\right)\left(\frac{u}{t}\right)^k \\ &\quad + \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+1+k}{n}\frac{2(n+1)+1}{n+1-k}\left(\frac{u}{t}\right)^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n}(-1)^{n+1+k}\binom{n+1}{k}\binom{n+2+k}{n+1}\left(\frac{u}{t}\right)^k + \binom{2n+3}{n+1}\left(\frac{u}{t}\right)^{n+1} \\
&= \sum_{k=0}^{n+1}(-1)^{n+1+k}\binom{n+1}{k}\binom{n+2+k}{n+1}\left(\frac{u}{t}\right)^k.
\end{aligned}
$$

which shows the assertion holds for coefficients of $F_{n+1}(t)$.

We next show (2). By the recursive formula of F_n, we obtain

$$
\begin{aligned}
\int_0^t F_n\left(\frac{u}{t}\right)u^k du &= \int_0^t F_{n-1}\left(\frac{u}{t}\right)u^k du - \int_0^t\int_u^t F_{n-1}\left(\frac{u}{\tau}\right)\frac{\partial}{\partial\tau}P_n\left(\frac{\tau}{t}\right)u^k d\tau du \\
&= \int_0^t F_{n-1}\left(\frac{u}{t}\right)u^k du - \int_0^t d\tau\frac{\partial}{\partial\tau}P_n\left(\frac{\tau}{t}\right)\int_0^\tau F_{n-1}\left(\frac{u}{\tau}\right)u^k du
\end{aligned}
$$

This equals zero if $k < n$ by induction; and when $k = n$, this becomes

$$
\begin{aligned}
\int_0^t F_{n-1}\left(\frac{u}{t}\right)u^n du - \left[P_n\left(\frac{\tau}{t}\right)\int_0^\tau F_{n-1}\left(\frac{u}{\tau}\right)u^n du\Big|_{\tau=0}^t - \int_0^t P_n\left(\frac{u}{t}\right)u^n du\right] \\
= \int_0^t P_n\left(\frac{u}{t}\right)u^n du = 0.
\end{aligned}
$$

which is what we needed to prove.

Again we easily verify, by mathematical induction, that

$$
\int_0^1 F_n(u)du = \frac{(-1)^n}{n+1}.
$$

Thus we have proved, in combination with the previous equation, that the coefficients of F_n are another solution to equation (3).

Now if we write

$$
X_n(t) = \int_0^t F_n\left(\frac{u}{t}\right)dB(u),
$$

then by the above argument, $X_n(t)$ is again a Brownian motion. The differential of $X_n(t)$, by Itô's formula [1], is seen to be

$$
dX_n(u) = dB(u) + \int_0^u \frac{\partial}{\partial u}F_n\left(\frac{\tau}{u}\right)dB(\tau)du.
$$

Therefore

$$
\begin{aligned}
&\int_0^t P_{n+1}\left(\frac{u}{t}\right)dX_n(u) \\
&= \int_0^t P_{n+1}\left(\frac{u}{t}\right)dB(u) + \int_0^t\left\{P_{n+1}\left(\frac{u}{t}\right)\int_0^u\frac{\partial}{\partial u}F_n\left(\frac{\tau}{u}\right)dB(\tau)\right\}du \\
&= \int_0^t P_{n+1}\left(\frac{u}{t}\right)dB(u) + \int_0^t dB(\tau)\int_\tau^t P_{n+1}\left(\frac{u}{t}\right)\frac{\partial}{\partial u}F_n\left(\frac{\tau}{u}\right)du \\
&= \int_0^t P_{n+1}\left(\frac{u}{t}\right)dB(u) + \\
&\qquad \int_0^t dB(\tau)\left\{F_n\left(\frac{\tau}{u}\right)P_{n+1}\left(\frac{u}{t}\right)\Big|_{u=\tau}^t - \int_\tau^t F_n\left(\frac{\tau}{u}\right)\frac{\partial}{\partial u}P_{n+1}\left(\frac{u}{t}\right)du\right\} \\
&= \int_0^t F_n\left(\frac{u}{t}\right)dB(u) - \int_0^t\left\{\int_u^t F_n\left(\frac{u}{\tau}\right)\frac{\partial}{\partial\tau}P_{n+1}\left(\frac{\tau}{t}\right)d\tau\right\}dB(u) \\
&= X_{n+1}(t)
\end{aligned}
$$

This establishes (4).

To show (1), let us fix $t_0 > 0$ and let $z = \int_0^{t_0} u^{n+1} dX_n(u)$. Now $z \in \mathbf{M}(X_n;\, t)$ and note that for all t such that $0 < t \le t_0$,

$$E[X_{n+1}(t) \cdot z] = \int_0^t P_{n+1}\left(\frac{u}{t}\right) u^{n+1} du = 0.$$

This verifies (1). The proof of theorem is thus completed.

Remark 1 *This construction was suggested by P. Lévy in his book* [3] *and* $F_1(t)$ *was given there.*

Remark 2 *Although we have (1), for all* $n > 0$, *we notice,*

$$\mathbf{M}(B;\infty) = \mathbf{M}(X_n;\infty). \tag{5}$$

This equation has the following interpretation. For each finite time t, as we have already seen, $\mathbf{B}(X_n;t)$ contains less information than $\mathbf{B}(B;t)$. Nevertheless, $\mathbf{B}(X_n;t)$ will "catch up" with $\mathbf{B}(B;t)$ by increasing time to infinity.

Acknowledgements

I sincerely thank M. Yor for his suggestions and preprints which have greatly enriched my knowledge on this topic.

References

[1] K. Itô, Introduction to Probability Theory. vol.3. (in Japanese) Iwanami-Shoten, (1978).

[2] M. Hitsuda, Gaussian innovations and integral operators. Proceedings of the Sixth USSR-Japan Symposium. Kiev, Aug. 5-10, 1991. World Scientific.

[3] P. Lévy, Processes Stochastiques et mouvement brownien.Gauthier-Villars. (1965), p. 299–304.

[4] Th. Jeulin, M. Yor, Filtration des ponts browniens et équations différentielles linéaires. Séminaire de Probabilités XXIV. Lect. Notes in Maths. 1426. Springer (1990), p. 227–265.

[5] Th. Jeulin, M. Yor, Une décomposition non-canonique du drap brownien. Séminaire de Probabilités XXVI. Lect. Notes in Maths. 1526. Springer (1992), p. 322–347.

[6] Th. Jeulin, M. Yor, Moyennes mobiles et semimartingales. Séminaire de Probabilités XXVII. Lect. Notes in Maths. 1557. Springer (1993), p. 53–77.

[7] Th. Jeulin, M. Yor, Sur certaines décompositions non-canoniques des diffusions réelles. Manuscrit non publié (Décembre 1990).