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Non-linear Wiener-Hopf theory, 1: an appetizer

by

David Williams

The first use of non-linear Wiener-Hopf theory occurred in Neveu's very important paper (Neveu, 1987) which was further developed in (Kaj & Salminen, 1993). See also (Jansons & Rogers, 1992) for further probabilistic insight.

In case the Reader thinks that Section 2 is just too bizarre, I emphasize that the method gives useful information on integral curves for dynamical systems of which (4.1) is the simplest possible case.

1. The classical problem

1(a) *Infinitesimal generator of a Markov chain.* Let

$$\mathbf{g}(t) = \{g_{ij}(t) : i, j \in J\}$$

be the transition matrix at time t of a Markov chain Y on a finite set J:

$$g_{ij}(t) := \mathbb{P}_i[Y(t) = j].$$

Then,

$$\mathbf{g}(u+t) = \mathbf{g}(u)\mathbf{g}(t), \quad \dot{\mathbf{g}}(t) = \frac{d\mathbf{g}}{dt} = G\mathbf{g}(t), \qquad \text{where } G := \dot{\mathbf{g}}(0).$$

Here, G is the infinitesimal generator, or Q-matrix, of $\{\mathbf{g}(t)\}$.

1(b) *Time transformation of a Markov chain.* Suppose now that $X = \{X(t)\} = \{X_t\}$ is a Markov chain, possibly of finite lifetime, with Q-matrix Q on a finite set E. Let $V : E \to \mathbb{R} \setminus \{0\}$. We shall also write V for the diagonal matrix $\operatorname{diag}(V(i))$. Set $E^{\pm} := \{i \in E : \pm V(i) > 0\}$ and partition $V^{-1}Q$ as

$$V^{-1}Q = \begin{array}{c} \\ E^+ \\ E^- \end{array} \begin{array}{c} \begin{array}{cc} E^+ & E^- \end{array} \\ \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \end{array}.$$

For $t \geq 0$, define

$$\varphi(t) := \int_0^t V(X_s)ds, \quad \tau_t^{\pm} := \inf\{u : \pm\varphi(u) > t\}, \quad Y_t^{\pm} := X(\tau_t^{\pm}).$$

Note that $Y_t^+ \in \mathbb{E}^+$ (or Y_t^+ is in the coffin state). It is clear that Y^+ is a Markov chain on E^+; let G^+ be its Q-matrix. We think of a $^+$Universe in which $^+$Observers use $S^+(t) := \sup\{\varphi(s) : s \leq t\}$ as a clock; such observers can see our particle only for part of the time when it is in E^+. Define

$$g_{jk}^+(t) := \mathbb{P}_j(Y_t^+ = k) \quad (j, k \in E^+),$$
$$h_{ik}^+(t) := \mathbb{P}_i(Y_t^+ = k) \quad (i \in E^-,\ k \in E^+),$$

$\mathbb{P}_i$ referring to the situation when $X_0 = i$. With H^+ as the $E^- \times E^+$ matrix with $H^+_{ik} := h^+_{ik}(0)$, we have

$$\dot{\mathbf{g}}^+(t) = G^+\mathbf{g}^+(t), \quad \mathbf{h}^+(t) = H^+\mathbf{g}^+(t),$$

the latter being probabilistically obvious.

Decompositions according to the time and nature of the first jump from $E^\pm$ to $E^\mp$ yield:

$$\mathbf{g}^+(t) = e^{tA} + \int_0^t e^{(t-s)A}B\mathbf{h}^+(s)ds,$$

whence

$$\dot{\mathbf{g}}^+(t) = A\mathbf{g}^+(t) + B\mathbf{h}^+(t),$$

and

$$\mathbf{h}^+(t) = \int_0^\infty e^{uD}C\mathbf{g}^+(u+t)dt = e^{-tD}\int_t^\infty e^{vD}C\mathbf{g}^+(v)dv.$$

whence

$$\dot{\mathbf{h}}^+(t) = -C\mathbf{g}^+(t) - D\mathbf{h}^+(t).$$

We therefore have

$$G^+ = A + BH^+, \quad H^+G^+ = -C - DH^+.$$

With the obvious notations G^- and H^-, we have

$$V^{-1}Q\begin{pmatrix} I^+ & H^- \\ H^+ & I^- \end{pmatrix} = \begin{pmatrix} I^+ & H^- \\ H^+ & I^- \end{pmatrix}\begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix},$$

where $I^\pm$ is the identity matrix on $E^\pm$. Of course, this does not in itself tell us what $G^\pm$ and $H^\pm$ are.

For a survey of this type of problem and some continuous-state-space generalizations, see Williams (1991).

2. The non-linear version: the simplest case

2(a) *Infinitesimal-generator function of a continuous-parameter branching process.* Consider the following model. At time 0, there is one particle. Each particle dies at constant rate K, and at the moment of death gives birth to n particles ($n = 0, 1, 2\ldots$ or ∞) with probability p_n. The usual independence assumptions hold.

Let $N(t)$ be the number of particles alive at time $t \geq 0$, and set

$$g(t,\theta) := \mathbb{E}\theta^{N(t)} = \sum_{0\leq n\leq\infty} \theta^n \mathbb{P}\big(N(t) = n\big) \quad (0 \leq \theta < 1).$$

Then, by the branching property,

$$g(u+t,\theta) = \mathbb{E}\mathbb{E}\left(\theta^{N(t+u)} \,\middle|\, N(u)\right) = \mathbb{E}g(t,\theta)^{N(u)} = g\big(u, g(t,\theta)\big),$$

whence, on differentiating with respect to u and setting $u = 0$,

$$\dot{g}(t,\theta) = G\big(g(t,\theta)\big),$$

where G is the 'infinitesimal-generator function'

$$G(\theta) := \frac{\partial}{\partial u} g(u,\theta)\Big|_{u=0} = K \sum_{0 \le n \le \infty} p_n(\theta^n - \theta), \quad (0 \le \theta < 1).$$

As $\theta \uparrow 1$, $G(\theta) \uparrow G(1-) = -Kp_\infty \le 0$.

2(b) *Multi-time substitution.* In Our universe, we have a branching process of particles which (while alive) live on the real line. Each particle is either of type $+$ or of type $-$. A particle of type $+$ has constant velocity $b_+ > 0$, while a particle of type $-$ has constant velocity $-b_- < 0$. A particle of type $i = \pm$ dies at rate $K_i := \lambda_i + q_i + r_i$; at its moment of death it

either just dies with probability λ_i/K_i,

or gives birth to a particle of the opposite type with probability q_i/K_i,

or gives birth to two particles of its own type with probability r_i/K_i.

A particle's child is born at the same position as its parent. The usual independence assumptions hold. The λ, q, r parameters satisfy

$$\lambda_\pm \ge 0, \quad q_\pm > 0, \quad r_\pm > 0.$$

Particle k has lifespan denoted by $[\beta_k, \delta_k)$. We write $S_k^+(u)$ for the furthest right (the 'sup') which particle k *or any of its ancestors* has reached by time u.

Observers in the $^+$Universe can see only particles of type $+$, and they use our S_k^+ as a clock in which to observe particle k. This means that our particle k will feature as a $^+$Particle in the $^+$Universe if and only if $S_k^+(\delta_k) > S_k^+(\beta_k)$, and will then have $^+$Lifespan $\big[S_k^+(\beta_k), S_k^+(\delta_k)\big)$; it will be a $^+$Child of the (almost certainly) unique ancestor in our system with death time $S_k^+(\beta_k)$. A few moments' thought will convince you that, if we ignore the positions of the $^+$Particles, then the evolution of the *number* of $^+$Particles in $^+$Time is a branching process just like that in Section 2(a).

For $t \ge 0$, define

$$\begin{aligned} N^+(t) &:= \text{Number of } {}^+\text{Particles alive at } {}^+\text{Time } t \\ &= \#\left\{k : t \in \big[S_k^+(\beta_k), S_k^+(\delta_k)\big)\right\}. \end{aligned}$$

We wish to study, for $0 \le \theta < 1$,

$$\begin{aligned} g^+(t,\theta) &:= \mathbb{E}_+ \theta^{N^+(t)}, \\ h^+(t,\theta) &:= \mathbb{E}_- \theta^{N^+(t)}. \end{aligned}$$

Here, $\mathbb{E}_\pm$ is the expectation given that at time 0 in our universe, there is just one particle of type $\pm$ at position 0 on $\mathbb{R}$. We shall have

$$\dot{g}^+(t,\theta) := \frac{\partial}{\partial t} g(t,\theta) = G^+\big(g^+(t,\theta)\big),$$

where G^+ is the infinitesimal-generator function for N^+, and

$$h^+(t,\theta) = \mathbb{E}_-\left(\theta^{N^+(t)}\,\middle|\,N^+(0)\right) = \mathbb{E}_- g^+(t,\theta)^{N^+(0)} = H^+\big(g^+(t,\theta)\big).$$

It is therefore enough to find G^+ and H^+.

3. Calculations.

By decomposing the behaviour according to the first jump, we have

$$g^+(t,\theta) = \theta e^{-tK_+/b_+} + \int_0^{t/b_+} e^{-K_+ s}\left\{\lambda_+ + q_+ h^+(t-b_+s,\theta) + r_+ g^+(t-b_+s,\theta)^2\right\} ds,$$

equivalently,

$$e^{K_+t/b_+} g^+(t,\theta) = \theta + b_+^{-1}\int_0^t e^{K_+v/b_+}\left\{\lambda_+ + q_+h^+(v,\theta) + r_+g^+(v,\theta)^2\right\} dv,$$

whence

$$b_+\dot{g}^+(t,\theta) = \lambda_+[1-g^+(t,\theta)] + q_+[h^+(t,\theta)-g^+(t,\theta)] + r_+[g^+(t,\theta)^2 - g^+(t,\theta)].$$

Similarly,

$$h^+(t,\theta) = \int_0^\infty e^{-K_- t}\left\{\lambda_- + q_-g^+(t+b_-u,\theta) + r_-h^+(t+b_-u,\theta)^2\right\} du,$$

whence it follows easily that

$$-b_-\dot{h}^+(t,\theta) = \lambda_-[1-h^+(t,\theta)] + q_-[g^+(t,\theta)-h^+(t,\theta)] + r_-[h^+(t,\theta)^2 - h^+(t,\theta)].$$

Now recall that $h^+(t,\theta) = H^+\big(g^+(t,\theta)\big)$.

4. First conclusions. We have shown that $y = H^+(x)$ $(0 \le x < 1)$ *is an integral curve for the dynamical system*

$$\text{(4.1)(a)} \qquad +b_+x'(t) = \lambda_+[1-x(t)] + q_+[y(t)-x(t)] + r_+[x(t)^2 - x(t)],$$
$$\text{(4.1)(b)} \qquad -b_-y'(t) = \lambda_-[1-y(t)] + q_-[x(t)-y(t)] + r_-[y(t)^2 - y(t)].$$

This curve connects the point $(0, H^+(0))$ on the (closed) left-hand edge of the unit square to the point $(1, H^+(1-))$ on the right-hand edge. Note that H^+ is a convex function on $[0,1)$: indeed, all its derivatives are non-negative.

The symmetry of (4.1) shows that, with obvious notation relating to the $^-$Universe, *the curve* $x = H^-(y)$ $(0 \le y < 1)$ *is also an integral curve for the system (4.1)*, and it links the top and bottom edges of the unit square. Any point at which the two 'probabilistic' curves $y = H^+(x)$ and $x = H^-(y)$ cross within the unit square must be an equilibrium point of the system (4.1). But, of course, more is true: if these curves do cross, they dominate the topology of the system (4.1).

This note *is* only an appetizer; and the only remaining thing for it to do is to provide some pictures of some integral curves for the system (4.1) for certain sets of parameters. Of course, there are lots of fascinating questions, even for this simplest case. The n-dimensional and infinite-dimensional generalizations are still more interesting. But that is work to share with research students, one of whom, Owen Lyne, has already done nice work on related travelling-wave problems. If we put $u(t,a) = 1 - x(a-ct)$ and $v(t,a) = 1 - y(a-ct)$ in (4.1), we obtain a system with first equation

$$\frac{\partial u}{\partial t} = -(c+b_+)\frac{\partial u}{\partial a} - (\lambda_+ + q_+)u + q_+v + r_+u[1-u],$$

containing a drift term, a death term, a mutation term, and a logistic term.

5. Pictures. In all the pictures,

$$b_+ = b_- = 1.$$

The unit square and the two probabilistic curves within it are shown via solid lines.

Figure (a). Here, $\lambda_+ = 1$, $\lambda_- = 1$, $q_+ = 1$, $q_- = 1$, $r_+ = 2$, $r_- = 2$. The point $(\frac{1}{2}, \frac{1}{2})$ is an equilibrium point, through which our probabilistic curves pass with slopes $2 - \sqrt{3}$ and $2 + \sqrt{3}$. But we see that the two probabilistic curves are here part of the same orbit which encloses the point $(1,1)$. As the computer picture suggests, for these parameters, the point $(1,1)$ is a *centre*: all orbits close to it are periodic. The Reader will discover a simple proof of this assertion for these particular parameters. We have $\mathbb{P}_-[N^+(0) = \infty] > 0$.

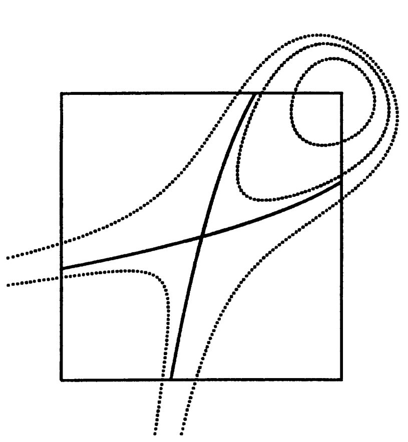
Figure (b). Here, $\lambda_+ = 0$, $\lambda_- = 0$, $q_+ = 1$, $q_- = 4$, $r_+ = 4$, $r_- = 4$. Our particles spend more time in state + than in state −, but the birth rates are high enough to guarantee that $\mathbb{P}_+[N^-(0) = \infty] > 0$. Orbits started near $(1,1)$ spiral in towards $(1,1)$.

Figure (c). Here, $\lambda_+ = 0$, $\lambda_- = 0$, $q_+ = 1$, $q_- = 4$, $r_+ = 0.4$, $r_- = 0.4$. Large-deviation theory makes it *intuitively clear* that since $r_+ + r_- < (\sqrt{q_+} - \sqrt{q_-})^2$, we must have $\mathbb{P}_-[N^+(0) = \infty] = 0$. But it is *topologically obvious* that, since there are no equilibria in the interior of the unit square when $\lambda_+ = \lambda_- = 0$, and since $\mathbb{P}_+[N^-(0) = 0] > 0$, we must have $\mathbb{P}_-[N^+(0) = \infty] = 0$. In this case there are infinitely many integral curves connecting the top and bottom of the unit square, one going steeply (initial slope 5) up from $(0,0)$ and the others contained in the 'triangle' formed by the probabilistic curves.

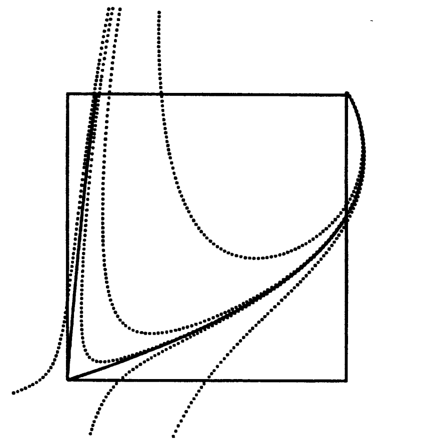
Figure (d). Here, $\lambda_+ = 1$, $\lambda_- = 1$, $q_+ = 1$, $q_- = 1$, $r_+ = 1$, $r_- = 1$. This case is 'critical' in many ways.

Some integral curves for the system (4.1)

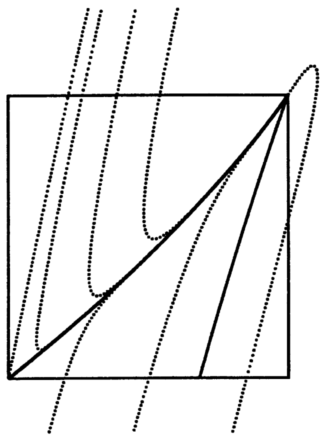
In each case, $b_+ = b_- = 1$



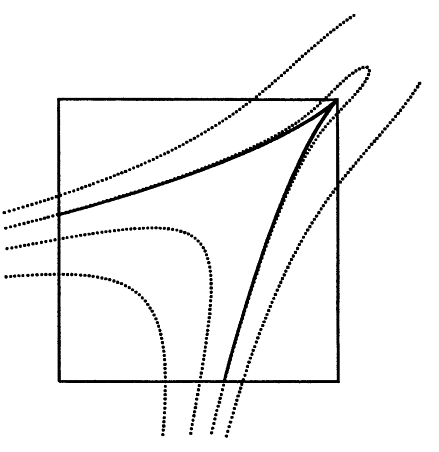
(a) $\lambda_+ = 1,\ \lambda_- = 1,\ q_+ = 1,\ q_- = 1.$
$r_+ = 2,\ r_- = 2.$



(b) $\lambda_+ = 0,\ \lambda_- = 0,\ q_+ = 1,\ q_- = 4.$
$r_+ = 4,\ r_- = 4.$



(c) $\lambda_+ = 0,\ \lambda_- = 0,\ q_+ = 1,\ q_- = 4.$
$r_+ = 0.4,\ r_- = 0.4.$



(d) $\lambda_+ = 1,\ \lambda_- = 1,\ q_+ = 1,\ q_- = 1.$
$r_+ = 1,\ r_- = 1.$

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