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JEAN BERTOIN
MARIA-EMILIA CABALLERO

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On the rate of growth of subordinators with slowly varying Laplace exponent

Jean Bertoin[1] and Ma.-Emilia Caballero[2]

(1) *Laboratoire de Probabilités (CNRS), Université Paris VI, t. 56 4, Place Jussieu 75252 Paris, France*
(2) *Instituto de Matemáticas U.N.A.M., México 04510 D.F. Mexico*

ABSTRACT. Results of Fristedt and Pruitt [6, 7] on the lower functions of a subordinator are improved in the case when the Laplace exponent is slowly varying. This yields laws of the iterated logarithm for the local times of a class of Markov processes. In particular, this extends recent results of Marcus and Rosen [9] on certain Lévy processes close to a Cauchy process.

1 Introduction and main results

Consider a subordinator $\sigma = (\sigma_t, t \geq 0)$, that is σ is a right-continuous increasing process with stationary independent increments and $\sigma_0 = 0$. Denote its Laplace exponent by Φ,

$$\mathbf{E}(\exp -\lambda\sigma_t) = \exp -t\Phi(\lambda) \qquad (\lambda, t \geq 0)$$

and its inverse by S,

$$S_t = \sup\{s : \sigma_s \leq t\} \qquad (t \geq 0).$$

Fristedt and Pruitt [6] proved the following law of the iterated logarithm for S. Introduce the inverse function φ of Φ and put

$$h(x) = \frac{\log\ |\log x|}{\varphi(x^{-1}\log\ |\log x|)} \qquad (x \in (0, 1/e) \cup (e, \infty)).$$

The mapping $x \to h(x)$ increases both in the neighborhood of $0+$ and of ∞, and we denote its inverse by f. Then there exist two constants $c_0, c_\infty \in [1, 2]$ such that

$$\limsup_{t\to 0+} S_t/f(t) = c_0 \qquad \text{and} \qquad \limsup_{t\to\infty} S_t/f(t) = c_\infty \qquad \text{a.s.} \tag{1}$$

Fristedt and Pruitt [7] also obtained precise estimates on the modulus of continuity of S. Specifically, denote by $\tilde{f}$ the inverse function of $\tilde{h}$, where

$$\tilde{h}(x) = \frac{|\log x|}{\varphi(x^{-1}\ |\log x|)} \qquad (0 < x < e)$$

There exist two constants $1 \leq \underline{c} \leq \overline{c} \leq 2$ such that

$$\liminf_{t\to 0+} \sup_{0\leq s\leq\sigma(1)} (S_{s+t} - S_s)/\tilde{f}(t) = \underline{c} \quad \text{a.s.} \tag{2}$$

$$\limsup_{t\to 0+} \sup_{0\leq s\leq\sigma(1)} (S_{s+t} - S_s)/\tilde{f}(t) = \overline{c} \quad \text{a.s.} \tag{3}$$

The constants c_0, c_∞, $\underline{c}$ and $\bar{c}$ do not seem to be known explicitly, except in the case when Φ is regularly varying with index $\alpha \in (0,1)$, see e.g. [1] and the references therein. The main results of this paper are that (1-3) can be made completely explicit when the Laplace exponent Φ is slowly varying. One can check that the argument of the proofs also applies when Φ is regularly varying with index 1; however this case has fewer applications than the preceding, and is left to the interested reader.

First, one has the following law of the iterated logarithm for S. Recall that $\Phi(\infty) < \infty$ only in the degenerate case when σ is a compound Poisson process.

Theorem 1 *Suppose that Φ is slowly varying at $0+$ (respectively, at ∞ and $\Phi(\infty) = \infty$). Then*

$$\limsup \frac{S_t\, \Phi\left(t^{-1} \log\, |\log \Phi(1/t)|\right)}{\log\, |\log \Phi(1/t)|} = 1 \qquad a.s.$$

as $t \to \infty$ (respectively, as $t \to 0+$).

Remark. There is an analogue of Theorem 1 for increasing random walks. This can be deduced from Theorem 1 considering a subordinator with Lévy measure the step distribution of the random walk, and applying the law of large numbers.

Next, we specify the modulus of continuity of S.

Theorem 2 *Suppose that Φ is slowly varying at ∞ and $\Phi(\infty) = \infty$. Then*

$$\lim_{t\to 0+} \sup_{0\le s\le 1} \frac{(S_{t+s} - S_s)\, \Phi\left(t^{-1} \log \Phi(1/t)\right)}{\log \Phi(1/t)} = 1 \quad a.s.$$

Presumably, Theorem 1 should follow (at least for large times) from a characterization due to Pruitt [11] of the lower functions of a general subordinator, but it does not seem straightforward. We will rather establish Theorems 1-2 using elementary lemmas in Fristedt and Pruitt [6, 7] and the observation that it is more fruitfull to work with the explicit functions

$$g(x) = \frac{\log\, |\log \Phi(1/x)|}{\Phi\left(x^{-1} \log\, |\log \Phi(1/x)|\right)} \text{ and } \tilde{g}(x) = \frac{\log\, \Phi(1/x)}{\Phi\left(x^{-1} \log\, \Phi(1/x)\right)} \tag{4}$$

than with the implicit functions f and $\tilde{f}$. The hint for this observation stems from a recent paper of Marcus and Rosen [9] where laws of the iterated logarithm are obtained for the local time of symmetric Lévy processes close to a Cauchy process. In section 3, we show that Theorem 1 can be applied both to give a short proof of the results of Marcus ans Rosen, and to extend them to a broader class of Lévy processes. (A related argument appears in [1] where the law of the iterated logarithm for a subordinator whose Laplace exponent is regularly varying with index $\alpha \in (0,1)$, is used to recover and extend earlier results of Marcus and Rosen [8]).

2 Proof of the theorems

To start with, we establish a simple lemma that holds even when Φ is not regularly varying. Recall that φ stands for the inverse function of Φ.

Lemma 1 *For every* $\gamma > 0$

$$\log \left| \log \Phi \left(\frac{\varphi(\gamma x^{-1} \log \ |\log x|)}{\log |\log x|} \right) \right| \sim \log |\log x|$$

both as $x \to 0+$ *and* $x \to \infty$.

Proof: First, we observe that since Φ increases

$$\begin{aligned} \Phi \left(\frac{\varphi(\gamma x^{-1} \log \ |\log x|)}{\log \ |\log x|} \right) &\leq \Phi \left(\varphi(\gamma x^{-1} \log \ |\log x|) \right) \\ &= \gamma x^{-1} \log \ |\log x| , \end{aligned}$$

provided that x being either small enough or large enough. On the other hand, recall that Φ is concave, so $\Phi(uv) \geq \Phi(u)v$ for all $u \geq 0$ and $v \in (0,1)$. As a consequence

$$\begin{aligned} \Phi \left(\frac{\varphi(\gamma x^{-1} \log \ |\log x|)}{\log \ |\log x|} \right) &\geq \Phi \left(\varphi(\gamma x^{-1} \log \ |\log x|) \right) / \log |\log x| \\ &= \gamma x^{-1} , \end{aligned}$$

provided that x is either small enough or large enough. ■

The next lemmas give respectively the upper and lower bounds in Theorem 1. Recall that the function g is defined in (4).

Lemma 2 *Suppose that* Φ *is slowly varying at* $0+$ *(respectively, at* ∞ *and* $\Phi(\infty) = \infty$*). Then*

$$\limsup S_t/g(t) \leq 1 \quad a.s.$$

as $t \to \infty$ *(respectively, as* $t \to 0+$*).*

Proof: It follows readily from Lemma 4 in Fristedt and Pruitt [6] that for every $\gamma > 1$ and $\delta \in (0, \gamma - 1)$

$$\limsup S_t / f_{\gamma,\delta}(t) \leq 1 \qquad \text{a.s.}$$

both as $t \to 0+$ and as $t \to \infty$, where $f_{\gamma,\delta}$ is the inverse function of

$$x \to \delta \frac{\log \ |\log x|}{\varphi \left(\gamma x^{-1} \log \ |\log x| \right)} .$$

So, all that we need is check that for every $\epsilon > 0$, there exists $\gamma > 1$ and $\delta \in (0, \gamma - 1)$ such that

$$(1 - \epsilon) \, f_{\gamma,\delta}(x) \leq g(x) \tag{5}$$

for all x small enough (respectively, large enough). In this direction, we observe first that

$$g\left(\delta\frac{\log\ |\log x|}{\varphi(\gamma x^{-1}\log\ |\log x|)}\right)\sim g\left(\frac{\log\ |\log x|}{\varphi(\gamma x^{-1}\log\ |\log x|)}\right),$$

because g is slowly varying. Then a few lines of calculation based on Lemma 1 and the hypothesis that Φ is slowly varying show that the right-hand-side is equivalent to

$$\frac{\log\ |\log x|}{\Phi\left(\varphi(\gamma x^{-1}\log\ |\log x|)\right)}=x/\gamma\,.$$

We deduce that (5) holds provided that $\gamma<(1-\epsilon)^{-1}$. ∎

Lemma 3 *Suppose that Φ is slowly varying at 0+ (respectively, at ∞ and $\Phi(\infty)=\infty$). Then*

$$\limsup S_t/g(t)\geq 1 \qquad a.s.$$

as $t\to\infty$ (respectively, as $t\to 0+$).

Proof: It follows now from Lemma 5 in Fristedt and Pruitt [6] that for every $\gamma<1$ and $\delta>\gamma$

$$\limsup S_t/f_{\gamma,\delta}(t)\geq 1 \quad \text{a.s}$$

both as $t\to 0+$ and as $t\to\infty$, where $f_{\gamma,\delta}$ has been defined in the proof of Lemma 2. So, all we need is to check that for every $\epsilon>0$, there exists $\gamma<1$ and $\delta>\gamma$ such that

$$(1+\epsilon)\,f_{\gamma,\delta}(x)\geq g(x) \tag{6}$$

for all x small enough (respectively, large enough). But the argument in Lemma 2 shows that

$$g\left(\delta\frac{\log\ |\log x|}{\varphi(\gamma x^{-1}\log\ |\log x|}\right)\sim x/\gamma\,,$$

and hence (6) holds provided that $\gamma>(1+\epsilon)^{-1}$. ∎

The proof of Theorem 2 is similar. First, one checks readily the following analogue of Lemma 1. For every $\gamma>0$

$$\log\Phi\left(\frac{\varphi(\gamma x^{-1}\ |\log x|)}{|\log x|}\right)\sim\log 1/x \qquad (x\to 0+) \tag{7}$$

(again, this holds even if Φ is not regularly varying). The upper-bound in Theorem 2 then follows from Lemma 5 of Fristedt and Pruitt [7] and (7) much in the same way as in Lemma 2. The lower-bound follows from Lemma 4 of [7] and (7) by an argument close to that in Lemma 3. We skip the details.

3 Applications to local times

We mentioned in the Introduction that the hint for Theorems 1-2 was the results of Marcus and Rosen [9] on the local time of certain symmetric Lévy processes. Conversely, it is interesting to discuss their results in our framework. In this direction, suppose that $X = (X_t, t \geq 0)$ is a standard Markov process started at a regular recurrent point, say 0. Then there exists a local time process at 0, $L = (L_t, t \geq 0)$, and the inverse local time $\sigma_\bullet = \inf\{s: L_s > \bullet\}$ is a subordinator. See Blumenthal and Getoor [2], section 5.3. The inverse S of σ obviously coincides with L, and thus Theorem 1 gives a law of the iterated logarithm for L as time goes to infinity, provided that

$$\text{the Laplace exponent } \Phi \text{ of } \sigma \text{ is slowly varying at } 0+. \tag{8}$$

Suppose now that the Markov process fulfills the duality conditions of chapter VI of Blumenthal and Getoor [2], and denote by $u^\lambda(x,y)$ the adequate version of the resolvent density. Then the local time L can be normalized such that

$$u^\lambda(0,0) = 1/\Phi(\lambda) \qquad (\lambda > 0),$$

and (8) holds if only if $u^\bullet(0,0)$ is slowly varying. When stronger dual conditions are fulfilled, namely when there exist semigroup densities $p_t(x,y)$ and $\widehat{p}_t(x,y)$ is duality, then

$$u^\lambda(0,0) = \int_0^\infty e^{-\lambda t} p_t(0,0)\, dt.$$

By a Tauberian theorem, we see that (8) holds if and only if the so called truncated Green function

$$G(t) = \int_0^t p_s(0,0)\, ds$$

is slowly varying at ∞, and in that case

$$G(t) \sim 1/\Phi(1/t) \qquad (t \to 0+).$$

Of course, the truncated Green function G is slowly varying at infinity whenever

$$p_\bullet(0,0) \quad \text{is regularly varying at } \infty \text{ with index } -1, \tag{9}$$

see e.g. Feller [5, Theorem VII. 9.1], but (9) is a strictly stronger requirement than (8).

Applying this to the case when X is a recurrent symmetric Lévy process having local time L for which (8) holds, we obtain the first part of Theorem 1.2 of Marcus and Rosen [9]. The second part, that is the law of the iterated logarithm for the difference $L - L^a$, where L^a is the local time at level $a \neq 0$, follows from the argument of section 4.2 in [1]. We point out that the result holds under the weaker assumption that the truncated Green function is slowly varying (this was conjectured by Marcus and Rosen) and that the symmetry condition can be dropped (actually, there are also some technical conditions in [9] which are now seen as unnecessary).

This reasoning also allows us to recover the law of the iterated logarithm for the local time process at level 1 for the two-dimensional Bessel process (see Meyre and

Werner [10], equation (1.c) on p. 51). More precisely, (9) holds when $X+1$ is a 2-dimensional Bessel process, and one then obtains

$$\limsup_{t\to\infty} \frac{L_t}{\log t \log_3 t} = 1 \quad \text{a.s.},$$

where $\log_3 = \log\log\log$.

Plainly, similar arguments apply when times tend to 0+, and Theorems 1-2 provide relevant informations on the local rate or growth of the local time of certain Markov processes.

We conclude this section with simple conditions that guaranty that the semigroup density at 0, $p_t(0,0)$, of a real-valued Lévy process X, is regularly varying with index -1. Denote the characteristic exponent by ψ, i.e.

$$\mathbf{E}(\exp i\lambda X_t) = \exp -t\psi(\lambda)$$

for every $t \geq 0$ and $\lambda \in \mathbf{R}$.

Proposition 1 *Assume that the real part $\Re\psi$ of ψ is regularly varying at ∞ with index 1, and that the imaginary part $\Im\psi$ satisfies*

$$\lim_{\lambda\to\infty} \Im\psi(\lambda)/\Re\psi(\lambda) = c \ \in (-\infty,\infty).$$

Then there exists a continuous version of the semigroup density $x \to p_t(0,x)$, and

$$p_t(0,0) \sim \frac{1}{\pi(1+c^2)} r(1/t) \qquad (t\to 0+),$$

where r is an asymptotic inverse of $\Re\psi$. In particular, $p_\bullet(0,0)$ is regularly varying at 0+ with index -1.

Proposition 2 *Assume that for some $t \geq 0$,*

$$\int_{-\infty}^{\infty} \exp\{-t\Re\psi(\lambda)\}\, d\lambda < \infty.$$

Then there exists a continuous version of the semigroup density $x \to p_t(0,x)$. Suppose moreover that $\Re\psi$ is regularly varying at 0+ with index 1 and that

$$\lim_{\lambda\to 0+} \Im\psi(\lambda)/\Re\psi(\lambda) = c \ \in (-\infty,\infty).$$

Then

$$p_t(0,0) \sim \frac{1}{\pi(1+c^2)} r(1/t) \qquad (t\to\infty)$$

where r is an asymptotic inverse of $\Re\psi$. In particular, $p_\bullet(0,0)$ is regularly varying at ∞ with index -1.

The proofs of Propositions 1 and 2 are similar, we shall focus on the latter which is slightly more delicate than the former.

Proof of Proposition 2. The first assertion follows immediately from Fourier inversion, and more precisely, since the density must be real,

$$p_t(0,0) = \frac{1}{\pi}\int_0^\infty \exp\{-t\Re\psi(\lambda)\}\ \cos\{t\Im\psi(\lambda)\}\ d\lambda\,. \tag{10}$$

Then put $R(\lambda) = \min\{\Re\psi(\mu),\ 0 \le \mu \le \lambda\}$ and recall from Theorem 1.5.3 in [3] that $R \sim \Re\psi$. Denote by r the inverse of R, so that r is an asymptotic inverse of $\Re\psi$ and its regularly varying at 0+ with index 1, see Theorem 1.5.12 in [3]. On the one hand, we have by an Abelian theorem

$$\int_0^\infty \exp\{-tR(\lambda)\}\ d\lambda = \int_0^\infty \exp\{-\lambda t\}\ dr(\lambda) \sim r(1/t) \quad (t \to \infty)\,. \tag{11}$$

On the other hand,

$$\int_0^\infty \exp\{-tR(\lambda)\}\ d\lambda = r(1/t)\int_0^\infty \exp\{-tR(\lambda r(1/t))\}\ d\lambda$$

and we know that $tR(\lambda r(1/t))$ converges pointwise to λ as $t \to \infty$. We deduce from (11) that

$$\lim_{t\to\infty}\int_0^\infty \exp\{-tR(\lambda r(1/t))\}\ d\lambda\ =\ 1\ =\ \int_0^\infty e^{-\lambda}\, d\lambda$$

and this implies that the family of nonnegative functions

$$\lambda \to \exp\{-tR(\lambda r(1/t))\} \qquad (t \ge 1)$$

is uniformly integrable, see e.g. Theorem I.21 in Dellacherie-Meyer [4].

Then we re-express (10) as

$$\pi p_t(0,0)/r(1/t) = \int_0^\infty \exp\{-t\Re\psi(\lambda r(1/t))\}\ \cos\{t\Im\psi(\lambda r(1/t))\}\ d\lambda$$

By hypothesis, the integrand converges pointwise to $\exp\{-\lambda\}\cos\{c\lambda\}$ as $t \to \infty$ and its absolute value is bounded by $\exp\{-tR(\lambda r(1/t)\}$ which is uniformly integrable. Thus the integral converges to

$$\int_0^\infty e^{-\lambda}\cos(\lambda c)\, d\lambda = \frac{1}{1+c^2}\,,$$

see e.g. Theorem I.21 in [4]. This proves our assertions. ∎

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