SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

JEAN JACOD
ANATOLI VLADIMIROVICH SKOROHOD
Jumping filtrations and martingales with finite variation

JUMPING FILTRATIONS AND MARTINGALES WITH FINITE VARIATION

J. JACOD and A.V. SKOROHOD

ABSTRACT: On a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t \geq 0}$ is called a *jumping filtration* if there is a sequence (T_n) of stopping times increasing to $+\infty$, such that on each set $\{T_n \leq t < T_{n+1}\}$ the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{T_n}$ coincide up to null sets. The main result is that $(\mathcal{F}_t)$ is a jumping filtration iff all martingales have a.s. locally finite variation.

1 - INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a probability space. By definition, a (right-continuous) filtration $(\mathcal{F}_t)_{t \geq 0}$ is called a *jumping filtration* if there exists a *localizing sequence* $(T_n)_{n \in \mathbb{N}}$ (i.e. a sequence of stopping times increasing a.s. to $+\infty$) with $T_0 = 0$ and such that for all $n \in \mathbb{N}$, $t > 0$:

the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{T_n}$ coincide up to null sets on $\{T_n \leq t < T_{n+1}\}$. (1)

The sequence (T_n) is then called a *jumping sequence*. Note that it is by no means unique. Our aim is to prove the

THEOREM 1: *A filtration is a jumping filtration iff all its martingales are a.s. of locally finite variation* (here and throughout the paper, martingales are supposed to be càdlàg).

The necessary condition is easy (see Section 2) and not surprising, in view of the following known fact: consider a marked point process, that is an increasing sequence (T_n) of times, and associated marks X_n taking values in some measurable space $(E, \mathcal{E})$, and suppose that $T_n \uparrow \infty$ a.s. Let $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by some initial σ-field $\mathcal{G}$ and the marked point process

(*i.e.* the smallest filtration such that $\mathcal{G} \subseteq \mathcal{F}_0$ and each T_n is a stopping time and X_n is $\mathcal{F}_{T_n}$-measurable). Then one knows (see [1]) that $(\mathcal{F}_t)$ is a jumping filtration with jump times (T_n), and if further $(E, \mathcal{E})$ is a Blackwell space all martingales have a.s. locally finite variation.

In fact, *any* jumping filtration is generated by some marked point process, with a "very large" set of marks: take times T_n as in (1), and $(E, \mathcal{E}) = \Pi_{n \in \mathbb{N}}(E_n, \mathcal{E}_n)$ where $E_n = \Omega \cup \{\Delta\}$ (Δ is an extra point) and $\mathcal{E}_n$ is the σ-field of E_n generated by $\mathcal{F}_{T_n}$, and $X_n(\omega)$ is the point of E with coordinates Δ, except the nth coordinate which is ω.

So Theorem 1 implies that if all martingales are a.s. of locally finite variation, the filtration is indeed generated by an initial σ-field $\mathcal{F}_0$ and a marked point process.

When the filtration is quasi-left continuous, the sufficient condition is relatively simple to prove, and some additional results are available: this is done in Section 3. The general case needs a systematic use of stochastic integrals w.r.t. random measures: some auxiliary results about these are gathered in Section 4, and the proof is given in Section 5.

2 - THE NECESSARY CONDITION

Assume here that $(\mathcal{F}_t)$ is a jumping filtration, with jumping sequence (T_n). For the necessary part of Theorem 1 it suffices to prove that a uniformly integrable martingale M which is 0 on $[0, T_n]$ and constant on $[T_{n+1}, \infty)$ for some n is a.s. of locally finite variation.

Set $T = T_n$ and $S = T_{n+1}$, and call G a regular version of the law of the pair (S, M_S), conditional on $\mathcal{F}_T$. By hypothesis, for each t there is an $\mathcal{F}_T$-measurable variable N_t such that $M_t = N_t$ a.s. on $\{T \leq t < S\}$. We have the following string of a.s. equalities (the third one comes from the martingale property; further $(u, x) \to |x|$ is G-integrable for a.a. ω, because M is uniformly integrable, and $G'(t) = G((t, \infty] \times \mathbb{R})$):

$$N_t G'(t) 1_{\{T \leq t\}} = E[N_t 1_{\{T \leq t < S\}} | \mathcal{F}_T] = E[M_t 1_{\{T \leq t < S\}} | \mathcal{F}_T]$$

$$= E[M_S 1_{\{T \leq t < S\}} | \mathcal{F}_T] = 1_{\{T \leq t\}} \int G(du, dx) \, x \, 1_{\{u > t\}} \qquad (2)$$

The right-hand side of (2), which we denote by A_t, is a.s. càdlàg with locally finite variation, as a function of t; further, the left-hand side of (2)

is a.s. equal to $M_t G'(t)$ on the set $\{T \leq t < S\}$, so outside a null set we have $M_t G'(t) = A_t$ for all t with $t < S$. Since $M_t = M_S$ for $t \geq S$ and G' is non-increasing, it follows that $t \to M_t$ is a.s. of locally finite variation if $S = \infty$ or if $G'(S) > 0$ or if $G'(S) = 0$ and $G'(S-) > 0$; by definition of G', at least one of these properties holds, hence the result.

3 - THE QUASI-LEFT CONTINUOUS CASE

Recall that the filtration is called *quasi-left continuous* if $\mathcal{F}_{T-} = \mathcal{F}_T$ (up to null sets) for all predictable times T, or equivalently if all martingales are quasi-left continuous. In this case, the proof of the sufficient condition in Theorem 1 is simple, and provides additional information about the existence of a minimal jumping sequence. More precisely, we have:

THEOREM 2: a) *If the filtration $(\mathcal{F}_t)$ is quasi-left continuous and all martingales are a.s. of locally finite variation, then $(\mathcal{F}_t)$ is a jumping filtration. Furthermore there is a jumping sequence $(T_n)_{n \in \mathbb{N}}$ such that*

(i) T_n *is totally inaccessible when $n \geq 1$ and $T_n < T_{n+1}$ if $T_n < \infty$.*

(ii) *Every totally inaccessible time T satisfies $[\![T]\!] \subseteq \cup_{n \geq 1} [\![T_n]\!]$ a.s.*

(iii) *Any other jumping sequence (T'_n) satisfies $\cup [\![T'_n]\!] \supseteq \cup [\![T_n]\!]$ a.s.*

(iv) *Local martingales jump only at the times T_n.*

b) *If $(\mathcal{F}_t)$ is a jumping filtration, with a jumping sequence consisting in totally inaccessible times, then the filtration is quasi-left continuous.*

(iii) means that (T_n) is the unique minimal jumping sequence, while (ii) means that it is the "maximal" sequence of totally inaccessible times.

Proof. We first suppose all the assumptions in (a).

α) Let $\mathcal{T}$ denote the class of all totally inaccessible times. We prove first that for any sequence $(S_n)_{n \geq 1}$ in $\mathcal{T}$ and any $q \in \mathbb{N}$, we have

$$\text{the random set } U = [\![0,q]\!] \cap (\cup_{n \geq 1} [\![S_n]\!]) \text{ is a.s. finite.} \qquad (3)$$

Set $V = \{\omega:$ there are infinitely many s with $(\omega,s) \in U\}$. Suppose that (3) fails, that is $\varepsilon := P(V)/2 > 0$. Call $\pi(A)$ the projection of a subset A of $\Omega \times \mathbb{R}_+$ on Ω. Define by induction optional subsets U_n of U and stopping times $T_n \in \mathcal{T}$ as such: set $U_1 = U$; then if U_n is known the optional section

theorem yields a stopping time T_n such that $[\![T_n]\!] \subseteq U_n$ (hence $T_n \in \mathcal{T}$) and $P(\pi(U_n) \cap \{T_n = \infty\}) \leq \varepsilon 2^{-n}$; then set $U_{n+1} = U_n \setminus [\![T_n]\!]$. Clearly $V \subseteq \pi(U_n)$ for all n, hence $P(V \cap \{T_n = \infty\}) \leq \varepsilon 2^{-n}$ and thus $A := \cap \{T_n < \infty\}$ satisfies $P(V \setminus A) \leq \varepsilon$ and $P(A) \geq \varepsilon > 0$.

Now call M^n the purely discontinuous martingale having a jump of size $+1$ at time T_n if $T_n < \infty$, and which is continuous elsewhere. The bracket of M^n is $[M^n, M^n]_t = 1_{\{T_n \leq t\}}$, and the M^n's are pairwise orthogonal because they have no common jumps. Then the series $\sum \frac{1}{n} M^n$ converges in L^2 to a square-integrable martingale whose variation on $[0,q]$ is bigger than $\sum \frac{1}{n} 1_{\{T_n \leq q\}}$. In particular this variation is infinite on the set A (since $T_n < \infty \Rightarrow T_n \leq q$), so $P(A)=0$ by hypothesis, hence a contradiction and (3) is proved.

β) Next we construct the sequence $(T_n)_{n \in \mathbb{N}}$ by induction. Set $T_0 = 0$. Suppose that T_n is known, and call $\mathcal{T}_n$ the (non-empty) set of all $T \in \mathcal{T}$ with $T \geq T_n$, and $T > T_n$ if $T_n < \infty$. Then define T_{n+1} to be the essential infimum of all T in $\mathcal{T}_n$. Since $S, S' \in \mathcal{T}_n \Rightarrow S \wedge S' \in \mathcal{T}_n$, there is a decreasing sequence $(S_p)_{p \geq 1}$ in $\mathcal{T}_n$, with limit T_{n+1}. In view of (3), we must have $S_p = T_{n+1}$ for p large enough (depending on ω), a.s.: hence $T_{n+1} \in \mathcal{T}_n$ and $T_{n+1} > T_n$ if $T_n < \infty$, and we have (i).

Using (3) once more, we get $\lim_n T = +\infty$ a.s. Since any $T \in \mathcal{T}$ has $T \geq T_{n+1}$ on the set $\{T > T_n\}$ by the definition of T_{n+1}, we have (ii). All local martingale having only totally inaccessible jumps, (iv) follows from (ii).

γ) Next we prove that (T_n) is a jumping sequence. Let $n \in \mathbb{N}$, $t \geq 0$ and $A \in \mathcal{F}_t$, and set $T = T_n$, $S = T_{n+1}$. We consider the martingale $N_s^A = P(A \cap \{T \leq t < S\} | \mathcal{F}_s)$, and also the point process $X_s = 1_{\{S \leq s\}}$ with its compensator Y. Since $A \cap \{T \leq t < S\} \in \mathcal{F}_{S \wedge t}$, we have

$$N_s^A = N_{s \wedge S \wedge t}^A, \qquad N_S^A = 1_B. \qquad (4)$$

Then $M_s = N_s^A - N_{s \wedge T}^A$ is null on $[0,T]$ and constant on $[S \wedge t, \infty)$, and so by (iv) has only one jump at time S, which is $\Delta M_S = \Delta N_S^A 1_{\{S \leq t\}} = -N_{S-}^A 1_{\{t \leq S\}}$. Thus, with the predictable process $H_u = -N_{u-}^A 1_{\{u \geq t\}}$, we obtain $M = X' - Y'$ with $X'_s = \int_0^s H_u dX_u$ and $Y'_s = \int_0^s H_u dY_u$ being the compensator of X'. Hence

$$N_s^A = N_T^A + \int_0^{s \wedge t} N_{u-}^A \, dY_u \qquad \text{a.s. if } T \leq s < S. \qquad (5)$$

Now, observing that $Y_s = 0$ for $s \leq T$ and with $\mathcal{E}(Y)$ denoting the Doléans

exponential of Y, we deduce $N_s^A = N_T^A \, \mathcal{E}(Y)_{S \wedge t}$ if $T \leq s < S$. Similarly $N_t^\Omega = N_T^\Omega \, \mathcal{E}(Y)_{S \wedge t}$ if $T \leq s < S$, hence

$$N_t^A \, N_T^\Omega = N_t^\Omega \, N_T^A \quad \text{a.s. on} \quad \{T \leq t < S\}. \tag{6}$$

Note that $A' = \{N_T^A = N_T^\Omega > 0\}$ is $\mathcal{F}_T$-measurable, and $N_t^A = 1_{A \cap \{T \leq t < S\}}$ and $N_t^\Omega = 1_{\{T \leq t < S\}}$: we readily deduce $A \cap \{T \leq t < S\} = A' \cap \{T \leq t < S\}$ a.s., hence (1).

δ) Now we prove (iii). Let (T_n') be another jumping sequence. If (iii) were not true, there would exist a pair n,p of integers such that $P(T_n' < T_p < T_{n+1}') > 0$. According to a (trivial) extension of Proposition (3.40) of [1], there is a $(0,\infty]$-valued $\mathcal{F}_{T_n}$-measurable variable R such that $S = T_n' + R$ has $S = T_p < \infty$ on the set $\{T_n' < T_p < T_{n+1}'\}$. Now, S is clearly a predictable time, and $P(T_p = S < \infty) > 0$ contradicts the property $T_p \in \mathcal{J}$. ∎

ε) It remains to prove (b). So now we assume that $(\mathcal{F}_t)$ is a jumping filtration, with a jumping sequence (T_n) having $T_n \in \mathcal{J}$ for $n \geq 1$. It is enough to show that if M is a bounded martingale and T is a finite predictable time, then $\Delta M_T = 0$ a.s. on each set $A = \{T_n < T < T_{n+1}\}$. (1) implies $\mathcal{F}_T \cap A = \mathcal{F}_{T-} \cap A = \mathcal{F}_{T_n} \cap A$ up to null sets. Further, $P(B \backslash A) = 0$ if $B = \{T_n < T \leq T_{n+1}\}$, so $\mathcal{F}_T \cap B = \mathcal{F}_{T-} \cap B$ up to null sets as well. But $B \in \mathcal{F}_{T-}$, hence $\Delta M_T 1_B$ is measurable w.r.t. the completion of $\mathcal{F}_{T-}$, and $\Delta M_T 1_B = 1_B E(\Delta M_T | \mathcal{F}_{T-})$ a.s. Since $E(\Delta M_T | \mathcal{F}_{T-}) = 0$ a.s. (M is a martingale and T is predictable), we obtain $\Delta M_T = 0$ a.s. on B. ∎

4 - RANDOM MEASURES AND MARTINGALES WITH FINITE VARIATION

1) Let us begin with two auxiliary results, which are more or less known. We consider two measurable spaces $(G,\mathcal{G})$ and $(H,\mathcal{H})$, and a positive transition measure $\eta(x;dy)$ from $(G,\mathcal{G})$ into $(H,\mathcal{H})$. The first lemma concerns the atoms of maximal mass of $\eta(x,.)$:

LEMMA 3: *Assume that $(H,\mathcal{H})$ is a Polish space with its Borel σ-field and that $\eta(x,H) \leq 1$ for all $x \in G$. Then if $\alpha(x) = \sup(\nu(x,\{y\}): y \in H)$:*

a) *α is $\mathcal{G}$-measurable.*

b) *There is a measurable function $\zeta: (G,\mathcal{G}) \to (H,\mathcal{H})$ such that $\alpha(x) = \eta(x,\{\zeta(x)\})$.*

c) *There is a $\mathcal{G} \otimes \mathcal{H}$-measurable set B such that $\frac{1}{2} \leq \int \eta(x,dy) 1_B(x,y) \leq \frac{3}{4}$ if $\alpha(x) \leq \frac{1}{4}$ and $\eta(x,E) > \frac{3}{4}$.*

Proof. There is a bi-measurable bijection φ from H into a Borel subset H' of [0,1) containing 0, and we set $\eta'(x,A) = \eta(x,\varphi^{-1}(A\cap H'))$ for every Borel subset of [0,1). Then y' is an atom of $\eta'(x,.)$ iff $y'=\varphi(y)$ where y is an atom of $\eta(x,.)$, with the same mass. Set $A(n,m)=[m2^{-n},(m+1)2^{-n})$.

a) The functions $f_n(x) = \sup(\eta'(x,A(n,m))\colon 0\leq m\leq 2^n-1)$ are measurable and decreases to $\alpha(x)$, hence the result.

b) Set $M_n(x) = \inf(m\colon \eta'(x,A(n,m))\geq\alpha(x))$. If $\alpha(x)=0$ then $M_n(x)=0$ and $A(n,M_n(x))$ decreases to $\{0\}$. If $\alpha(x)>0$, for all n large enough we have for all m: either $A(n,m)$ contains exactly one atom of $\eta'(x,.)$ of mass $\alpha(x)$, or it contains no such atom and $\eta'(n,A(n,m))<\alpha(x)$; thus for n large enough we have $A(n+1,M_{n+1}(x))\subseteq A(n,M_n(x))$. Hence for all x the sequence $A(n,M_n(x))$ converges as $n\to\infty$ to a singleton, say $\{\zeta'(x)\}$, with $\zeta'(x)=0$ if $\alpha(x)=0\in H'$ and $\zeta'(x)\in H'$ otherwise (because $\eta'(x,\{\zeta'(x)\})=\alpha(x)$). Then $\zeta = \zeta'\circ\varphi^{-1}$ satisfies the requirements.

c) Set $U(x) = 1\wedge\inf(t\geq 0\colon\eta'(x,[0,t])\geq 1/2)$. If $\alpha(x)\leq 1/4$ and $\eta'(x,[0,1]) = \eta(x,H) > 3/4$ we have $1/2 \leq \eta'(x,[0,U(x)]) \leq 3/4$. Then $B = \{(x,y)\colon \varphi(y)\in[0,U(x)]\}$ answers the question. ∎

The second lemma is a variation on the fact that if $L^2(\mu)\subseteq L^1(\mu)$ for a measure μ, then μ is of finite total mass, and it results from discussions with J. Azéma and Ph. Biane.

LEMMA 4: *Assume that there is a $\mathcal{G}\otimes\mathcal{H}$-measurable partition $(F_n)_{n\geq 1}$ of $G\times H$ such that $\int\eta(x,dy)1_{F_n}(x,y) \leq 1$ for all n. There is a $\mathcal{G}\otimes\mathcal{F}$-measurable function U with $0<U\leq 1$ and $\int\eta(x,dy)U^2(x,y) \leq 1$ for all $x\in G$, and*

$$\int\eta(x,dy)U(x,y) = \infty \quad \Leftrightarrow \quad \eta(x,F) = \infty. \tag{7}$$

Proof. We define by induction the sequence $\gamma_n(x)$, with $\gamma_0(x)=0$ and

$$\gamma_{n+1}(x) = \inf(m\colon \sum_{i\colon\gamma_n(x)<i\leq m}\int\eta(x,dy)1_{F_i}(x,y) \geq 1).$$

Set $N(x) = \inf(n\colon \gamma_n(x)=\infty)$ and $K_n = \cup_{i\geq 1}[F_i\cap\{(x,y)\colon\gamma_{n-1}(x)<i\leq\gamma_n(x)\}]$ if $n\geq 1$. Finally set $\delta = \sum_{n\geq 1} n^{-2}$ and $U = (1\vee 2\delta)^{-1/2}\sum_{n\geq 1}\frac{1}{n}1_{K_n}$.

The K_n's constitute a measurable partition of $G\times F$, hence U is measura-

ble and $0 < U \le 1$. By construction $\int \eta(x,dy) 1_{K_n}(x,y) \le 2$, so $\int \eta(x,dy) U^2(x,y) \le$ 1. Further the integral $\int \eta(x,dy) 1_{K_n}(x,y)$ is bigger than 1 if $n < N(x)$ and null if $n > N(x)$, while $\eta(x,F) = \infty \Leftrightarrow N(x) = \infty$, so (7) follows. ∎

2) Now we turn to random measures. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, and $\mathcal{P}$ denotes the predictable σ-field on $\Omega \times \mathbb{R}_+$. Let E be a Polish space with its Borel σ-field $\mathcal{E}$, and $\tilde{\Omega} = \Omega \times \mathbb{R}_+ \times E$, and $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}$. An *integer-valued random measure* is a random measure μ on $\mathbb{R}_+ \times E$ of the form

$$\mu(\omega; dt, dx) = \sum_{s > 0, \gamma_s(\omega) \in E} \varepsilon_{(s, \gamma_s(\omega))}(dt, dx), \tag{8}$$

where γ is an optional process taking values in $E \cup \{\Delta\}$, and for which there is a $\tilde{\mathcal{P}}$-measurable partition (G_n) of $\tilde{\Omega}$ with $E[\sum_{s > 0} 1_{G_n}(.,s,\gamma_s)] < \infty$. It is known that there is such a partition with $E[\sum_{s > 0} 1_{G_n}(.,s,\gamma_s)] \le 1$ for all n.

We denote by ν the (predictable) *compensator* of μ, and we use all notation of [1], Chapter III: in particular if W is a $\tilde{\mathcal{P}}$-measurable function on $\tilde{\Omega}$ we set (with $+\infty$ whenever an integral is not well defined):

$$\hat{W}_t(\omega) = \int_E W(\omega, t, x) \nu(\omega, \{t\}, dx), \qquad a_t = \hat{1}_t = \nu(., \{t\} \times E), \tag{9}$$

$$W * \mu_t = \int_{[0,t] \times E} W(.,s,x) \mu(.; ds, dx), \quad \text{and similarly fo } W * \nu, \tag{10}$$

$$\left. \begin{aligned} C^\infty(W)_t &= (W - \hat{W})^2 * \nu_t + \sum_{s \le t} (1 - a_s)(\hat{W}_s)^2, \\ C^0(W)_t &= |W - \hat{W}| * \nu_t + \sum_{s \le t} (1 - a_s) |\hat{W}_s|. \end{aligned} \right\} \tag{11}$$

Recall that one may define the stochastic integral process $W * (\mu - \nu)$ of W w.r.t. $\mu - \nu$ (for a $\tilde{\mathcal{P}}$-measurable W) iff one may write $W = W' + W''$, with W', W'' $\tilde{\mathcal{P}}$-measurable and $C^\infty(W')_t + C^0(W'')_t < \infty$ a.s. for all $t < \infty$. Further, if $\mathcal{L}^2$ (resp. $\mathcal{L}^1$) is the set of all $\tilde{\mathcal{P}}$-measurable functions W such that $C^\infty(W)_\infty$ (resp $C^0(W)_\infty$) is integrable, we have ([1], Proposition (3.71)):

$$\left. \begin{aligned} W \in \mathcal{L}^2 &\Leftrightarrow W * (\mu - \nu) \text{ is a square-integrable martingale} \\ W \in \mathcal{L}^1 &\Leftrightarrow W * (\mu - \nu) \text{ has integrable variation over } \mathbb{R}_+. \end{aligned} \right\} \tag{12}$$

Finally, we also set

$$\alpha_t(\omega) = \sup_{x \in E} \nu(\omega; \{t, x\}), \qquad J = \{a > 0\}, \qquad K_\varepsilon = \{\alpha > \varepsilon\}. \tag{13}$$

By Lemma 3, α is a predictable process, and J, K_ε are predictable sets.

Note also that $0 \le \alpha \le a \le 1$.

Our main aim in this section is to prove the following two theorems, in which $\mathscr{L}_b^i$ denotes the set of all bounded functions in $\mathscr{L}^i$ $(i=1,2)$, and

$$F_t^\varepsilon = 1_{(K_\varepsilon)^c} * \nu_t + \sum_{s \in K_\varepsilon, s \le t} (1 - \alpha_s). \tag{14}$$

THEOREM 5: *Let $\varepsilon \in (0,1)$. There is equivalence between:*

a) $\mathscr{L}_b^2 \subseteq \mathscr{L}_b^1$.

b) $\mathscr{L}_b^1$ *equals the set of all bounded $\widetilde{\mathscr{P}}$-measurable functions on $\widetilde{\Omega}$.*

c) $\mathscr{L}_b^2$ *equals the set of all bounded $\widetilde{\mathscr{P}}$-measurable functions on $\widetilde{\Omega}$.*

d) $E(F_\infty^\varepsilon) < \infty$.

Proof. We begin with some remarks. If $\varepsilon < \eta$ then $K_\eta \subseteq K_\varepsilon$; if further $s \in K_\varepsilon \setminus K_\eta$, then $1 - \alpha_s \le a_s / \varepsilon$ and $a_s \le (1 - \alpha_s)/(1-\eta)$. Hence

$$F^\varepsilon \le (1 + \frac{1}{\varepsilon})F^\eta, \qquad F^\eta \le (1 + \frac{1}{1-\eta})F^\varepsilon, \tag{15}$$

and (d) does not depend on $\varepsilon \in (0,1)$. In the rest of the proof we take $\varepsilon = 1/4$ and write $K = K_{1/4}$, $F = F^{1/4}$. Next, apply Lemma 3 to $(G, \mathscr{G}) = (\Omega \times \mathbb{R}_+, \mathscr{P})$ and $(H, \mathscr{H}) = (E, \mathscr{E})$, with the measure $\eta((\omega, t), dx) = \nu(\omega, \{t\} \times dx)$: we obtain a predictable E-valued process ζ such that $\alpha_t = \nu(\{t, \zeta_t\})$ and a $\widetilde{\mathscr{P}}$-measurable set B such that $1/2 \le 1_B \le 3/4$ when $\alpha \le 1/4$ and $a > 3/4$. Then the sets $A = \{(\omega, t, \zeta_t(\omega)): (\omega, t) \in K\}$ and $C = [(K^c \cap \{0 < a \le 3/4\}) \times E] \cup [B \cap ((K^c \cap \{a > 3/4\}) \times E)]$ are $\widetilde{\mathscr{P}}$-measurable and satisfy for some predictable process β:

$$\left. \begin{array}{l} A \subseteq K \times E, \qquad \widehat{1}_A = \alpha 1_K , \\[2mm] C \subseteq (J \setminus K) \times E, \quad \widehat{1}_C = \beta 1_{J \setminus K} \text{ with } \beta = a \text{ if } a \le 3/4, \quad 1/2 \le \beta \le 3/4 \text{ if } a > 3/4. \end{array} \right\}$$

This, (14) and the definition of $K = K_{1/4}$ yield

$$F_\infty \le 1_{J^c} * \nu_\infty + 4 \sum_{s \in K} \alpha_s (1 - \alpha_s) + 8 \sum_{s \in J \setminus K} \beta_s (1 - \beta_s), \tag{16}$$

(b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (d). Consider the measure η on $(\widetilde{\Omega}, \widetilde{\mathscr{P}})$ defined by $\eta(W) = E[W 1_{J^c} * \nu_\infty]$. Lemma 4 implies the existence of a $\widetilde{\mathscr{P}}$-measurable function U with $0 < U \le 1$ and $\eta(U^2) < \infty$ and such that $\eta(U) < \infty$ implies $\eta(\widetilde{\Omega}) < \infty$. But (11) yields $E[C^\infty(U 1_{J^c})_\infty] = \eta(U^2)$, so $U 1_{J^c} \in \mathscr{L}_b^2$ by definition of $\mathscr{L}_b^2$, hence (a) implies

$U1_{J^c} \in \mathcal{L}_b^1$ and by (11) again $\eta(U) = E[C^0(U1_{J^c})_\infty] < \infty$. Therefore

$$E(1_{J^c} * \nu_\infty) < \infty. \qquad (17)$$

Next we consider the measure η on $(\Omega \times \mathbb{R}_+, \mathcal{P})$ defined by $\eta(H) = E[\sum_{s \in K} \alpha_s(1-\alpha_s)H_s]$. If H is a predictable process, (11) yields

$$C^\infty(H1_A)_t = \sum_{s \in K, s \leq t} \alpha_s(1-\alpha_s)H_s^2, \qquad C^0(H1_A)_t = \sum_{s \in K, s \leq t} 2\alpha_s(1-\alpha_s)|H_s|, \qquad (18)$$

hence by the same argument as above, (a) and Lemma 4 imply $\eta(\Omega \times \mathbb{R}_+) < \infty$, that is

$$E[\sum_{s \in K} \alpha_s(1-\alpha_s)] < \infty. \qquad (19)$$

Finally, with the measure defined by $\eta(H) = E[\sum_{s \in J \setminus K} \beta_s(1-\beta_s)H_s]$ (and $(\beta, C, J \setminus K)$ instead of (α, A, K) in (18)) one obtains similarly

$$E[\sum_{s \in J \setminus K} \beta_s(1-\beta_s)] < \infty. \qquad (20)$$

Putting together (17), (19) and (20), we deduce (d) from (16).

(d) $\Rightarrow$ (b): Let W be a $\tilde{\mathcal{P}}$-measurable function bounded by a constant δ. First $|\hat{W}| \leq \delta a$, and since $\sum_s a_s(1-a_s) \leq F_s$ (because $1-a \leq 1-\alpha$) we have

$$E[\sum_s (1-a_s)|\hat{W}_s|] < \infty. \qquad (21)$$

From the definition of A, W can be written as $W = U + H1_A$ with $U = W1_{A^c}$ and H a predictable process. We have $|W - \hat{W}| \leq 2\delta$ and $|\hat{U}| \leq \delta \hat{1}_{A^c}$, and $W - \hat{W} = H(1-\alpha) - \hat{U}$ on A, and $\hat{1}_{A^c} = 1-\alpha$ on K, so that with $V = |W - \hat{W}|$:

$$\hat{V} \leq 2\delta \hat{1}_{A^c} + \delta(1-\alpha) + |\hat{U}| \leq 4\delta(1-\alpha) \text{ on } K \qquad (22)$$

Hence with V as above $V * \nu_\infty = (V1_{K^c}) * \nu_\infty + \sum_{s \in K} \hat{V}_t \leq 4\delta F_\infty$, and adding this to (21) yields $E[C^0(W)_\infty] < \infty$, hence $W \in \mathcal{L}_b^1$.

(d) $\Rightarrow$ (c): This is proved as the previous implication, with $V = (W - \hat{W})^2$ satisfying $\hat{V} \leq 8\delta^2(1-\alpha)$ on K instead of (22).

(c) $\Rightarrow$ (d): By hypothesis $E[C^\infty(1)_\infty] < \infty$, hence (17). We also have $E[C^\infty(1_A)_\infty] < \infty$ which gives (19), and $E[C^\infty(1_B)_\infty] < \infty$ which gives (20). ∎

THEOREM 6: *Let $\varepsilon \in (0,1)$. There is equivalence between:*

a) *All stochastic integrals* $W*(\mu-\nu)$ *are a.s. of locally finite variation.*

b) *We have* $F_t^\varepsilon < \infty$ *a.s. for all* $t<\infty$.

Proof. (b) $\Rightarrow$ (a). By localization we can assume $E(F_\infty^\varepsilon)<\infty$. According to (3.69), (3.70) and (3.71) of [1], any stochastic integral $W*(\mu-\nu)$ is the sum of a local martingale with locally finite variation and another stochastic integral $W'*(\mu-\nu)$ with W' bounded. Then (d) $\Rightarrow$ (c) of Theorem 5 shows the result.

(a) $\Rightarrow$ (b). If $C^\infty(W)_t<\infty$ (or $E[C^\infty(W)_t]<\infty$) for all t, (a) and (12) yield $C^0(W)_t<\infty$ a.s. for all t; there is even a localizing sequence (T_n) with $E[C^0(W)_{T_n}]<\infty$, so we have *locally* (a) of Theorem 5. However the localizing sequence *a priori* depends on W, so we cannot apply (a) $\Rightarrow$ (d) of Theorem 5.

Below we assume (a), and we fix $t>0$. In order to prove $F_t^\varepsilon<\infty$ a.s. it is enough by (15) and (16) to prove, with $K=K_{1/4}$:

$$1_{J^c}*\nu_t < \infty, \qquad \sum_{s\in K, s\le t} \alpha_s(1-\alpha_s) < \infty, \qquad \sum_{s\in J\setminus K, s\le t} \beta_s(1-\beta_s) < \infty. \qquad (23)$$

We can apply Lemma 4 with $(G,\mathcal{G}) = (\Omega,\mathcal{F})$, $(H,\mathcal{H}) = (\mathbb{R}_+\times E, \mathcal{R}_+\otimes\mathcal{E})$ and $\eta(\omega,.) = (1_{J^c\cap[0,t]}\cdot\nu)(\omega,.)$: there is an $\mathcal{F}\otimes\mathcal{R}_+\otimes\mathcal{E}$-measurable function U with $0<U\le1$ and $\int\eta(\omega;ds,dx)U^2(\omega,s,x) \le 1$ and (7). If $\rho(d\omega,ds,dx) = P(d\omega)\eta(\omega;ds,dx)$ there is a $\tilde{\mathcal{P}}$-measurable partition (D_n) of $\tilde{\Omega}$ with $\rho(D_n)<\infty$, so $W = \rho(U|\tilde{\mathcal{P}})$ is well defined. We then have $0\le W\le1$, $W^2 \le \rho(U^2|\tilde{\mathcal{P}})$, and $\rho(U^2) = E[\int\eta(.;ds,dx)U^2(.,s,x)] \le 1$, hence $E[C^\infty(W1_{J^c\cap[0,t]})_\infty] = \rho(W^2) \le 1$ and (a), (12) and a localization argument imply $C^0(W1_{J^c\cap[0,t]})_\infty < \infty$ a.s. Hence there is a localizing sequence (T_n) such that

$$\rho(U1_{[0,T_n]}) = \rho(W1_{[0,T_n]}) = E[(W1_{J^c})*\nu_{T_n\wedge t}] < \infty,$$

hence $\int\eta(ds,dx)U(s,x) < \infty$ a.s. on $\cup_n\{T_n\ge t\} = \Omega$. Then (7) yields $\eta(\mathbb{R}_+\times E)<\infty$ a.s., which gives the first part of (23).

We apply the same argument with $\eta(\omega,ds) = \sum_{r\in K, r\le t} \alpha_r(1-\alpha_r)\varepsilon_r(ds)$, and $(G,\mathcal{G}) = (\Omega,\mathcal{F})$ and $(D,\mathcal{D}) = (\mathbb{R}_+,\mathcal{R}_+)$. Let U be as in Lemma 4. Then set $\rho(d\omega,ds) = P(d\omega)\eta(\omega,ds)$ and $H = \rho(U|\mathcal{P})$, which has $0\le H\le1$ and $\rho(H^2)\le1$. Then $W = H1_A$ satisfies (18), so $E[C^\infty(W)_\infty] = \rho(H^2) \le 1$, hence we deduce exactly as above that $C^0(W)_\infty<\infty$ a.s., and that $\int\eta(ds)U(s) < \infty$ a.s., hence $\eta(\mathbb{R}_+)<\infty$ a.s. and this gives the second part of (23).

The third part of (23) is proved similarly, upon substituting $(\beta,J\setminus K,B)$ with (α,K,A). $\blacksquare$

3) So far the measure μ was fixed, but here we allow it to change. Denote by $\mathcal{A}$ the class of all integer-valued random measures, on all Polish spaces E, and by $\mathcal{A}_0$ the subclass of all measures in $\mathcal{A}$ such that the process α associated by (13) has $\alpha_t \leq 1/2$ identically. If $\mu \in \mathcal{A}$ has the compensator ν, let $\mathcal{M}(\mu)$ be the space of all local martingales of the form $W*(\mu-\nu)$.

PROPOSITION 7: *If $\mu \in \mathcal{A}$ there exists $\mu' \in \mathcal{A}_0$ with $\mathcal{M}(\mu') = \mathcal{M}(\mu)$.*

Proof. a) We start with $\mu \in \mathcal{A}$, on the Polish space E, and with compensator ν. As in the proof of Theorem 5, there is a predictable E-valued process ζ with $\alpha_t = \nu(\{t, \zeta_t\})$. Recall the process γ in (8), and in (13) set $K = K_{1/2}$.

The measure μ' will be on $\mathbb{R}_+ \times E'$, with $E' = E \times \{0\} + E_\Delta \times \{1\}$. We set $E'_\Delta = E' + \{\Delta'\}$, and

$$
\gamma'_t = \begin{cases} (\gamma_t, 0) & \text{if } t \notin K \text{ and } \gamma_t \in E \\ (\gamma_t, 1) & \text{if } t \in K \text{ and } \gamma_t \in E_\Delta \backslash \{\zeta_t\} \\ \Delta' & \text{otherwise.} \end{cases} \tag{24}
$$

This is an optional E'_Δ-valued process, with which one associates the random measure μ' by (8). Clearly $\mu' \in \mathcal{A}$, and we add a dash to all quantities related to μ': e.g. ν', α', etc...

b) We presently prove that $\mu' \in \mathcal{A}_0$. First, with any $\tilde{\mathcal{P}}'$-measurable function W' on $\tilde{\Omega}' = \Omega \times \mathbb{R}_+ \times E'$, we associate the $\tilde{\mathcal{P}}$-measurable function $f(W')(\omega, t, x) = W'(\omega, t, (x, 0))$. We have

$$
W'1_{K^c}*\mu' = f(W')1_{K^c}*\mu, \qquad W'1_{K^c}*\nu' = f(W')1_{K^c}*\nu \tag{25}
$$

(the first equality is obvious, and the second one follows by taking the compensators). Thus $\alpha'_t = \alpha_t \leq 1/2$ and $a'_t = a_t$ if $t \in K^c$. Next, since for every finite predictable time T the measure $\nu(\{T\} \times .)$ is characterized by the property $\nu(\{T\} \times A) = P(\gamma_T \in A | \mathcal{F}_{T-})$, and similarly for ν', up to changing ν' on a null set we can assume that we have identically:

$$
\left. \begin{array}{l} \nu'(\{t\} \times (E \times \{0\})) = 0, \qquad \nu'(\{t\} \times (C \times \{1\})) = \nu(\{t\} \times (C \backslash \{\zeta_t\})), \\ \nu'(\{t\} \times \{\Delta, 1\}) = 1 - \alpha_t. \end{array} \right\} \text{ if } t \in K. \tag{26}
$$

Then $\alpha'_t \leq a'_t = 1-\alpha_t < 1/2$ if $t \in K$. Hence $\alpha'_t \leq 1/2$ for all t, and $\mu' \in \mathcal{A}_0$.

c) It remains to prove $\mathcal{M}(\mu')=\mathcal{M}(\mu)$. Since stochastic integrals w.r.t. random measures are characterized by their jumps, it suffices to prove that if W is $\tilde{\mathcal{P}}$-measurable (resp. W' is $\tilde{\mathcal{P}}'$-measurable), we can find a $\tilde{\mathcal{P}}'$-measurable W' (resp. a $\tilde{\mathcal{P}}$-measurable W) such that for all t:

$$W(t,\gamma_t)1_E(\gamma_t)-\int_E \nu(\{t\}\times dx)W(t,x) = W'(t,\gamma'_t)1_{E'}(\gamma'_t)-\int_{E'} \nu'(\{t\}\times dy)W'(t,y). \qquad (27)$$

$\tilde{\mathcal{P}}$-measurable functions are functions of the form

$$W(t,x) = H_t 1_{\{\zeta_t\}}(x) + U(t,x), \quad \text{with} \quad U(t,\zeta_t) = 0, \qquad (28)$$

where H is predictable and U is $\tilde{\mathcal{P}}$-measurable, and the left-hand side of (27) is then

$$H_t 1_{\{\zeta_t\}}(\gamma_t) + U(t,\gamma_t)1_{E\setminus\{\zeta_t\}}(\gamma_t) - \alpha_t H_t - \hat{U}_t. \qquad (29)$$

$\tilde{\mathcal{P}}'$-measurable functions are functions of the form (with $(x,i)\in E'$):

$$W'(t,(x,i)) = G_t^i 1_{\{\zeta_t\}}(x) + G_t 1_{\{\Delta,1\}}(x,i) + U^i(t,x), \quad \text{with} \quad U^i(t,\zeta_t) = 0, \qquad (30)$$

where G, G^0, G^1 are predictable and U^0, U^1 are $\tilde{\mathcal{P}}$-measurable on $\tilde{\Omega}$. Due to (24), (25) and (26), the right-hand side of (27) is then

$$\left.\begin{array}{ll} G_t^0 1_{\{\zeta_t\}}(\gamma_t) + U^0(t,\gamma_t)1_{E\setminus\{\zeta_t\}}(\gamma_t) - \alpha_t G_t^0 - \hat{U}_t^0 & \text{if } t \notin K \\[2mm] G_t 1_{\{\Delta\}}(\gamma_t) + U^1(t,\gamma_t)1_{E\setminus\{\zeta_t\}}(\gamma_t) - (1-a_t)G_t - \hat{U}_t^1 & \text{if } t \in K \end{array}\right\} \qquad (31)$$

If we start with (28) and if we define W' by (30) with $G^1=0$, $G^0=H1_{K^c}$, $U^0=U1_{K^c}$, $G=-H1_K$, $U^1(t,x) = [U(t,x)-H_t 1_{\{x\neq\zeta_t\}}]1_K(t)$, a simple computation shows that (29) and (31) are equal. We also have equality between (29) and (31) if we start with (30) and define W by (28), with $H = H^0 1_{K^c}-G1_K$ and $U(t,x) = U^0(t,x)1_{K^c}(t) + [U^1(t,x)-G_t 1_{\{x\neq\zeta_t\}}]1_K(t)$. Therefore (27) holds. ∎

COROLLARY 8: *Let $\mu\in\mathcal{A}$. There is equivalence between:*

a) All elements of $\mathcal{M}(\mu)$ are a.s. of locally finite variation.

b) For every $\mu'\in\mathcal{A}_0$ such that $\mathcal{M}(\mu')=\mathcal{M}(\mu)$ we have $1*\nu'_t<\infty$ a.s. for all $t<\infty$ (or equivalently $1*\mu'_t<\infty$ a.s. for all $t<\infty$).

Proof. (a) $\Rightarrow$ (b): Let $\mu' \in \mathcal{A}_0$ with $\mathcal{M}(\mu')=\mathcal{M}(\mu)$. Then (a) $\Rightarrow$ (b) of Theorem 6 applied to μ' implies $F_t'^{1/2}<\infty$ a.s., where $F'^{1/2}$ is associated with μ' by (14). Since $\mu' \in \mathcal{A}_0$ we have $K_{1/2}'=\varnothing$, so $1*\nu_t'<\infty$ a.s. It is also well known that $1*\nu_t'<\infty$ a.s. for all $t<\infty$ is equivalent to $1*\mu_t'<\infty$ a.s. for all $t<\infty$.

(b) $\Rightarrow$ (a): This readily follows from (b) $\Rightarrow$ (a) of Theorem 6 and from the existence of $\mu' \in \mathcal{A}_0$ with $\mathcal{M}(\mu')=\mathcal{M}(\mu)$. ∎

5 - THE SUFFICIENT CONDITION OF THEOREM 1

For the proof of the sufficient condition in Theorem 1, we need two preliminary lemmas. If $\mu \in \mathcal{A}$ is given by (8), we set $D(\mu) = \{(\omega,t): \gamma_t(\omega) \in E\}$.

LEMMA 9: *For any sequence* (μ_n) *in* $\mathcal{A}_0$ *there exists* $\mu \in \mathcal{A}_0$ *such that* $D(\mu) = \cup_n D(\mu_n)$ *and* $\cup_n \mathcal{M}(\mu_n) \subseteq \mathcal{M}(\mu)$.

Proof. For each n, μ_n is a random measure on the Polish space E^n, with the associated process γ^n (see (8)). Set $E = \Pi E_\Delta^n$, $E_\Delta = E+\{\Delta\}$, and define an E_Δ-valued optional process γ by

$$\gamma_t = \begin{cases} (\gamma_t^1, \ldots, \gamma_t^n, \ldots) & \text{if } t \in \cup D(\mu_n) \\ \Delta & \text{otherwise.} \end{cases}$$

The associated measure μ belongs to $\mathcal{A}_0$ (indeed if $x=(x_1, x_2, ..)$ is an atom of $\nu(\{t\}\times.)$, there is at least one n such that $x_n \neq \Delta$ and x_n is an atom of $\nu_n(\{t\}\times.)$, and $\nu(\{t,x\}) \leq \nu_n(\{t,x_n\})$, so $\alpha_t \leq \sup_n \alpha_t^n \leq 1/2$). By construction $D(\mu) = \cup_n D(\mu_n)$. Finally if $M = W_n*(\mu_n-\nu_n) \in \mathcal{M}(\mu_n)$ one easily checks that $M = W*(\mu-\nu) \in \mathcal{M}(\mu)$ if $W(\omega,t,(x_1,...)) = W_n(\omega,t,x_n)1_{E^n}(x_n)$. ∎

LEMMA 10: *Assume that all martingales are a.s. of locally finite variation.*

a) *If* $\mu \in \mathcal{A}_0$ *the set* $D(\mu)$ *has almost all its* $\mathbb{R}_+$-*sections locally finite.*

b) *There exists* $\mu \in \mathcal{A}_0$ *such that any other* $\mu' \in \mathcal{A}_0$ *has* $D(\mu') \subseteq D(\mu)$ *a.s.*

Proof. a) The $\mathbb{R}_+$-section of $D(\mu)$ through ω is locally finite iff $1*\mu_t(\omega)<\infty$ for all $t<\infty$, so the claim follows from Corollary 8.

b) We construct by induction an increasing sequence of stopping times $(T_n)_{n \geq 0}$ and a sequence $(\mu_n)_{n \geq 1}$ of elements of $\mathcal{A}_0$ with the following:

$$T_n < \infty \;\Rightarrow\; T_n < T_{n+1} \quad \text{a.s.,} \qquad [\![T_n]\!] \subseteq D(\mu_n) \quad \text{a.s.}$$
$$\text{for all } \mu \in \mathscr{A}_0, \quad \text{we have } \;]\!]T_{n-1}, T_n[\![\cap D(\mu) = \varnothing \quad \text{a.s.}$$

$$(32)$$

We start with $T_0 = 0$. Suppose that we know (T_n, μ_n) with (32) for $n \leq p$. Set $S(\mu) = \inf(t > T_p : t \in D(\mu))$ if $\mu \in \mathscr{A}_0$, and $T_{p+1} = \text{ess inf}(S(\mu) : \mu \in \mathscr{A}_0)$. (a) implies $S(\mu) > T_p$ a.s. on the set $A = \{T_p < \infty\}$. By Lemma 9 if $\mu, \mu' \in \mathscr{A}_0$ there is $\mu'' \in \mathscr{A}_0$ with $D(\mu'') = D(\mu) \cup D(\mu')$, hence $S(\mu'') = S(\mu) \wedge S(\mu')$. Therefore there exists a sequence (ρ_n) in $\mathscr{A}_0$ such that $S(\rho_n)$ decreases a.s. to T_{p+1}. Applying again Lemma 9, we obtain $\mu_{p+1} \in \mathscr{A}_0$ with $D(\mu_{p+1}) = \cup_n D(\rho_n)$, so $T_{p+1} = S(\mu_{p+1})$ a.s. and thus $T_{p+1} > T_p$ a.s. on A. We have $[\![T_{p+1}]\!] \subseteq D(\mu_{p+1})$ a.s., and for any $\mu \in \mathscr{A}_0$ we have $T_{p+1} \leq S(\mu)$ a.s., so the last property in (32) is satisfied for $n = p+1$.

So far, we have constructed the sequences (T_n), (μ_n) with (32). Taking the measure μ associated with the sequence (μ_n) in Lemma 9 gives (b). ∎

Proof of the sufficient condition of Theorem 1. We assume that all martingales are a.s. of locally finite variation. Let $\mu \in \mathscr{A}_0$ be the measure constructed in Lemma 10, and $T_n = \inf(t : 1 * \mu_t = n)$. Since $1 * \mu_t < \infty$ a.s. for all $t < \infty$, we have outside a null set: $T_0 = 0$, $T_n \uparrow \infty$, $T_n < T_{n+1}$ if $T_n < \infty$. We will prove that $(\mathscr{F}_t)$ is a jumping filtration with jumping sequence (T_n). To this effect, it suffices to prove that for $t \geq 0$, $A \in \mathscr{F}_t$, $n \in \mathbb{N}$ fixed, there exists $A' \in \mathscr{F}_{T_n}$ with

$$A \cap \{T_n \leq t < T_{n+1}\} = A' \cap \{T_n \leq t < T_{n+1}\} \quad \text{a.s.} \qquad (33)$$

The proof is similar to part (γ) of the proof of Theorem 2. We set $T = T_n$, $S = T_{n+1}$, and consider the martingale $N_s^A = P(A \cap \{T \leq t < S\} | \mathscr{F}_s)$. Let $\rho \in \mathscr{A}$ be the random measure associated with the jumps of the pair (N^A, N^Ω): it is given by (8) with $E = \mathbb{R}^2 \backslash \{0\}$ and $\gamma_t = (\Delta N_t^A, \Delta N_t^\Omega)$, and we know that both N^A and N^Ω belong to $\mathscr{M}(\rho)$. By Proposition 7 there is $\rho' \in \mathscr{A}_0$ with $N^A, N^\Omega \in \mathscr{M}(\rho')$. By Lemma 9 there is $\mu' \in \mathscr{A}_0$ with $N^A, N^\Omega \in \mathscr{M}(\mu')$ and $D(\mu') = D(\mu) \cup D(\rho')$, while Lemma 10 yields $D(\mu') \subseteq D(\mu)$ a.s., so in fact $D(\mu') = D(\mu)$ a.s. Therefore we also have $T_n = \inf(t : 1 * \mu_t' = n)$ a.s., so up to substituting μ with μ' we can and will assume that $N^A, N^\Omega \in \mathscr{M}(\mu)$.

Set $M_s = N_s^A - N_{s \wedge T}^A$, so $M \in \mathscr{M}(\mu)$. As for Theorem 2, we have (4) and $\Delta M_s = -N_{S-}^A 1_{\{t \leq S\}}$. Thus $D(\mu) \cap \{\Delta M \neq 0\} \subseteq [\![S]\!]$ and $\Delta M_s = -N_{s-}^A 1_{\{T < s \leq t \wedge S\}}$ for all $t \in D(\mu)$, outside a null set. In other words, if γ is associated with μ by (8) and if $U(\omega, s, x) = -N_{t-}^A(\omega) 1_{\{T(\omega) < s \leq t \wedge S(\omega)\}}$, outside a null set we have

$\Delta M_t(\omega) = U(\omega, t, \gamma_t(\omega)) 1_E(\gamma_t(\omega))$ for all $(\omega, t) \in D(\mu)$. Since U is $\tilde{\mathcal{P}}$-measurable, we deduce from Theorems (3.45) and (4.47) of [1] that, since $M \in \mathcal{M}(\mu)$:

$$M = W*(\mu - \nu), \quad \text{with} \quad W = U + \frac{\hat{U}}{1-a} 1_{\{a<1\}}.$$

A simple computation shows that $W(s,x) = -\dfrac{1}{1-a_s} 1_{\{a_s<1\}} N_{s-}^A 1_{(T, t \wedge S]}(s)$. If $F = 1*\nu$ we then deduce that

$$M_s = \int_T^{s \wedge t} N_{r-}^A \frac{1}{1-a_r} 1_{\{a_r<1\}} \, dF_r \quad \text{if} \quad T \le s < S.$$

The process $Y_s = \displaystyle\int_{s \wedge T}^{s \wedge t} \frac{1}{1-a_r} 1_{\{a_r<1\}} \, dF_r$ is increasing and finite-valued, and $N_s^A = N_T^A + M_s$ if $s \ge T$, hence (5) holds. Similarly (5) holds for N^Ω, so we deduce (6), and $A' = \{N_T^A = N_T^\Omega > 0\}$ satisfies (33).

REMARK: When the σ-field $\mathcal{F}_\infty$ is separable, the proof is much simpler. Indeed, in this case there is a sequence $(M^n)_{n \in \mathbb{N}}$ of martingales which "generates" (in the stochastic integrals sense) the space of all local martingales. Therefore if μ is the integer-valued random measure on $E = \mathbb{R}^{\mathbb{N}}$ associated with the jumps of the infinite-dimensional process $(M^n)_{n \in \mathbb{N}}$, $\mathcal{M}(\mu)$ is the space of of local martingales, so we do not need Lemmas 9 and 10. ∎

REFERENCE

[1]   J. JACOD: Calcul stochastique et problèmes de martingales. Lect. Notes In Math., 714. Springer Verlag: Berlin, 1979.

J. Jacod: Laboratoire de Probabilités (CNRS, URA 224), Université Paris 6, Tour 56, 4 Place Jussieu, F-75252 PARIS Cedex

A.V. Skorohod: Institute of Mathematics, Ukrainian Acad. Sciences, KIEV