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Exact Rates of Convergence to the Local Times of Symmetric Lévy Processes

Michael B. Marcus*and Jay Rosen†

1 Introduction

Let $X = \{X(t), t \in \mathbf{R}^+\}$ be a symmetric real-valued Lévy process with characteristic function

(1.1) $$Ee^{i\lambda X(t)} = e^{-t\psi(\lambda)}$$

and Lévy exponent

(1.2) $$\psi(\lambda) = 2\int_0^\infty (1 - \cos u\lambda)\, d\nu(u)$$

for ν a Lévy measure, i.e. $\int_0^\infty (1 \wedge u^2)\, d\nu(u) < \infty$. We also include the case $\psi(\lambda) = \lambda^2/2$ which gives us standard Brownian motion.

Such Lévy processes X have an almost surely jointly continuous local time which we denote by $L = \{L_t^x, (t,x) \in \mathbf{R}^+ \times \mathbf{R}\}$, and normalize by requiring that

$$E^0\left(\int_0^\infty e^{-t}\, dL_t^x\right) = u^1(x)$$

where

$$u^1(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos x\lambda}{1 + \psi(\lambda)}\, d\lambda$$

is the 1-potential density for X. We set

(1.3) $$\sigma^2(x) = \frac{2}{\pi}\int_0^\infty \frac{1 - \cos x\lambda}{\psi(\lambda)}\, d\lambda.$$

It follows from Pitman [9] that $\psi(\lambda)$ is regularly varying at infinity of order $1 < \beta \leq 2$, if and only if $\sigma^2(x)$ is regularly varying at zero of order $\beta - 1$, and we have

(1.4) $$\sigma^2(x) \sim c_\beta \frac{1}{x\psi(\frac{1}{x})} \qquad as\ x \to 0$$

with c_β depending only on β. Throughout this paper we use the notation $f \sim g$ to mean that $\lim f/g = 1$.

Since L_t^x is jointly continuous we have that

(1.5) $$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\int_0^t 1_{[x,x+\epsilon]}(X_s)\, ds = L_t^x$$

almost surely for each x, t.

The object of this paper is to determine the exact rates of convergence in (1.5).

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Theorem 1 *Let $X = \{X(t), t \in \mathbf{R}^+\}$ be a real valued symmetric Lévy process with $\sigma^2(x)$ concave on $[0,\delta]$ and regularly varying at zero of order $\beta - 1$ where $1 < \beta \le 2$, and let $\{L_t^x, (t,x) \in \mathbf{R}^+ \times \mathbf{R}\}$ be the local time of X. Then*

$$\limsup_{\epsilon \to 0} \sup_{x \in \mathbf{R}^1} \frac{\left|\frac{1}{\epsilon}\int_0^t 1_{[x,x+\epsilon]}(X_s)\,ds - L_t^x\right|}{\sigma(\epsilon)\sqrt{2\log(1/\epsilon)}} = \sqrt{\frac{2}{\beta+1}} \sup_{x \in \mathbf{R}^1} L_t^x \tag{1.6}$$

almost surely for almost every $t \in \mathbf{R}^+$.

Let us note that the exact uniform modulus of continuity for L_t^x is given by

$$\limsup_{\epsilon \to 0} \sup_{|x-y| \le \epsilon} \frac{|L_t^y - L_t^x|}{\sigma(\epsilon)\sqrt{2\log(1/\epsilon)}} = \sqrt{2 \sup_{x \in \mathbf{R}^1} L_t^x}$$

almost surely for every $t \in \mathbf{R}^+$, which immediately implies (by (2.6) below) that the left hand side of (1.6) is bounded above by $\sqrt{2\sup_{x\in\mathbf{R}^1} L_t^x}$. However, this is not the the actual value, $\sqrt{\frac{2}{\beta+1}\sup_{x\in\mathbf{R}^1} L_t^x}$, which appears on the right hand side.

There is an analogous local theorem, which applies with even less restrictive conditions on $\sigma^2(x)$.

Theorem 2 *Let $X = \{X(t), t \in \mathbf{R}^+\}$ be a real valued symmetric Lévy process with $\sigma^2(x)$ regularly varying at zero of order $\beta - 1$ where $1 < \beta \le 2$, and let $\{L_t^x, (t,x) \in \mathbf{R}^+ \times \mathbf{R}\}$ be the local time of X. Then for each $x \in \mathbf{R}^1$*

$$\limsup_{\epsilon \to 0} \frac{\left|\frac{1}{\epsilon}\int_0^t 1_{[x,x+\epsilon]}(X_s)\,ds - L_t^x\right|}{\sigma(\epsilon)\sqrt{2\log\log(1/\epsilon)}} = \sqrt{\frac{2}{\beta+1} L_t^x} \tag{1.7}$$

almost surely for almost every $t \in \mathbf{R}^+$.

In case X_t is a symmetric stable process with $\psi(t) = t^\beta$ these theorems take a more explicit form.

Theorem 3 *Let $X = \{X(t), t \in \mathbf{R}^+\}$ be the symmetric stable process of order β where $1 < \beta \le 2$, and let $\{L_t^x, (t,x) \in \mathbf{R}^+ \times \mathbf{R}\}$ be the local time of X. Then*

$$\limsup_{\epsilon \to 0} \sup_{x \in \mathbf{R}^1} \frac{\left|\frac{1}{\epsilon}\int_0^t 1_{[x,x+\epsilon]}(X_s)\,ds - L_t^x\right|}{\sqrt{2\epsilon^{\beta-1}\log(1/\epsilon)}} = \sqrt{\frac{2a_\beta}{\beta+1} \sup_{x \in \mathbf{R}^1} L_t^x} \tag{1.8}$$

almost surely for almost every $t \in \mathbf{R}^+$, and for each $x \in \mathbf{R}^1$

$$\limsup_{\epsilon \to 0} \frac{\left|\frac{1}{\epsilon}\int_0^t 1_{[x,x+\epsilon]}(X_s)\,ds - L_t^x\right|}{\sqrt{2\epsilon^{\beta-1}\log\log(1/\epsilon)}} = \sqrt{\frac{2a_\beta}{\beta+1} L_t^x} \tag{1.9}$$

almost surely for almost every $t \in \mathbf{R}^+$, where

$$a_\beta = \frac{1}{\Gamma(\beta)\sin(\frac{\pi}{2}(\beta-1))}. \tag{1.10}$$

The above theorems were first established for Brownian local time by Khoshnevisan [2], who asked us whether we could generalize this to local times of other Markov processes.

To prove the above theorems we use Lemma 4.3 in [7], a consequence of an isomorphism theorem of Dynkin, which enables us to obtain results for the local times of symmetric Markov processes from analogous results about their associated Gaussian processes. The mean zero Gaussian process $\{G(x), x \in \mathbf{R}\}$ with covariance $g(x,y)$ is said to be associated with the Markov process X, if $g(x,y)$ is the 1–potential of X. In [7] we pointed out that it is useful to study local times of symmetric Markov processes through their associated Gaussian processes because there are many tools available to us in the theory of Gaussian processes. This is the approach we use in this paper.

We now present the results about Gaussian processes which we will need.

Theorem 4 *Let $G = \{G_x, x \in \mathbf{R}^1\}$ be a real valued Gaussian process with stationary increments and incremental variance $\sigma^2(x) = E(G_{y+x} - G_y)^2$ which is concave on $[0,\delta]$ and regularly varying at zero of order $\beta - 1$ where $1 < \beta \leq 2$. Then for any compact interval I*

$$\limsup_{\epsilon \to 0} \sup_{x \in I} \frac{|\frac{1}{\epsilon}\int_x^{x+\epsilon} G_y^2\, dy - G_x^2|}{\sigma(\epsilon)\sqrt{2\log(1/\epsilon)}} = 2\sqrt{\frac{1}{\beta+1}} \sup_{x\in I} |G_x|, \tag{1.11}$$

a.s. Furthermore, for $\sigma^2(x)$ regularly varying at zero of order $\beta - 1$ where $1 < \beta \leq 2$ we have that for each $x \in \mathbf{R}^1$

$$\limsup_{\epsilon \to 0} \frac{|\frac{1}{\epsilon}\int_x^{x+\epsilon} G_y^2\, dy - G_x^2|}{\sigma(\epsilon)\sqrt{2\log\log(1/\epsilon)}} = 2\sqrt{\frac{1}{\beta+1}} |G_x|, \tag{1.12}$$

a.s.

This in turn will follow from the next theorem, which we prove in section 2.

Theorem 5 *Let $G = \{G_x, x \in \mathbf{R}^1\}$ be a real valued Gaussian process with stationary increments and incremental variance $\sigma^2(x) = E(G_{y+x} - G_y)^2$ which is concave on $[0,\delta]$ and regularly varying at zero of order $\beta - 1$ where $1 < \beta \leq 2$. Then for any compact interval I*

$$\limsup_{\epsilon \to 0} \sup_{x \in I} \frac{|\frac{1}{\epsilon}\int_x^{x+\epsilon} G_y\, dy - G_x|}{\sigma(\epsilon)\sqrt{2\log(1/\epsilon)}} = \sqrt{\frac{1}{\beta+1}}, \tag{1.13}$$

a.s. Furthermore, for $\sigma^2(x)$ regularly varying at zero of order $\beta - 1$ where $1 < \beta \leq 2$ we have that for each $x \in \mathbf{R}^1$

$$\limsup_{\epsilon \to 0} \frac{|\frac{1}{\epsilon}\int_x^{x+\epsilon} G_y\, dy - G_x|}{\sigma(\epsilon)\sqrt{2\log\log(1/\epsilon)}} = \sqrt{\frac{1}{\beta+1}}, \tag{1.14}$$

a.s.

For purposes of comparison and later reference we note here the exact uniform and local moduli of continuity for the Gaussian processes considered in Theorem 5:

(1.15)
$$\limsup_{\epsilon\to 0}\sup_{\substack{|x-y|\le\epsilon\\ x,y\in I}}\frac{|G_y-G_x|}{\sigma(\epsilon)\sqrt{2\log(1/\epsilon)}}=1,$$

a.s., and for each $x\in\mathbf{R}^1$

(1.16)
$$\limsup_{\epsilon\to 0}\sup_{y:|y-x|\le\epsilon}\frac{|G_y-G_x|}{\sigma(\epsilon)\sqrt{2\log\log(1/\epsilon)}}=1,$$

a.s. The uniform modulus limit (1.15) follows from Theorem 7, [4]. The condition on $-\log\sigma(x)$ in that theorem is satisfied because of the monotone density theorem for regularly varying functions, (see Theorem 1.7.2, [1]). The local modulus limit follows from Kono's Theorems 5 and 6, [3], taking into account the remarks made in the proof of Theorem 5.5, [5].

Theorem 2 is quite general. It applies to the local times of Lévy processes in the domain of attraction of a stable process of order β, $1<\beta\le 2$. Theorem 1 applies to the local times of a more restricted class of Lévy processes which nevertheless is quite large as can be seen from the following theorem proved in [8].

Theorem 6 *Let $h(x)$ be any function which is regularly varying and increasing as $x\to\infty$, and let $1<\beta\le 2$. Then we can find a Lévy process with $\sigma^2(x)$ concave such that*
$$\sigma^2(x)\sim|x|^{\beta-1}h(\ln 1/x)\qquad \textit{as } x\to 0.$$

2 Proofs

We first prove Theorem 5 and then indicate how Theorems 1–4 will follow from the methods of [5]–[7].

Proof of Theorem 5: We first prove (1.13). To do this we only need to make a few modifications to Khoshnevisan's proof of the same result for Brownian motion, Theorem 2.1(a), [2]. Let

$$I(h,x)=\frac{1}{h}\int_x^{x+h}G_y\,dy-G_x=\frac{1}{h}\int_x^{x+h}(G_y-G_x)\,dy,$$

and note that

(2.1)
$$\begin{aligned}&E\left(\{I(h,x)\}^2\right)\\&=\frac{1}{h^2}\int_x^{x+h}\int_x^{x+h}E\left\{(G_y-G_x)(G_z-G_x)\right\}dy\,dz\\&=\frac{2}{h^2}\iint_{x\le y\le z\le x+h}E\left\{(G_y-G_x)(G_z-G_x)\right\}dy\,dz\\&=\frac{2}{h^2}\iint_{x\le y\le z\le x+h}\frac{1}{2}\left(E\left\{(G_y-G_x)^2\right\}+E\left\{(G_z-G_x)^2\right\}\right.\\&\qquad\left.-E\left\{(G_z-G_y)^2\right\}\right)dy\,dz\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h^2}\int_0^h\int_0^z \left(\sigma^2(y)+\sigma^2(z)-\sigma^2(z-y)\right)\,dy\,dz \\
&= \frac{1}{h^2}\int_0^h\int_0^z \sigma^2(z)\,dy\,dz \\
&= \frac{1}{h^2}\int_0^h z\sigma^2(z)\,dz \\
&\sim \frac{1}{\beta+1}\sigma^2(h)
\end{aligned}
$$

where we have used the fact that $z\sigma^2(z)$ is regularly varying with index β.

We now follow the proof in [2]. The only non-trivial change occurs in obtaining an upper bound for

$$P\left(\max_{k\le[\rho^n]}|I(\rho^{-n},k\rho^{-n})|\le\sigma(\epsilon)\sqrt{2\log(1/\epsilon)}\sqrt{\frac{\theta}{\beta+1}}\right).$$

In [2], Khoshnevisan uses the fact that for Brownian motion

$$\mathcal{I}=\{I(\rho^{-n},k\rho^{-n});0\le k\le[\rho^n]\}$$

is a set of independent random variables. However, in our case, due to the concavity of the incremental variance, it is easy to see that our $\mathcal{I}$ is a set of negatively correlated mean-zero Gaussian random variables and the inequality derived in [2] using independence, follows in our case from Slepian's lemma. This gives the lower bound in (1.13). For the upper bound we just follow [2] and use (1.15).

We now obtain (1.14). Khoshnevisan just states the corresponding result for Brownian motion in Theorem 2.2,(a), [2] and says that the proof is similar to his proof for the uniform case. We agree, and so, as for (1.13), we will only show how to handle the lower bound in (1.14) without the assumption of independence. We take $x=0$ and set

$$I(h)=\frac{1}{h}\int_0^h G_y\,dy-G_0=\frac{1}{h}\int_0^h(G_y-G_0)\,dy.$$

We compute for $t>s>0$ small

$$
\begin{aligned}
(2.2)\qquad & E\left(I(s)I(t)\right) \\
&= \frac{1}{st}\int_0^t\int_0^s E\left\{(G_y-G_0)(G_z-G_0)\right\}\,dy\,dz \\
&= \frac{1}{st}\int_0^t\int_0^s \frac{1}{2}\left(E\left\{(G_y-G_0)^2\right\}+E\left\{(G_z-G_0)^2\right\}-E\left\{(G_z-G_y)^2\right\}\right)\,dy\,dz \\
&= \frac{1}{2st}\int_0^t\int_0^s\left(\sigma^2(y)+\sigma^2(z)-\sigma^2(z-y)\right)\,dy\,dz \\
&= \frac{1}{2st}\left(t\int_0^s\sigma^2(y)\,dy+s\int_0^t\sigma^2(z)\,dz-\int_0^t\int_0^s\sigma^2(z-y)\,dy\,dz\right) \\
&= \frac{1}{2st}\left(t\int_0^s\sigma^2(y)\,dy+s\int_0^t\sigma^2(z)\,dz-\int_0^t\int_{-z}^{s-z}\sigma^2(v)\,dv\,dz\right) \\
&\le \frac{1}{2st}\left(t\int_0^s\sigma^2(y)\,dy+s\int_0^t\sigma^2(z)\,dz-\int_0^{s-t}\int_{-v}^{s-v}\sigma^2(v)\,dz\,dv\right)
\end{aligned}
$$

where the next to last line came from the change of variables $(y,z)\mapsto(v,z)$ with $v=y-z$, and the last line came from the observation that

$$\{(v,z)|0\le v\le s-t,\ -v\le z\le s-v\}\subseteq\{(v,z)|0\le z\le t,\ -z\le v\le s-z\}.$$

Since

$$\int_0^{s-t}\int_{-v}^{s-v}\sigma^2(v)\,dz\,dv = s\int_0^{t-s}\sigma^2(v)\,dv$$

we see from (2.2) that

$$E\left(I(s)I(t)\right) \leq \frac{1}{2st}\left(t\int_0^s \sigma^2(y)\,dy + s\int_{t-s}^t \sigma^2(z)\,dz\right) \tag{2.3}$$

We now take $\theta < 1$ and set

$$X_k = \frac{I(\theta^k)}{(E\{I(\theta^k)^2\})^{1/2}} \sim \sqrt{\beta+1}\frac{I(\theta^k)}{\sigma(\theta^k)} \tag{2.4}$$

by (2.1). Note that by Theorem 1.5.6 of [1] we have that for $j < k$ and θ sufficiently small

$$\sigma^2(z) \leq 2\sigma^2(\theta^j)$$

for all $\theta^j - \theta^k \leq z \leq \theta^j$. Hence by (2.3), and using Theorem 1.5.6 of [1] once again, we see that for all $j < k$ and $\delta > 0$

$$\begin{aligned}
&E(X_jX_k)\\
&\leq c(\theta^{k-j}\frac{\sigma(\theta^j)}{\sigma(\theta^k)} + \frac{\sigma(\theta^k)}{\sigma(\theta^j)})\\
&\leq c(\theta^{k-j}(\frac{1}{\theta^{k-j}})^{\frac{\beta-1}{2}+\epsilon} + (\theta^{k-j})^{\frac{\beta-1}{2}+\epsilon})\\
&\leq \delta
\end{aligned}$$

for θ sufficiently small.

Let $U_1, U_2, \ldots$ and Z be a set of independent $N(0,1)$ random variables and set

$$Y_k = \sqrt{1-\delta}U_k + \sqrt{\delta}Z$$

Note also that X_k, Y_k are mean- zero Gaussian random variables with $E(X_k^2) = E(Y_k^2) = 1$ and $E(X_jX_k) \leq E(Y_jY_k)$ for $j \neq k$. Hence by (2.4), Slepian's lemma and the independence of the U_k's we see that for all $0 < \delta < 1/2$

$$\begin{aligned}
&P(\limsup_{k\to\infty}\frac{I(\theta^k)}{\sigma(\theta^k)\sqrt{2\log\log(\theta^{-k})}} \geq \frac{1-2\delta}{\sqrt{\beta+1}})\\
&\geq P(\limsup_{k\to\infty}\frac{X_k}{\sqrt{2\log(k)}} \geq 1-\delta)\\
&= \lim_{n\to\infty} P(\sup_{k\geq n}\frac{X_k}{\sqrt{2\log(k)}} \geq 1-\delta)\\
&\geq \lim_{n\to\infty} P(\sup_{k\geq n}\frac{Y_k}{\sqrt{2\log(k)}} \geq 1-\delta)\\
&= P(\limsup_{k\to\infty}\frac{Y_k}{\sqrt{2\log(k)}} \geq 1-\delta)\\
&= P(\limsup_{k\to\infty}\frac{U_k}{\sqrt{2\log(k)}} \geq \sqrt{1-\delta})\\
&= 1
\end{aligned}$$

and this gives us the lower bound in (1.14). The upper bound follows easily from (2.1) and interpolation using (1.16) as in the proof of (1.13). □

Proofs of Theorems 1-4: The passage from Theorem 5 to Theorem 4 is simple and follows methods worked out in section 2, [5]. Given (1.11), the next step is to apply Theorem 4.3, [5] which enables us to transfer results about Gaussian processes to results about the local times of the associated Markov processes. However, the Gaussian process in Theorem 4 is not the Gaussian process associated with X. The Gaussian process associated with X has incremental variance

$$\tilde{\sigma}^2(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos x\lambda}{1+\psi(\lambda)}\,d\lambda.$$

(Clearly, $\tilde{\sigma}^2(x) \sim \sigma^2(x)$ as $x \to 0$). The extension of Theorem 4 to these processes is handled exactly the same way as the transition from Theorem 2.4 to Theorem 2.5 in [6]. As in [6], we first consider Gaussian processes with incremental variance (1.3) because it is easier to find examples of such functions which are concave. One can also see from section 3 of [6] how theorem 4.3 of [7] is used. Thus we get from (1.11) that

$$\limsup_{\epsilon\to 0}\sup_{x\in I}\frac{|\frac{1}{\epsilon}\int_x^{x+\epsilon} L_t^y\,dy - L_t^x|}{\sigma(\epsilon)\sqrt{2\log(1/\epsilon)}} = \sqrt{\frac{2}{\beta+1}\sup_{x\in I} L_t^x} \tag{2.5}$$

almost surely for almost every $t \in \mathbf{R}^+$. Now, since $\{L_t^y, (t,y) \in \mathbf{R}^+ \times \mathbf{R}\}$ is continuous almost surely for the Lévy processes which we are considering, we have with probability one that

$$\int_x^{x+\epsilon} L_t^y\,dy = \int_0^t 1_{[x,x+\epsilon]}(X_s)\,ds \tag{2.6}$$

for all x and ϵ. Using (2.6) in (2.5) gives us (1.6), and a similar argument takes us from (1.12) to (1.7). Theorem 3 consists of special cases of Theorems 1 and 2. The constant α_β in Theorem 3 is determined in [6].

Remark 1: The only new ingredient in this note is Theorem 5, since its application to local times is immediate following the methods worked out in [5]-[7]. Furthermore, concerning Theorem 5, the reader no doubt realizes that this was essentially proved by Khoshnevisan in [2] since the extension from Brownian motion to the more general cases we consider only requires a few modifications. However, in [2], because he doesn't use the isomorphism theorem, Khoshnevisan has to consider much more than Brownian motion. In fact, he deals with explicit representations for Brownian local time. Thus it seems that even if one only wants to obtain results for Brownian local time, the methods used in this paper are more efficient. However, we must qualify this statement since our Theorems 1-3 are only for almost all t whereas the result in [2] for Brownian local time holds for all t.

Remark 2: It is possible to prove results similar to Theorem 5 for a large class of Gaussian processes and thereby extend Theorems 1 and 2. In particular, the upper bounds in Theorems 1 and 2 can be obtained under much more general conditions than the ones given. When $\sigma^2(x)$ is slowly varying at zero the situation changes. In some cases the denominator has a different form than in (1.6) and (1.7). This is because (1.15) and (1.16) have different denominators for certain slowly varying $\sigma^2(x)$, see [5]. We have not pursued these points because our primary concern has been to demonstrate the usefullness of Dynkin's isomorphism theorem rather than in carrying on an exhaustive study of the rate of convergence in (1.5).

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