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# On the maximum of a diffusion process in a drifted Brownian environment

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## 1. Introduction

In this paper we investigate asymptotic behavior of the tail of the distribution of the maximum of a diffusion process in a drifted Brownian environment. This problem is a diffusion analogue of the Afanas'ev problem([1]). Our result is naturally compatible with that of Afanas'ev[1].

Let $\{W(x), x \in \mathbf{R}, P\}$ be a Brownian environment, namely, let $\{W(t), t \geq 0, P\}$ and $\{W(-t), t \geq 0, P\}$ be independent Brownian motions in one-dimension with $W(0) = 0$. We consider a diffusion process $X(t, W)$ defined formally by

$$X(t, W) = \text{Brownian motion} - \frac{1}{2}\int_0^t \{W'(X(s, W)) + c\}ds,$$

where $c$ is a positive constant. The precise meaning of $X(t, W)$ is simply a diffusion process with generator

$$\frac{1}{2}e^{W(x)+cx}\frac{d}{dx}(e^{-W(x)-cx}\frac{d}{dx}),$$

starting at 0. Such a diffusion process can be constructed from a Brownian motion through changes of scale and time. For a fixed environment $W = (W(x), x \in \mathbf{R})$ we denote by $P_W$ the probability law of the process $\{X(t, W)\}$ and put

$$\mathcal{P} = \int P(dW)P_W.$$

Thus $\mathcal{P}$ is the full law of $\{X(t, \cdot)\}$. We often write $X(t) = X(t, \cdot)$. Since $c > 0$, $\max_{t\geq 0} X(t)$ is finite ($\mathcal{P}$-a.s.). The problem is the following : How fast does $\mathcal{P}\{\max_{t\geq 0} X(t) > x\}$ decay as $x \to \infty$ ? Since

$$\mathcal{P}\{\max_{t\geq 0} X(t) > x\} = E\{A(A + B)^{-1}\}, \tag{1.1}$$

where

$$A = \int_{-\infty}^{0} e^{W(t)+ct}\,dt, \qquad B = \int_{0}^{x} e^{W(t)+ct}\,dt, \tag{1.2}$$

the problem is nothing but to find the asymptotics of $E\{A(A+B)^{-1}\}$ as $x \to \infty$ . The result varies according as $c > 1$ , $c = 1$ , $0 < c < 1$ , as will be stated in the following theorem.

THEOREM. (i)*If* $c > 1$ , *then*

$$\mathcal{P}\{\max_{t\geq 0} X(t) > x\} \sim \frac{2c-2}{2c-1}\exp\{-(c-\frac{1}{2})x\}, \ x \to \infty.$$

*(ii) If* $c = 1$ , *then*

$$\mathcal{P}\{\max_{t\geq 0} X(t) > x\} \sim (2/\pi)^{1/2} x^{-1/2} \exp\{-x/2\}, \ x \to \infty.$$

*(iii) If* $0 < c < 1$ , *then*

$$\mathcal{P}\{\max_{t\geq 0} X(t) > x\} \sim const. x^{-3/2} \exp\{-c^2 x/2\}, \ x \to \infty,$$

*where*

$$const. = 2^{5/2-2c}\Gamma(2c)^{-1} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty z(a+z)^{-1} a^{2c-1} e^{-a/2} y^{2c} e^{-\lambda z} u \sinh u \, da\, dy\, dz\, du\,,$$

$\lambda = (1+y^2)/2 + y\cosh u$ .

## 2. Proof of the theorem

Since A and B are independent, the right hand side of (1.1) equals $E\{Af(A)\}$ where $f(a) = E\{(a+B)^{-1}\}, a \geq 0$ . Fixing $x > 0$ , we consider the time reversal $\widehat{W}(t) = W(x-t) - W(x), 0 \leq t \leq x$ . Since $\{\widehat{W}(t), 0 \leq t \leq x\}$ is also a Brownian motion, we have

$$\begin{aligned} f(a) &= E\{(a + \int_0^x \exp\{\widehat{W}(t) + ct\}\,dt)^{-1}\} \\ &= E\{(a + e^{-W(x)} \int_0^x \exp\{W(x-t) + ct\}\,dt)^{-1}\} \\ &= E\{(ae^{W(x)-cx} + \int_0^x e^{W(t)-ct}\,dt)^{-1} e^{W(x)-cx}\} \\ &= e^{(1/2-c)x} E\{(ae^{W(x)-cx} + \int_0^x e^{W(t)-ct}\,dt)^{-1} e^{W(x)-x/2}\} \\ &= e^{(1/2-c)x} E\{(ae^{W(x)-(c-1)x} + \int_0^x e^{W(t)-(c-1)t}\,dt)^{-1}\} \\ &= e^{(1/2-c)x} E\{(a + \int_0^x e^{W(t)+(c-1)t}\,dt)^{-1} e^{W(x)+(c-1)x}\}. \end{aligned} \tag{2.1}$$

In deriving the fifth equality in the above we used the formula of Cameron-Martin-Maruyama-Girsanov; the last equality was derived by using $\widehat{W}(t)$ as in the case of the first equality. From the fifth equality of (2.1) we obtain the following lemma.

LEMMA 1. *For any $c>0$ and $x>0$*

$$\mathcal{P}\{\max_{t\geq 0} X(t) > x\} = e^{(1/2-c)x} E\{A(Ae^{W(x)-(c-1)x} + \int_0^x e^{W(t)-(c-1)t} dt)^{-1}\}, \tag{2.2}$$

*where $A$ is given by (1.2).*

The following lemma due to Yor will also be used.

LEMMA 2(Yor[2]). *For any $\nu > 0$ we have*

$$\int_0^\infty \exp(W(t) - \frac{\nu t}{2}) dt \overset{d}{=} 2/Z_\nu, \tag{2.3}$$

*where $\overset{d}{=}$ means equality in distribution and $Z_\nu$ is a gamma variable of index $\nu$, that is,*

$$P\{Z_\nu \in dt\} = \Gamma(\nu)^{-1} t^{\nu-1} e^{-t} dt \qquad (t>0).$$

2.1. Proof of (i)

When $c > 1$, Lemma 1 implies

$$\lim_{x\to\infty} e^{-(1/2-c)x} \mathcal{P}\{\max_{t\geq 0} X(t) > x\} = E\{A(\int_0^\infty e^{W(t)-(c-1)t} dt)^{-1}\} .$$

It is easy to see that the above expectation is finite. To obtain its exact value we use Lemma 2. We thus obtain (i).

2.2. Proof of (ii)

For $x > 0$ we put

$$\varphi(x) = E\{\log \int_0^x e^{W(t)} dt\}, \quad \psi(x) = \frac{d}{dx}\varphi(x).$$

Then it is easy to see that

$$\psi(x) = E\{(\int_0^x e^{W(t)} dt)^{-1} e^{W(x)}\} = E\{(\int_0^x e^{W(t)} dt)^{-1}\} ;$$

in fact, the second equality is a consequence of the last equality of (2.1) with $a = 0$ and $c = 1$. Thus $\psi(x)$ is monotone decreasing in $x$.

LEMMA 3. *When $c = 1$, we have*

$$E\{A(\int_0^x e^{W(t)+t} dt)^{-1}\} \sim \sqrt{2/\pi}\, x^{-1/2} e^{-x/2} \quad as \quad x \to \infty . \tag{2.4}$$

Proof. Since $E\{A\} = 2$ in case $c = 1$, the left hand side of (2.4) equals $2E\{(\int_0^x e^{W(t)+t} dt)^{-1}\}$ which also equals $2e^{-x/2}E\{(\int_0^x e^{W(t)} dt)^{-1} e^{W(x)}\}$ by virtue of (2.1) with $a = 0$ and $c = 1$. Thus we have

$$E\{A(\int_0^x e^{W(t)+t} dt)^{-1}\} = 2e^{-x/2}\psi(x). \tag{2.5}$$

On the other hand, using the scaling property $\{W(t)\} \stackrel{d}{=} \{\sqrt{x}\,W(t/x)\}$ we have

$$\varphi(x) = E\{\log \int_0^1 e^{\sqrt{x}W(t)}\,dt\} + \log x\,,$$

and hence

$$\begin{aligned}\lim_{x\to\infty} x^{-1/2}\varphi(x) &= \lim_{x\to\infty} E\{\frac{1}{\sqrt{x}}\log \int_0^1 e^{\sqrt{x}W(t)}\,dt\}\\ &= E\{\max_{0\le t\le 1} W(t)\} = \sqrt{2/\pi}\,,\end{aligned}$$

which combined with the monotonicity of $\psi(x) = \varphi'(x)$ implies

$$\psi(x) \sim (2\pi x)^{-1/2} \quad \text{as } x \to \infty \quad . \tag{2.6}$$

This together with (2.5) proves the lemma.

LEMMA 4. *For $x > 0$ we have*

$$E\{(\int_0^x e^{W(t)}\,dt)^{-2} e^{W(x)}\} \le \psi(x/2)^2 \quad . \tag{2.7}$$

Proof. The left hand side of (2.7) is dominated by

$$\begin{aligned}&E\{(\int_0^{x/2} e^{W(t)}\,dt)^{-1}(\int_{x/2}^x e^{W(t)}\,dt)^{-1} e^{W(x)}\}\\ =\;& E\{(\int_0^{x/2} e^{W(t)}\,dt)^{-1}(\int_{x/2}^x e^{W(t)-W(x/2)}\,dt)^{-1} e^{W(x)-W(x/2)}\}\\ =\;& E\{(\int_0^{x/2} e^{W(t)}\,dt)^{-1}\}E\{(\int_0^{x/2} e^{W(t)}\,dt)^{-1} e^{W(x/2)}\}\\ =\;& \psi(x/2)^2\;;\end{aligned}$$

in deriving the second equality in the above we used the fact that $\{W(t+\frac{x}{2}))-W(\frac{x}{2}), t \ge 0\}$ is a Brownian motion independent of $\{W(t), 0 \le t \le x/2\}$ .

The proof of (ii) is now given as follows. By (1.1) we have

$$\begin{aligned}0 &\le E\{A(\int_0^x e^{W(t)+t}\,dt)^{-1}\} - \mathcal{P}\{\max_{t\ge 0} X(t) > x\}\\ &= E\{AB^{-1} - A(A+B)^{-1}\}\\ &\le E\{2^{-1}A^{3/2}B^{-3/2}\} = 2^{-1}E\{A^{3/2}\}E\{B^{-3/2}\}.\end{aligned} \tag{2.8}$$

We prove

$$E\{A^{3/2}\} < \infty, \tag{2.9}$$

$$E\{B^{-3/2}\} < \text{const.}\, x^{-3/4} e^{-x/2}\,. \tag{2.10}$$

(2.9) follows immediately from Lemma 2 ; a direct proof can also be given as follows. Using Hölder's inequality we have

$$\begin{aligned} E\{A^{3/2}\} &= E\{(\int_0^\infty e^{W(t)-4t/5}e^{-t/5}\,dt)^{3/2}\} \\ &\le (5/3)^{1/2}E\{\int_0^\infty \exp\{\frac{3}{2}(W(t)-\frac{4t}{5})\}dt\} = (5/3)^{1/2}\cdot(40/3). \end{aligned}$$

(2.10) can be proved by making use of the CMMG formula, the Schwarz inequality, Lemma 4 and then (2.6) ; in fact, putting $B_0 = \int_0^x e^{W(t)}dt$ we have

$$\begin{aligned} E\{B^{-3/2}\} &= E\{B_0^{-3/2}e^{W(x)-x/2}\} \\ &\le e^{-x/2}E\{B_0^{-1}e^{W(x)}\}^{1/2}E\{B_0^{-2}e^{W(x)}\}^{1/2} \\ &\le e^{-x/2}\psi(x)^{1/2}\psi(x/2) \\ &\le \text{const.}\, e^{-x/2}x^{-1/4}\cdot x^{-1/2}. \end{aligned}$$

The assertion (ii) of our theorem follows from Lemma 3, (2.8), (2.9) and (2.10).

2.3. Proof of (iii)

The proof of (iii) relies essentially on the following Yor's formula.

Yor's formula([3: the formula(6.e)]). *For any bounded Borel functions $f$ and $g$ we have*

$$\begin{aligned} &E\{f(\int_0^t e^{2W(s)}\,ds)g(e^{W(t)})\} \\ &= c_t\int_0^\infty dy\int_0^\infty dz\, g(y)f(1/z)\exp\{-z(1+y^2)/2\}\psi_{yz}(t), \end{aligned}$$

*where*

$$\begin{aligned} c_t &= (2\pi^2 t)^{-1/2}\exp\{\pi^2/2t\}, \\ \psi_r(t) &= \int_0^\infty \exp\{-u^2/2t\}e^{-r(\cosh u)}(\sinh u)\sin(\pi u/t)\,du. \end{aligned}$$

To proceed to the proof of (iii) we put

$$f(a,z) = a(a+4z)^{-1}, \quad g(y) = y^{2c},$$

$$B^{(\nu)}(t) = \int_0^t e^{2(W(s)+\nu s)}\,ds.$$

Using first the CMMG formula and then Yor's formula we have

$$\begin{aligned} &E\{a(a+\int_0^x e^{W(t)+ct}\,dt)^{-1}\} = E\{a(a+4B^{(2c)}(x/4))^{-1}\} \\ &= E\{a(a+4B^{(0)}(x/4))^{-1}\exp(2cW(x/4)-\tfrac{c^2x}{2})\} \\ &= \exp(-c^2x/2)E\{f(a,B^{(0)}(x/4))g(e^{W(x/4)})\} \\ &= \exp(-c^2x/2)c_{x/4}\int_0^\infty dy\int_0^\infty dz\, g(y)f(a,1/z)\exp\{-z(1+y^2)/2\}\psi_{yz}(x/4). \end{aligned}$$

Since Lemma 2 implies

$$P\{A \in da\} = 2^{2c}\Gamma(2c)^{-1}a^{-2c-1}e^{-2/a}da \quad (a > 0),$$

we have

$$\begin{aligned} &\mathcal{P}\{\max_{t\geq 0} X(t) > x\} \\ (2.11) \qquad &= 2^{2c+1/2}\Gamma(2c)^{-1}\pi^{-1}\exp(2\pi^2/x)x^{-1/2}\exp(-c^2x/2) \\ &\quad \times \int_0^\infty dy \int_0^\infty dz \int_0^\infty du\, y^{2c}h(z)e^{-\lambda z}\exp(-2u^2/x)(\sinh u)\sin(4\pi u/x), \end{aligned}$$

where

$$\begin{aligned} h(z) &= \int_0^\infty az(az+4)^{-1}a^{-2c-1}e^{-2/a}\,da, \\ \lambda &= (1+y^2)/2 + y\cosh u. \end{aligned}$$

LEMMA 5. *Let $0 < c < 1$ and put*

$$F(y,z,u) = y^{2c}h(z)e^{-\lambda z}u\sinh u.$$

*Then we have*

$$M = \int_0^\infty \int_0^\infty \int_0^\infty F(y,z,u)\,dy\,dz\,du < \infty,$$

Proof. By a change of variable $\cosh u = v$, we have

$$M = \int_0^\infty dy \int_0^\infty dz \int_1^\infty dv\; y^{2c}h(z)e^{-\lambda z}\log(v+\sqrt{v^2-1}),$$

where $\lambda = (1+y^2)/2 + yv$. Since

$$h(z) = 2^{-2c-1}z\int_0^\infty u^{2c-1}e^{-u}(u+\frac{z}{2})^{-1}du,$$

it is easy to see that

$$(2.12) \qquad h(z) \longrightarrow 2^{-2c}\Gamma(2c) \qquad \text{as } z\to\infty \quad ,$$

$$(2.13) \qquad h(z) \sim_{\text{as } z\downarrow 0} \begin{cases} 2^{-2c-1}\Gamma(2c-1)z & \text{if } c > 1/2, \\ 2^{-2}z\log 1/z & \text{if } c = 1/2, \\ 2^{-4c}\int_0^\infty a^{2c-1}(a+1)^{-1}da\cdot z^{2c} & \text{if } 0 < c < 1/2. \end{cases}$$

Therefore for any $\varepsilon > 0$ and $\alpha > 0$ we have

$$\begin{aligned} M_1 &= \int_0^\infty dy \int_1^\infty dz \int_1^\infty dv\, y^{2c}h(z)e^{-\lambda z}\log(v+\sqrt{v^2-1}) \\ &\leq \text{const.} \int_0^\infty \int_1^\infty y^{2c}v^\varepsilon \lambda^{-1}e^{-\lambda}\,dydv \\ &\leq \text{const.} \int_0^\infty \int_1^\infty y^{2c}v^\varepsilon \lambda^{-\alpha}\,dydv \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{2c-\varepsilon-1}(1+y^2)^{-\alpha+1+\varepsilon}z^\varepsilon(1+z)^{-\alpha}\,dy\,dz \end{aligned}$$

(by putting $v = (2y)^{-1}(1+y^2)z$ with $y$ fixed ),

which is finite if $\varepsilon > 0$ is sufficiently small and $\alpha > 0$ sufficiently large. Note that const. in the above may vary from place to place and depend on $\varepsilon$ and $\alpha$ . Next we prove that

$$(2.14) \qquad M_2 = \int_0^\infty dy \int_0^1 dz \int_1^\infty dv\, y^{2c} h(z) e^{-\lambda z} \log(v + \sqrt{v^2-1}) < \infty \quad .$$

Assume $1/2 < c < 1$ . Then by (2.13)

$$\begin{aligned} M_2 &\leq \text{const.} \int_0^\infty dy \int_0^1 dz \int_1^\infty dv\, y^{2c} z e^{-\lambda z} v^\varepsilon \\ &\leq \text{const.} \int_0^\infty \int_1^\infty \lambda^{-2} y^{2c} v^\varepsilon \, dy\, dv \qquad (\text{ we used } \int_0^1 z e^{-\lambda z} dz \leq \lambda^{-2} \text{ )} \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{2c-1-\varepsilon}(1+y^2)^{-1+\varepsilon} z^\varepsilon (1+z)^{-2}\, dy\, dz \end{aligned}$$

(by putting $v = (2y)^{-1}(1+y^2)z$ with $y$ fixed )

which is finite for sufficiently small $\varepsilon > 0$ by virtue of $1/2 < c < 1$ . When $c = 1/2$ , (2.13) implies

$$M_2 \leq \text{const.} \int_0^\infty dy \int_0^1 dz \int_1^\infty dv\, y z^{1-\varepsilon} e^{-\lambda z} v^\varepsilon$$

for $0 < \varepsilon < 1$ . Since $\int_0^1 z^{1-\varepsilon} e^{-\lambda z} dz \leq \text{const.}\lambda^{-2+\varepsilon}$ , we have

$$\begin{aligned} M_2 &\leq \text{const.} \int_0^\infty \int_1^\infty \lambda^{-2+\varepsilon} y v^\varepsilon \, dy\, dv \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{-\varepsilon}(1+y^2)^{-1+2\varepsilon} z^\varepsilon (1+z)^{-2+\varepsilon}\, dy\, dz < \infty \end{aligned}$$

provided that $\varepsilon > 0$ is small enough. Finally assume $0 < c < 1/2$ . Then by (2.13)

$$\begin{aligned} M_2 &\leq \text{const.} \int_0^\infty dy \int_0^1 dz \int_1^\infty dv\, y^{2c} z^{2c} e^{-\lambda z} v^\varepsilon \\ &\leq \text{const.} \int_0^\infty \int_1^\infty \lambda^{-1-2c} y^{2c} v^\varepsilon \, dy\, dv \\ &\leq \text{const.} \int_0^\infty \int_0^\infty y^{2c-\varepsilon-1}(1+y^2)^{-2c+\varepsilon} z^\varepsilon (1+z)^{-1-2c}\, dy\, dz < \infty \end{aligned}$$

provided that $\varepsilon > 0$ is small enough. Thus (2.14) is proved.

We can now complete the proof of (iii) as follows. From (2.11) we have

$$(2.15) \qquad \mathcal{P}\{\max_{t \geq 0} X(t) > x\} = 2^{2c+5/2}\Gamma(2c)^{-1} \exp(2\pi^2/x) x^{-3/2} \exp(-c^2 x/2) M(x) ,$$

where

$$M(x) = \int_0^\infty \int_0^\infty \int_0^\infty F(y,z,u) \sin(4\pi u/x)/(4\pi u/x) \exp(-2u^2/x)\, dy\, dz\, du .$$

By Lemma 5 we have $\lim_{x\to\infty} M(x) = M$ which equals

$$2^{-4c} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty z(a+z)^{-1} a^{2c-1} e^{-a/2} y^{2c} e^{-\lambda z} u \sinh u \, da\, dy\, dz\, du\,.$$

Thus the assertion (iii) follows from (2.15).

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