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Estimates of the Hausdorff dimension of the boundary

of positive Brownian sheet components.

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The Brownian sheet is examined. We provide upper and lower bounds for the Hausdorff dimension of the boundary of time components for which the Brownian sheet is strictly positive or strictly negative. In particular we show that this dimension lies strictly between 1 and 3/2. It is also shown that there exist random time points which are boundary points for both positive and negative components.

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Recent work of Dalang and Walsh (1992a,b) has investigated the structure of neighbourhoods of random times $\underline{t}$ which are boundary points of sets $\{\underline{s} : W(\underline{s}) > 1\}$. Ehm (1981) and Rosen (1983) show that the level sets of the Brownian sheet have dimension 3/2. Kendall (1980) and Dalang and Walsh (1992a,b) show that typical points of the level set of, say, $\{\underline{s} : W(\underline{s}) = 1\}$ do not belong to the boundary of a single component of $\{\underline{s} : W(\underline{s}) > 1\}$. Thus there is a difference between the union of the boundaries of the individual components of $\{t: W(t) > 1\}$ and the boundary of the union of these components. In this note we show that in fact such time points have Hausdorff dimension strictly between 1 and 3/2.

Theorem One

The Hausdorff dimension of the boundary of every component of $\{t: W(t) > 1\}$ is in the interval $[5/4, 3/2)$.

This result can be seen as on the one hand adding to the results quoted above, while also showing that there are more boundary points than are simply required of boundaries of open sets in two dimensions.

We also show

Theorem Two

There exist random time points which are boundary points of at least one component of $\{t: W(t) > 1\}$ and at least one component of $\{t: W(t) < 1\}$.

For notational convenience we examine boundaries of components of $\{\underline{s}: W(\underline{s}) > 1\}$. It will be clear that the results that are derived hold for the boundaries of components of $\{\underline{s}: W(\underline{s}) > c\}$ or $\{\underline{s}: W(\underline{s}) < c\}$ for any real c.

The paper is organized as follows In Section One we introduce notation and definitions that will be used throughout. We also quote certain results. The next section establishes that with probability one there exist components of the time set {t: W(t) > 1} whose boundary has Hausdorff dimension at least 5/4. Section Three is devoted to proving a technical result used in Section Two. The proof of the lower bound in Theorem One is completed in Section Four where it is argued that every component of {t: W(t) > 1} must have dimension at least 5/4. Section Five completes the proof of Theorem One by supplying the upper bound for boundary dimension. The argument is essentially that found in Dalang and Walsh (1992a). The paper concludes with Section Six where Theorem Two is proven.

The clarity and indeed the contents of this paper owe a great deal to conversations with Robert Dalang. It is a pleasure to thank him and Tufts University for their hospitality.

Section One

We show that the Hausdorff dimension of the set of points $\{\underline{t}: \underline{t}$ is a boundary point of a component of $\{\underline{s}: W(\underline{s}) > 1\}\}$ is at least 5/4 by using two facts:

1 Frostman's Theorem: The capacitory dimension of a set is equal to the Hausdorff dimension. See e.g. Kahane (1985), page 133, or Taylor (1961).

2 A compact set F has positive α capacity if we can find n_i increasing to infinity and points

$$x_1^{n_i}, x_2^{n_i}, \ldots, x_{n_i}^{n_i} \in F$$

so that

$$\limsup_{i \to \infty} \frac{1}{n_i^2} \sum_{1 \le k < j \le n_i} \frac{1}{|x_k^{n_i} - x_j^{n_i}|^\alpha} \le M < \infty.$$

See e.g. Landkoff (1972), pages 160-162.

Therefore to show our set has Hausdorff dimension at least 5/4 it will be sufficient to show that for each M and d, the capacitory dimension is at least 5/4 - f(M,d), where as M tends to infinity and d tends to zero, f(M,d) tends to zero.

Definitions:

Given a time point $\underline{t} = (t_1,t_2)$, we define $B^{1,t}(s) = W(t_1+s,t_2) - W(t_1,t_2)$, $s \ge 0$. Similarly $B^{2,t}(s) = W(t_1,t_2+s) - W(t_1,t_2)$, $s \ge 0$.

$W^{\underline{t}}(s_1,s_2) = W(t_1+s_1,t_2+s_2) - W(t_1,t_2+s_2) - W(t_1+s_1,t_2) + W(t_1,t_2)$. In this paper we will not require Brownian motion to have variance t at time t, so with this loose terminology, the above two processes are Brownian motions. A Brownian motion with unit speed will be called a standard Brownian motion. Given a time point $\underline{t}$, $F(\underline{t})$ will denote σ-{W($\underline{s}$): $\underline{s} \le \underline{t}$}. Given a time rectangle R, G(R) =
$\sigma-\{W(t_1,t_2) - W(t_1,s_2) - W(s_1,t_2) + W(s_1,s_2): (t_1,t_2), (s_1,s_2) \in R\}$. Given a time region R equal to the finite union of rectangles R_i, G(R) will denote the sigma-field generated by the sigma-fields $G(R_i)$. Given two times $\underline{s} < \underline{t}$, G($\underline{s}$,$\underline{t}$) will denote the σ-field G($[0,t_1]\times[s_2,t_2] \cup [s_1,t_1]\times[0,t_2]$). Note that for any $\underline{t} > \underline{s}$, $F(\underline{s})$ is independent of $G(\underline{s},\underline{t})$.

A random variable $\underline{T} \in R_+^2$ is a *stopping point* if it satisfies the condition: for each $\underline{t} \in R_+^2$, the event $\{\underline{T} \leq \underline{t}\}$ is $F_{\underline{t}}$ measureable.

The following result is clear, it can for instance be proven by the method used by Walsh (1984) in proving Theorem 1.6.

Proposition 1.1

Let $\underline{T}$ be a stopping point. Then

i) $W^{\underline{T}}$ is a Brownian sheet independant of $F_{\underline{T}}$.

ii) $\frac{1}{\sqrt{T_2}} B^{1,\underline{T}}$ is a standard Brownian motion independant of $F_{\underline{T}}$.

iii) $\frac{1}{\sqrt{T_1}} B^{2,\underline{T}}$ is a standard Brownian motion independant of $F_{\underline{T}}$.

iv) All three processes above are independant.

In fact part (i) is contained in Theorem 1.6 of Walsh (1984) which applies to weak stopping points.

Definition of $H^h(\underline{t},r)$.

We now define a stopping point which will be fundamental. While the definition is natural, it requires a few distinct steps to describe. It should be remembered that we are attempting to construct an increasing curve, C, starting from a given $\underline{t}$, on which W is greater than 1, provided that W($\underline{t}$) is greater than 1+r. In the following $\underline{t} = (t_1,t_2)$ will be a fixed time point or possibly a fixed stopping point. ∞ will for our purposes simply be a graveyard time point.

Step One: Define the stopping time T_1 to equal $\inf\{s \geq 0\colon B^{1,\underline{t}}(s) = -r \text{ or } dr\}$. If T_1 is not in $\frac{r^2}{t_2}(\frac{1}{M},M)$ then $H^h(\underline{t},r) = \infty$. If $T_1 \in \frac{r^2}{t_2}(\frac{1}{M},M)$ and $B^{1,\underline{t}}(T_1) = dr$, then $H^h(\underline{t},r) = (t_1+T_1,t_2)$. Otherwise we use Step Two.

Step Two: Define the stopping time T_2 to equal $\inf\{s \geq 0\colon B^{2,\underline{t}} = -r \text{ or } (1+d)r\}$. If T_2 is not in $\frac{r^2}{t_1}(\frac{1}{M},M)$ or $B^{2,\underline{t}}(T_2) = -r$ then $H^h(\underline{t}) = \infty$. If not we go to Step Three.

Step Three: We define T_3 to equal
$\inf\{s > 0\colon W(s+t_1+T_1,t_2+T_2) = W(t_1,t_2) + dr$.
If $W(t_1+s,T_2) > W(t_1,t_2)$ for $s \in [0, T_3]$ and $T_3 \in \frac{r^2}{t_2}(\frac{1}{M},M)$, then we define $H^h(\underline{t},r)$ to be $(T_3+T_1,T_2)+(t_1,t_2)$; otherwise it is equal to ∞.

The suffix h for H^h denotes the priviliged position given the horizontal time direction. We similarly define $H^v(\underline{t},r)$ by reversing the roles of the first and second time co-ordinates. We say $H^j(\underline{t},r)$ is successful if it is not ∞. The utility of the definition lies in the fact that H^j is a stopping point if $\underline{t}$ is, and also if W($\underline{t}$) > 1+r and H^j is successful then there is an increasing path from $\underline{t}$ to $H^j(\underline{t},r)$ on which the value of W is always above 1 and such that (over this path) the difference between W and 1 increases by dr. We now record some fundamental properties of the stopping points H^j. For a standard linear Brownian motion B, starting from 0, we define $T_c = \inf\{t\colon B(t) = c \text{ or } -1\}$. We define the constants

$v(d,M) = P[\, T_d \in (\frac{1}{M},M),\, B(T_d) = d\,]$,

$u(d,M) = P[\, T_{1+d} \in (\frac{1}{M},M),\, B(T_{1+d}) = 1+d\,]$,

$c(d,M) = v(d,M) + (1-v(d,M))u(d,M)$,

It should be noted that as M tends to infinity v(d,M) tends to 1/(1+d) and u(d,M) tends to 1/(2+d). And so c(d,M) tends to $\frac{2}{2+d}$.

Lemma 1.1

There exists a constant C such that for all stopping points $\underline{t}$ in $(1,\infty)^2$ and j = h or v,

$$|\, P[H^j \text{ is successful } | F_{\underline{t}}] - c(d,M) \,| < Cr^{1/6}$$

Proof

Without loss of generality we consider H^h. The chance that $H^h(\underline{t},r) = (t_1+T_1,t_2)$ is precisely equal to v(d,M). The chance that, in defining H^h we proceed to step 3 is equal to (1-v(d,M))u(d,M). Therefore the lemma will be proven if we can show that the chance that we proceed to step 3 but H^h is unsuccessful is less than $Cr^{1/6}$ for suitable r. This last event is contained in the union of events

a $\quad \sup_{s_i \le Mr^2} |W^{\underline{t}}(s_1,s_2)| \le r^{3/2}$.

b $\quad B^3(s) = W(T^{1,r}+s, T^{2,r}) - W(T^{1,r},T^{2,r})$ hits $r^{3/2}$ before it hits $-r^{4/3}$.

c $\quad$ The time for B^3 defined above to hit either $-r^{4/3}$ or $r^{4/3}$ is greater than r^2.

Standard inequalities for the Brownian sheet and Brownian motion yield the desired inequalities. See e.g. Ito and McKean (1965).

□

We now define a succession of stopping points $U^j(\underline{t})$ for a time point $\underline{t}$. It should be borne in mind that we will be interested in points $\underline{t}$ such that $W(\underline{t}) \in (1+2^{-n/2}, 1+2\,2^{-n/2})$. Our goal will be to construct an increasing path on which W's value increases from close to 1 to above 2, without going below 1.

$U^1(\underline{t}) = H^h(\underline{t}, 2^{-n/2})$,

For j even, $U^j(\underline{t}) = \infty$ if $U^{j-1}(\underline{t}) = \infty; = H^v(U^{j-1}(\underline{t}), 2^{-n/2}(1+d)^{j-1})$ otherwise.

For j odd, $U^j(\underline{t}) = \infty$ if $U^{j-1}(\underline{t}) = \infty; = H^h(U^{j-1}(\underline{t}), 2^{-n/2}(1+d)^{j-1})$ otherwise.

As before we say U^j is successful if it is not equal to ∞. If $U_j(\underline{t})$ is successful, then there is an increasing time path from $\underline{t}$ to $U^j(\underline{t})$ along which W is strictly greater than 1, and such that $W(U^j(\underline{t})) - 1$ will be of the order $2^{-n/2}(1+d)^j$.

In the following we will be interested in $U^j(\underline{t})$ for $\underline{t}$ so that $W(\underline{t}) \in (1+2^{-n/2}, 1+2\,2^{-n/2})$. Accordingly we record some simple facts.

i $\quad$ If $W(\underline{t}) \in (1+2^{-n/2}, 1+2\,2^{-n/2})$, then, provided U^{N^n} is successful for $N^n = \frac{n}{2}\log_{1+d}(2)$, we have $W(U^{N^n}) \in (2,3)$. Here and in the following we round N^n up to the nearest integer.

ii $\quad$ Under the above circumstances for $\underline{t} \in [1,2]^2$, it must be the case that $U^r \in [1, 2\,2^{-n}(1+d)^{2r}M]^2$.

We define one last stopping point $V(\underline{t})$ for $\underline{t} \in [1,2]^2$. By observation (ii) above, if $U^{N^n}(\underline{t})$ is successful, then $U^{N^n} = (U_1,U_2)$ must be in $[1,8M]^2$. We define $V(\underline{t})$ to equal $(16M,U_2)$ if

i W(s,U_2) > 1 on [U_1, 16M]

and

ii W(16M, U_2) > 2.

Otherwise $V(\underline{t}) = \infty$. The use of this definition of V will emerge in the next section.

We now state some simple lemmas whose proofs are left to the reader.

The lemma below follows from Lemma 1.1 and the definitions of this section.

Lemma 1.2

There exist K_M and k_M so that for all $\underline{t} \in [1,2]^2$,

$$k_M(c(d,M))^r \le P[U^r(\underline{t}) < \infty] \le K_M(c(d,M))^r$$

for integer r $\in [1, N^n]$ Also we may choose k_M and K_M so that

$$k_M(c(d,M))^{N^n} \le P[V(\underline{t}) < \infty] \le K_M(c(d,M))^{N^n}$$

Lemma 1.3

There exists an integer k, depending only on M so that for any $\underline{t}$ and r, the event $\{U^{r-k} < \infty\}$, is measurable with respect to $G(\underline{t},\underline{t}+(1+d)^{2r}(2^{-n},2^{-n})) = G([0,t_1]\times[0,t_2+(1+d)^{2r}2^{-n})]\cup[0,t_1+(1+d)^{2r}2^{-n})]\times[0,t_2])$.

Section Two

In this section we obtain (modulo a technical lemma) the capacitance estimates required for the lower bound in Hausdorff dimension. We prove

Proposition 2.1

For every $\varepsilon > 0$, there exist components of the set $\{\underline{s} : W(\underline{s}) > 1\}$ whose boundary dimension is at least $5/4 - \varepsilon$.

Before proving this proposition we need some technical groundwork. Let n be an even integer and let $D_n = \{(\frac{j}{2^n},\frac{k}{2^n}): j, k \in Z\} \cap [1,2]^2$.

Define $K_n = \{\underline{t} = (t_1,t_2) \in D_n : W(t_1 + 2^{-n},t_2) \ge 1 + 2^{-n/2},\ W(t_1 + 2^{-(n+1)},t_2) \le 1\}$. For $\underline{t} \in K_n$ we define $L(\underline{t})$ to equal (s,t_2) where $s = \sup\{\ t \le t_1 + 2^{-n} : W(s,t_2) = 1\}$. Define $B_n = \{\underline{t} \in K_n : V(\underline{t} + (2^{-n},0)) < \infty\}$ and $B_n' = \{L(\underline{t}) : \underline{t} \in B_n\}$. We are directly interested in the set B_n', since its members are boundary points of Brownian sheet components. However, as the following lemma shows, for capacitory purposes, we may deal with the set B_n.

Lemma 2.1

For any positive α

$$\frac{1}{|B_n|^2}\sum_{x\neq y, x,y\in B_n}\frac{1}{|\underline{x}-\underline{y}|^{\alpha}} \geq 2^{\alpha}\frac{1}{|B_n'|^2}\sum_{x\neq y, x,y\in B_n'}\frac{1}{|\underline{x}-\underline{y}|^{\alpha}}$$

Proof

The above inequality simply follows form the inequality for each x, y $\in B_n$, $|L(x)-L(y)| \geq \frac{1}{2}|x-y|$.

□

The lemma below will perhaps reveal the motivation behind our definition of the final stopping point $V(\underline{t})$.

Lemma 2.2

There exists an a.s. finite number of components of $\{W > 1\}$, $C_1, C_2, \ldots C_N$ such that for every n, every point in B_n' is a boundary point of C_j for some j.

Proof

If $\underline{t} \in B_n'$, then there exists an increasing path from $\underline{t}$ to the line segment $[1,8M]\times\{16M\}$ on which W > 1 (except for the point $\underline{t}$ at which w equals 1) and such that W takes value at least 2 on the line segment $[1,8M]\times\{16M\}$. The Brownian motion W(s,16M) has only finitely many excursions from value 1 to value 2 beginning in the finite time interval [1, 8M]. But the number of components of $\{W > 1\}$ which intersect the line segment $[1,8M]\times\{16M\}$ at points where W is greater than 2 must be less than this a.s. finite number of excursions.

□

Lemma 2.3

Let $x(d,M) = -\frac{\log_2(c(d,M))}{\log_2(1+d)}$. There exist finite, strictly positive constants k and K such that for all n, $k2^{+3n/2}2^{-x(d,M)n/2} < E[|B_n|] = E[|B_n'|] < K2^{+3n/2}2^{-x(d,M)n/2}$.

Proof

It follows from Lemma 1.2 that for any of the 2^{2n} $\underline{t}$s in D_n, $P[V(\underline{t}+(2^{-n},0)) < \infty]$ is of the order $(c(d,M))^{N^*}$ which equals 1/2 to the power $\frac{n}{2}\left[\frac{\log_2(c(d,M))}{\log_2(1+d)}\right]$.

This event is independent of the event $\underline{t} \in K_n$, which has probability of the order of $2^{-n/2}$ and the result follows.

□

It should be noted that as M tends to infinity and then d tends to zero, x(d,M) tends to 1/2. Throughout this section we will assume that d and M have been chosen and fixed so that x(d,M) < 1/2 + ε.

The lemma below requires some solid work and its proof is postponed to the next section.

Lemma 2.4

Let $\underline{t}$ and $\underline{s}$ be elements of D_n with $|\underline{t}-\underline{s}|_{\max} = \max\{ |t_1-s_1|, |t_2-s_2|\} \in [2^{-i}, 2^{-i+1})$ and $|\underline{t}-\underline{s}|_{\min} = \min\{ |t_1-s_1|, |t_2-s_2|\} \in [2^{-j}, 2^{-j+1})$, then there exists finite K so that

$$P[\underline{t} \text{ and } \underline{s} \in B_n] \le K 2^{-n/2} 2^{-x(d,M)n/2} 2^{-(n-i)/2} 2^{-(n-j)x(d,M)/2}$$

Given this lemma we obtain the following capacity estimate for B_n.

Proposition 2.2

For every $\alpha < 3/2 - x(d,M)/2$, there exists a finite constant K_α so that

$$\frac{1}{E[|B_n|]^2} E\left[\sum_{x \ne y, x, y \in B_n'} \frac{1}{|\underline{x}-\underline{y}|^\alpha} \right] \le K_\alpha$$

Proof

The expectation of the sum on the left hand side is bounded by

$$K \sum_{\underline{s},\underline{t} \in D_n} \sum_{i=0}^{n} \sum_{j=i}^{n} \sum_{\substack{|\underline{t}-\underline{s}|_{\min} \in [2^{-j}, 2^{-j+1}) \\ |\underline{t}-\underline{s}|_{\max} \in [2^{-i}, 2^{-i+1})}} \frac{P[\underline{s}, \underline{t} \in B_n]}{2^{-i\alpha}}$$

for some constant K not depending on n. Using Lemma 2.3, the above is bounded by

$$K' \sum_{\underline{t} \in B_n} \sum_{i=0}^{n} \sum_{j=i}^{n} \sum_{\substack{|\underline{t}-\underline{s}|_{\min} \in [2^{-j}, 2^{-j+1}) \\ |\underline{t}-\underline{s}|_{\max} \in [2^{-i}, 2^{-i+1})}} 2^{i\alpha} 2^{-n/2} 2^{-x(d,M)n/2} 2^{-n/2} 2^{i/2} 2^{-x(d,M)(n-j)/2}$$

Summing over $\underline{t} \in B_n$ yields a factor of 2^{2n}, while summing over $|\underline{t}-\underline{s}|_{\min} \in [2^{-j}, 2^{-j+1})$ $|\underline{t}-\underline{s}|_{\max} \in [2^{-i}, 2^{-i+1})$ yields a factor of $2^{n-i} 2^{n-j}$. Therefore the sum in the lefthand side of the statement of the Proposition is majorized by

$$K 2^{-n} 2^{2n} 2^{-x(d,M)n} \sum_{i=0}^{n} 2^{(n-i)} 2^{i\alpha} 2^{i/2} \sum_{j=i}^{n} 2^{(n-j)} 2^{x(d,M)j/2}$$

which is majorized by

$$K 2^{+n} 2^{-x(d,M)n} \sum_{i=0}^{n} 2^{(n-i/2)} 2^{i\alpha} \sum_{j=i}^{n} 2^{(n-j)} 2^{x(d,M)j/2}.$$

summing over j reduces the above to

$$K 2^{+n} 2^{-x(d,M)n} \sum_{i=0}^{n} 2^{(n-i/2)} 2^{i\alpha} 2^{(n-i)} 2^{x(d,M)i/2}.$$

Because $\alpha < 3/2 - x(d,M)/2$ this sum over i is bounded by $K_\alpha' 2^{3n} 2^{-x(d,M)n} \le K_\alpha (E[|B_n|])^2$ for finite K_α' and K_α not depending on n. The proposition follows.

□

Corollary 2.4

There exists c > 0 and finite K such that with probability at least c for each n

$$\frac{1}{|B_n'|^2} \sum_{x \ne y, x, y \in B_n'} \frac{1}{|\underline{x}-\underline{y}|^\alpha} < K$$

Proof

First note that by Lemma 2.1, it is sufficient to show that there exist c and K so that with probability at least c

$$\frac{1}{|B_n|^2} \sum_{x \neq y, x, y \in B_n} \frac{1}{|\underline{x}-\underline{y}|^\alpha} < K$$

for each n. Also note that $\underline{t}, \underline{s} \in D_n$ implies that for positive α, $\frac{1}{|\underline{s}-\underline{t}|^\alpha} \leq 2^{-\alpha/2}$.

Therefore

$$E[|B_n|^2] \leq 2^{\alpha/2} E\left[\sum_{x \neq y, x, y \in B_n} \frac{1}{|\underline{x}-\underline{y}|^\alpha}\right] + E[|B_n|] \leq K(E[|B_n|])^2.$$

A simple Cauchy-Schmidt argument (see e.g. Kahane (1985), page 8) yields the conclusion that $P[|B_n| > c(K)E[|B_n|]] > c(K)$ for c strictly positive depending on K but not on n. Furthermore Proposition 2.2 guarantees that for G sufficiently large,

$$P\left[\frac{1}{E[|B_n|]^2} \sum_{x \neq y, x, y \in B_n} \frac{1}{|\underline{x}-\underline{y}|^\alpha} \leq G\right] > 1-c(K)/2.$$

Therefore with probability at least c(K)/2

$$\frac{1}{|B_n|^2} \sum_{x \neq y, x, y \in B_n} \frac{1}{|\underline{x}-\underline{y}|^\alpha} < \frac{G}{(c(K))^2}$$

□

Proof of Proposition 2.1

Recall that d and M have been chosen to ensure that 3/2 - x(d,M)/2 > 5/4 - ε so we can assume that α > 5/4 − ε as well. Let $H_M(\omega)$ be the union of the components of {W > 1} which intersect the line segment $\{6M^2\} \times [0, 4M^2]$ at points where W > 3/2. By Lemma 2.2 there are only finitely many such components and this is a closed set. Therefore its boundary is just the union of the boundaries of the individual components. For each n, $B_n' \subset \delta H_M$ and (by Proposition 2.2) with probability at least c > 0,

$$\frac{1}{|B_n'|^2} \sum_{x \neq y, x, y \in B_n'} \frac{1}{|\underline{x}-\underline{y}|^\alpha} < K$$

Therefore with probability at least c > 0,

$$\frac{1}{|B_n'|^2} \sum_{x \neq y, x, y \in B_n'} \frac{1}{|\underline{x}-\underline{y}|^\alpha} < K$$

occurs for infinitely many n. Thus (by Landkoff (1972)), with probability at least c, the capacitory dimension of δH_M is at least 5/4 − ε. The 0-1 law of Orey and Pruitt (1973), page 141, implies that

P[there exists components of { W > 1} whose boundaries

have Hausdorff dimension at least α] = 1

Proposition 2.1 follows.

□

Section Three

This section is devoted to proving Lemma 2.4 of the previous section. We start by proving the lemma for the easiest case: $\underline{s} < \underline{t}$. Though more complicated, the proof for the other cases can be broken down into the same basic steps.

It is readily seen that Lemma 2.4 is equivalent to

Lemma 2.4a

Let $\underline{t}$ and $\underline{s}$ be elements of D_n with $|\underline{t}-\underline{s}|_{\max} = \max\{ |t_1-s_1|, |t_2-s_2|\} \in 2^{-n}[(1+d)^i, (1+d)^{i+1})$, $\min\{ |t_1-s_1|, |t_2-s_2|\} \in 2^{-n}[(1+d)^j, (1+d)^{j+1})$. Then there exists K such that

$$P[\underline{t} \text{ and } \underline{s} \in B_n] \leq K 2^{-n/2} 2^{-x(d,M)n/2}(1+d)^{-i/2} c(d.M)^{-j/2}$$

Proof of Lemma 2.4a for the case $\underline{s} < \underline{t}$.

Let $\underline{s}$ and $\underline{t}$ satisfy

i $|\underline{s} - \underline{t}|_{\min} = |s_1-t_1| \in 2^{-n}[(1+d)^j,(1+d)^{j+1})$

ii $|\underline{s} - \underline{t}|_{\max} = |s_2-t_2| \in 2^{-n}[(1+d)^i,(1+d)^{i+1})$

The event $\{\underline{s}, \underline{t} \in B_n\}$ is contained in the intersection of four events:

A1: $\{U^{j/2-k}(\underline{s}) < \infty\}$ if j/2-k ≥ 0; = Ω the whole probability space if j/2-k < 0,

A2: $\{V(\underline{t}) < \infty\}$,

A3 $\{\underline{s} \in K_n\}$,

A4 $\{\underline{t} \in K_n\}$.

In defining event A1, the constant k is the constant of Lemma 1.3. Accordingly, the event A1 is measurable with respect to the σ-field $G(\underline{s},\underline{s}+(2^{-n}(1+d)^{j-1}, 2^{-n}(1+d)^{j-1}))$ = $G([s_1,s_1+2^{-n}(1+d)^{j-1}]\times[0,s_2+2^{-n}(1+d)^{j-1}] \cup [0,s_1+2^{-n}(1+d)^{j-1}]\times[s_2,s_2+2^{-n}(1+d)^{j-1}])$. Therefore it is independent of the event A2, and, using Corollary 1.3, we obtain the inequality

$$P[A1 \cap A2] < K(c(d,M))^N (c(d,M))^{j/2-k}. < K'(c(d,M))^N (c(d,M))^{j/2}.$$

Also the event A2∩A1 is independent of the random variable W($\underline{s}$). The event A3 is contained in the event $\{W(\underline{s} + (2^{-n},0)) \in (2^{-n/2}, 2\,2^{-n/2})\}$. These observations imply that P[A3 | A1 ∩ A2] $\leq K 2^{-n/2}$. Similarly, the event A4 is contained in the event $\{W(\underline{t} + (2^{-n},0)) \in (2^{-n/2}, 2\,2^{-n/2})\}$. Let the σ-field G equal σ-$\{G([0,s_1+2^{-(j+1)}]\times[0,s_2+2^{-(j+1)}]), G([0,t_1]\times[t_2,\infty], [t_1,\infty]\times[0,t_2])\}$. The events A1, A2, A3 are measurable with respect to G; in addition the random variable W(1, $s_2 + 2^{-n}(1+d)^{j-1}$) - W(1, $s_2 + 2^{-n}(1+d)^{j-1} + 2^{-n}(1+d)^{i-1}$) is independent of G and "contributes" to $W(\underline{t} + (2^{-n},0))$. We conclude that P[A4 | A1 ∩ A2 ∩ A3] $\leq K(1+d)^{-i/2}$ for some K not depending on n,i, j. The result follows.

□

This approach can be followed in the general case once we have proven

Proposition 3.1

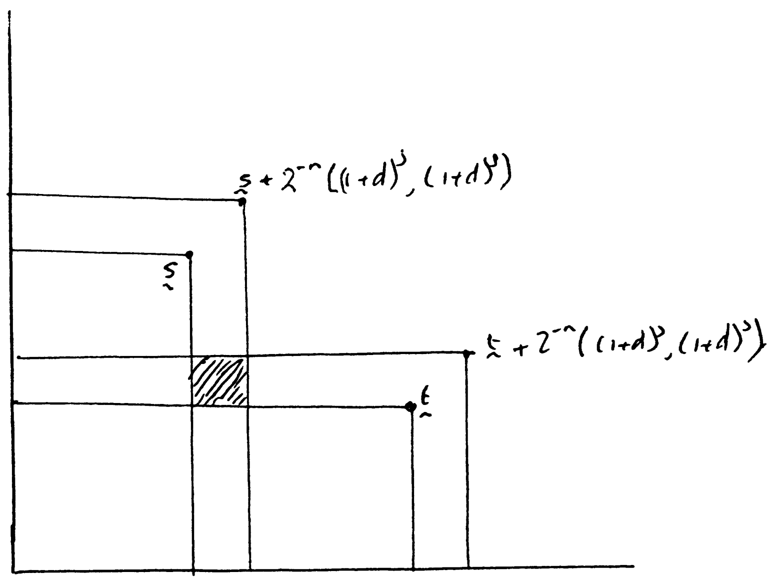
Let $\underline{s}$ and $\underline{t}$ be as in Lemma 2.3. Let k be as in Lemma 1.3. Then

$$P[U^{j/2-k}(\underline{t}),\ U^{j/2-k}(\underline{s}) < \infty] < K'(c(d,M))^{j/2}(c(d,M))^{j/2}.$$

for some K independent of n and j.

In the following for the sake of concreteness we will deal with the case $s_1 < t_1$, $s_2 > t_2$, $t_1 - s_1 > s_2 - t_2$. It will be clear that only minor relabeling in our arguments will establish the desired bounds for the other cases.

We wish to bound $P[U^{j/2-k}(\underline{t}), U^{j/2-k}(\underline{s}) < \infty]$. $U^{j/2-k}(\underline{t})$ is measurable with respect to $G(\underline{t},\underline{t}+2^{-n}((1+d)^j,(1+d)^j)))$ and $U^{j/2-k}(\underline{s})$ is measurable with respect to $G(\underline{s},\underline{s}+(2^{-n}((1+d)^j,(1+d)^j)))$. The problem is that these sigma fields are not independent. They both contain the sigma-field generated by the white noise of the shaded region in the diagram below.



Lemma 3.1

Let c be a fixed positive constant and let B be a standard Brownian motion. Suppose $Z(t) = (1-\lambda)B(t) + f(t)$ where f is a process satisfying $|f(t)| < \lambda$ for all $t \geq 0$. Define $T^Z = \inf\{t: Z(t) = c \text{ or } -1\}$ and $T^B = \inf\{t: B(t) = c \text{ or } -1\}$ and the events

$$A(B,x,M) = \{B(T^B) = x,\ T_B > M\} \quad A'(B,x,M) = \{B(T^B) = x,\ T_B < M\}$$

$$A(Z,x,M) = \{Z(T^Z) = x,\ T_Z > M\} \quad A'(Z,x,M) = \{Z(T^Z) = x,\ T_Z < M\}$$

There exists a constant K (=K(c)) so that for all λ, M and x ∈ { -1, c}

$$P\left[A(B,x,M)\ \ A(Z,x,M)\right] + P\left[A(Z,x,M)\ \ A(B,x,M)\right] \le K\lambda^{1/2}$$

and

$$P\left[A'(B,x,M)\ \ A'(Z,x,M)\right] + P\left[A'(Z,x,M)\ \ A'(B,x,M)\right] \le K\lambda^{1/2}$$

Proof
For brevity we will only prove that

$$P\left[A'(B,c,M)\ \ A'(Z,c,M)\right] \le K\lambda^{1/2}$$

for suitable K, the other proofs are similar. We need only concern ourselves with λ much smaller than c.

The event $\{A'(B,c,M)\ \ A'(Z,c,M)\}$ is contained in the union of the events

A $\{T^B \in [M-\lambda^{2/3}, M]\}$

and

B $\{\sup_{0 \le s \le \lambda^{2/3}} B(T^B + s) - B(T^B) < \frac{\lambda(2+c)}{1-\lambda}\}$.

Our desired inequality follows from standard Brownian inequalities. See e.g. Ito and McKean (1965).

□

It is necessary to introduce the sigma fields

$$G_l = \sigma\{\, G(\underline{s}, U^{l-1}(\underline{s})),\ G(\underline{t}, U^{l-1}(\underline{t})),\ \}$$

$$G_{l,+} = \sigma\{\, G(\underline{s}, U^{l-1}(\underline{s})),\ G(\underline{t}, U^{l}(\underline{t})),\ \}$$

If it were possible to prove inequalities such as

$$|P[U^l(\underline{t}) < \infty \mid G_l] - c(d,M)| < k2^{-l/6}$$

$$|P[U^l(\underline{s}) < \infty \mid G_{l,+}] - c(d,M)| < k2^{-l/6}$$

then the bound claimed in Proposition 3.1 would follow easily. Unfortunately this is not true in total generality as there may be path wildness.

It is true that, for instance, given a stopping point $\underline{T} > \underline{t}$, the Brownian motion

$$B(r) = \frac{1}{\sqrt{T_2}} W(T_1+r, T_2) - W(T_1, T_2)$$

is independent of σ-$\{G(\underline{t},\underline{T}),\ G(\underline{s}, U_{l+1})\}$. Our problem is that the Brownian motion

$$\frac{1}{\sqrt{T_1}} W(T_1, T_2+r) - W(T_1, T_2)$$

is not independant of σ-$\{G(\underline{t},\underline{T}),\ G(\underline{s}, U_{l-1})\}$. In fact

$$W(T_1, T_2+r) - W(T_1, T_2) =$$

$$W(T_1, T_2+r) - W(T_1, T_2) - \left[W([U^{l-1}(\underline{s})]_1, T_2+r) - W([U^{l-1}(\underline{s})]_1, T_2)\right] +$$

$$W([U^{l-1}(\underline{s})]_1, T_2+r) - W([U^{l-1}(\underline{s})]_1, T_2) - \Big[W(s_1, T_2+r) - W(s_1, T_2)\Big] +$$
$$W(s_1, T_2+r) - W(s_1, T_2)$$

The first and third processes are independant of σ-$\{G(\underline{t}, \underline{T}), G(\underline{s}, U_{l-1})\}$. Therefore if we can control the middle process we will be in a position to apply Lemma 3.1. We hope this will motivate the following.

We proceed to define "wild" sets we wish to discard from consideration. For $0 \le l, h \le j/2-k$ we define

$$A(l, \underline{s}) = \{ \text{ there exists } r, r' \in (t_2, t_2 + 4M(2^{-n/2}(1+d)^l)^2 :$$
$$W([U_{l-1}(\underline{s})]_1, r) - W(s_1, r) - W([U_{l-1}(\underline{s})]_1, r') + W(s_1, r') > (2^{-n/2}(1+d)^l)^{3/2}$$
$$B(l, \underline{t}) = \{ \text{ there exists } r, r' \in (s_1, s_1 + 4M(2^{-n/2}(1+d)^l)^2 :$$
$$W(r, [U_l(\underline{t})]_2) - W(r, t_2) - W(r', [U_l(\underline{t})]_2) + W(r', t_2) > (2^{-n/2}(1+d)^l)^{3/2}$$

Lemma 3.2

There exist constants c and K so that $P[A(l, \underline{s}) \cup B(l, \underline{t})] < Ke^{-c2^{n/4}/(1+d)^{-l/2}}$. This lemma follows from bounds found in Orey and Pruitt (1973).

Lemma 3.3

There exists a constant c so that

$$| P[U^l(\underline{t}) < \infty \mid G_l] - c(d, M) | < c(2^{-n/2}(1+d)^l)^{-l/6}$$

on $A(l, \underline{s})^c$. Similarly

$$| P[U^l(\underline{s}) < \infty \mid G_{l,+}] - c(d, M) | < c(2^{-n/2}(1+d)^l)^{-l/6}$$

on $B(l, \underline{t})^c$.

Proof

We will just give the proof for $U^l(\underline{t})$ as that for $U^l(\underline{s})$ is almost identical. In addition we will assume for simplicity that l is even. The proof for l even is slightly more complicated but no new ideas are needed.

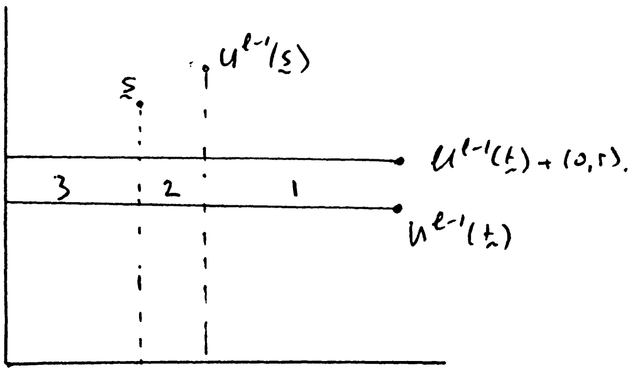
As l is even, $U^l(\underline{t}) = H^v(U^{l-1}(\underline{t}), 2^{-n}(1+d)^{l-1})$. Also for any stopping point $\underline{P}$, greater (in natural partial order) than $\underline{t}$, we have $B^{1,\underline{P}}$ is independent of G_l, our problem comes down to dealing with step one defining H^v. Consider the Brownian motion $Z(r) = W([U^{l-1}(\underline{t})]_1, [U^{l-1}(\underline{t})]_2+r) - W([U^{l-1}(\underline{t})]_1, [U^{l-1}(\underline{t})]_2)$. As noted above, this can be decomposed as

$$Z(r) = W([U^{l-1}(\underline{t})]_1, [U^{l-1}(\underline{t})]_2+r) - W([U^{l-1}(\underline{t})]_1, [U^{l-1}(\underline{t})]_2) =$$
$$\Big(W([U^{l-1}(\underline{t})]_1, [U^{l-1}(\underline{t})]_2+r) - W([U^{l-1}(\underline{t})]_1, [U^{l-1}(\underline{t})]_2) -$$
$$W([U^{l-1}(\underline{s})]_1, [U^{l-1}(\underline{t})]_2+r) - W([U^{l-1}(\underline{s})]_1, [U^{l-1}(\underline{t})]_2)\Big)$$

$$+ \Big(W([U^{l-1}(\underline{s})]_1, [U^{l-1}(\underline{t})]_2+r) - W([U^{l-1}(\underline{s})]_1, [U^{l-1}(\underline{t})]_2) -$$

$$W(s_1,[U^{l-1}(\underline{t})]_2+r) - W(s_1,[U^{l-1}(\underline{t})]_2)\Big)$$

$$+ W(s_1,[U^{l-1}(\underline{t})]_2+r) - W(s_1,[U^{l-1}(\underline{t})]_2)$$



Given $\sigma\{G(\underline{t},\underline{T}),G_l\}$ the first and third processes above, sum to a process equal in law to

$$[[U_{l-1}(\underline{t})]_1 - [U^{l-1}(\underline{s})]_1 + s_1]^{1/2} B(s)$$

where B is a standard Brownian motion independent of $\sigma\{G(\underline{t},\underline{T}),G_l\}$. The second process above is measureable with respect to this σ field. On the event $A(l,\underline{s})^c$, this process is bounded in magnitude by $(2^{-n/2}(1+d)^l)^{3/2}$. The result now follows from scaling and Lemma 3.1.

□

Proof of Proposition 3.1

Let $L(\omega) =$ inf $\{l \geq 0 : \omega \in A(l,\underline{t}) \cup B(l,\underline{s})\}$. We decompose the event $\{U_{j/2-k}(\underline{s}),\ U_{j/2-k}(\underline{t}) < \infty\}$ into

$$\bigcup_{l=0}^{j/2-k} \{U_{j/2-k}(\underline{s}),\ U_{j/2+k}(\underline{t}) < \infty,\ V = l\} \ \cup\ \{U_{j/2-k}(\underline{s}),\ U_{j/2+k}(\underline{t}) < \infty,\ V > j/2-k\}$$

$$\subset \bigcup_{l=0}^{j/2-k} \{U_{l-1}(\underline{s}),\ U_{l-1}(\underline{t}) < \infty,\ V = l\} \ \cup\ \{U_{j/2-k}(\underline{s}),\ U_{j/2+k}(\underline{t}) < \infty,\ V > j/2-k\}$$

By lemmas 3.2 and 3.3 the probability of the latter event is bounded by

$$K \sum_{l=0}^{j/2-k} (c(d,M)^{2(l-1)} e^{-c(2^{-n/2}(1+d)^l)^{l/2}} \qquad + \qquad (c(d,M))^{2(j/2-k)}$$

This is easily seen to be bounded by the appropriate quantity.

□

Proof of Lemma 2.4a

The case $\underline{s} < \underline{t}$ has already been dealt with. Of the remaining cases (since they are essentially the same) we will consider the case where $t_1 > s_1,\ t_2 < s_2$, $|t_1 - s_1| \geq |t_2 - s_2|$. Let $s_2 - t_2 \in 2^{-n/2}[(1+d)^j,(1+d)^{j+1})$. By definition,

$P[V(\underline{s}), V(\underline{t}) < \infty] \le P[U^{j/2-k}(\underline{s}), V(\underline{t}) < \infty]$. This latter expression is equal to

$$E\left[U^{j/2-k}(\underline{s}),\ U^{j/2-k}(\underline{t}) < \infty,\ \prod_{l=j/2-k+1}^{N} P[U^l < \infty \mid F(U^{j/2-k}(\underline{s})), F(U^{l-1}(\underline{t}))]\right]$$

$$\le\ E\left[U^{j/2-k}(\underline{s}),\ U^{j/2-k}(\underline{t}) < \infty,\ \prod_{l=j/2+k}^{N} P[U^l < \infty \mid F(U^{j/2-k}(\underline{s}))F(U^{l-1}(\underline{t}))]\right]$$

By Lemma 1.2, the above is majorized by

$$E\left[U^{j/2-k}(\underline{s}),\ U^{j/2-k}(\underline{t}) < \infty,\ \prod_{l=j/2+k}^{N} (c(d,M))(1+c2^{-n/6}(1+d)^{j/3})\right].$$

By Proposition 3.1, this is less than $K(c(d,M))^{j/2}(c(d,M))^N$.

The remainder of the proof follows as with the case $\underline{s} < \underline{t}$.

□

Section Four

Section Two shows that, for each $\varepsilon > 0$, with probability one there exist components of $\{W > 1\}$ whose boundary has Hausdorff dimension at least $5/4 - \varepsilon$. In this section we use standard properties of the Brownian sheet to show that every such component must have a boundary with dimension at least 5/4.

Proposition 4.1

The Hausdorff dimension of every component of $\{W > 1\}$ is at least 5/4.

To show the above it is sufficient to show that for every $\varepsilon > 0$ and every rational time point $\underline{q} \in Q_+ \times Q_+$, the component of $\{W > 1\}$ containing $\underline{q}$ (if it exists) has, with probability one, a boundary of dimension at least $5/4 - \varepsilon$. We will prove this fact for the point (1,1) but the reader will see that the proof works for any fixed time point.

We now state some propositions without proof. We give some remarks which will hopefully convince the reader that no new ideas are required to prove the stated propositions.

Given $\varepsilon > 0$, the arguments of Sections Two and Three can be refined to show that with probability $c(\varepsilon) > 0$, there is a component C of $\{W > 1\} \cap [1,2]^2$ such that δC has dimension at least $5/4 - \varepsilon$, and (2,2) is in C. If we denote this event by $A(\varepsilon)$, then we may even prove that

Proposition 4.2

For some $k(\varepsilon) > 0$, $P[A(\varepsilon) \mid W(2,2)] > k(\varepsilon)$ on the event $\{W(2,2) \in (2,3)\}$.

The arguments used in Section Two and Three for our Brownian sheet process work equally well with the process

$$B(s,t) = B_1(s) - B_2(t)$$

for two independant, not necessarily standard, Brownian motions. In fact the major problems of calculations dissappear as $W(t_1,t_2) - W(s_1,t_2) - W(t_1,s_2) + W(s_1,s_2)$ become stochastically insignificant. Similarly if we consider the process

$W^{c,\underline{t}}(\underline{s}) = \frac{1}{c}\left[W(\underline{t} + c^2\underline{s}) - W(\underline{t})\right]$ for c small, the terms $W^{c,\underline{t}}(s_1',s_2') - W(s_1,s_2') - W(s_1',s_2) + W(s_1,s_2)$are stochastically manageable and all estimates derived in Section Two and Three will hold uniformly for c $\in$ (0,1]. So no new ideas are required to prove ·

Proposition 4.3

Let $A(\varepsilon,\underline{t},c)$ be the event that there is a component C of $\{W > 1\} \cap [\underline{t},\underline{t}+(c^2,c^2)]$ such that

1 δC has dimension at least $5/4 - \varepsilon$,

2 $\underline{t}+(c^2,c^2) \in$ C For M > 1, there exists a constant $k(\varepsilon,M) > 0$ such that for all $\underline{t} \in [1,M]^2$ and c > 1 $P[A(\varepsilon,\underline{t},c) \mid W(\underline{t} + (c^2,c^2))] > k(\varepsilon,M)$ on $\{W(\underline{t}+(c^2,c^2)) \in (1+c, 1+2c)$.

Given (t_1,t_2), we define the Brownian sheet

$$W_{\underline{t}}'(s_1,s_2) = \frac{s_1}{t_1}\frac{s_2}{t_2}W(\frac{t_1^2}{s_1},\frac{t_2^2}{s_2})$$

and the let the stopping points $U^{1\prime}(\underline{t}), U^{2\prime}(\underline{t}), \ldots, U^{N\prime}(\underline{t}), V'(\underline{t})$ be defined for the sheet W' above.

Finally define the random points $U^{1\prime\prime}, U^{2\prime\prime}, \ldots, V''$ by,

$$\underline{U^{j}}'' = [(U^{j\prime\prime})_1,(U^{j\prime\prime})_2] = [\frac{t_1^2}{(U^{j\prime})_1},\frac{t_2^2}{(U^{j\prime})_2}],$$

These random points can play the same role as the points U^j in Section Two and Three, the only essential difference being that they decrease as j decreases. They will also be used in Section Six. Using these points instead of the U^j we can prove our final stated result

Proposition 4.4

Let $B(\varepsilon,\underline{t},c)$ be the event there is a component C of $\{W > 1\} \cap [\underline{t},\underline{t}+(c^2,c^2)]$ such that

1 δC has dimension at least $5/4 - \varepsilon$,

2 $\underline{t} \in C$

Let M be > 1. There exists a constant $k(\varepsilon,M) > 0$ such that for all stopping points $\underline{S} \in [1,M]^2$ and c > 1 $P[B(\varepsilon,\underline{S},c) \mid W(\underline{S})] > k(\varepsilon,M)$ on $\{W(\underline{S}) \in (1+c, 1+2c)\}$.

To finish the proof of Proposition 4.1 from Proposition 4.4, we require some fresh arguments.

We require some notation: Given time point $\underline{t}$, we define $C(\underline{t})$ to be the component of $\{W > 1\}$ containing $\underline{t}$, if it exists. Given in addition a time rectangle R containing $\underline{t}$, we define $T(\underline{t},R)$ to be $\delta(C(\underline{t}) \cap R)\quad \delta R$.

Before proving Proposition 4.1, we require two preliminary lemmas. In the following, given a process X and a point y,

$$T^X(y) = \inf\{ t > 1 : X(t) < y \}.$$

Throughout, B and V will denote independant standard Brownian motions.

Lemma 4.1

Let ε be a positive constant, less than two. Let λ be a positive constant less than 1/2. Define Z = B + λV. Let R be a fixed constant.

$$P\left[|T^{|B|}(\varepsilon) - T^{|Z|}(\varepsilon)| \geq \lambda^{1/4},\ T^{|B|}(\varepsilon) < R \right] < C(R)\lambda^{1/4}$$

Proof

Consider the events

1 $|T^B(\varepsilon) - T^B(\varepsilon \pm \lambda^{1/2})| \geq \lambda^{1/2}$,

2 $\sup_{s \in [1, R+1]} |V(s)| < \lambda^{-1/2}$.

Outside of these two events, the events $\{ |T^{|B|}(\varepsilon) - T^{|Z|}(\varepsilon)| \geq \lambda^{1/4} \}$ and $\{ T^{|B|}(\varepsilon) < R \}$ are incompatible. The lemma follows from simple bounds for Brownian motion.

□

Similar elementary considerations give the following, whose proof is therefore omitted.

Lemma 4.2

With the notation of the previous lemma, let F_R^B be $\sigma\{B(t): t \leq R\}$. Then

(a): For small λ,

$$P[T^{|Z|}(\varepsilon) - T^{|B|}(\varepsilon) \geq \lambda^3 \,|\, F_R^B] > 1/3$$

on $A^B(\varepsilon,\lambda) =$
$\{T^{|B|}(\varepsilon) \in [1,R], T^B(\varepsilon + \lambda^{1/2}) - T^B(\varepsilon) < \lambda^{1/2}, T^B(\varepsilon) - T^B(\varepsilon - \lambda^{4/3}) > \lambda^3\}$.

(b): $P[A^B(\varepsilon,\lambda)^c \quad \{T^{|B|}(\varepsilon) \text{ not in } [1,R]\}\ ] \leq K(R)\lambda^{1/4}$ for some finite K(R).

Proof of Proposition 4.1

Fix ε > 0, arbitrarily small. Choose M sufficiently large that $P[T^B(1) > M/2\ ] < \varepsilon/4$ for a standard Brownian motion. Here $T^B(1)$ is the stopping time of Lemma 4.1. Let d = k(ε,M) > 0. The main part of the proof consists of establishing that, given r (large), we can choose c (small) so that there are increasing stopping points $S_1, \ldots, S_r$ such that outside of a set of probability ε

1 For each i, $S_i > S_{i-1} + (2c, 2c)$.

2 For each i, $S_i \in [1, M)^2$.

3 For each i ∈ {1, 2, . . , r} S_i is in C(1,1).

Once this has been proven Proposition 4.4 yields the bound P[C(1,1) has a boundary of Hausdorff dimension at least 5/4 − ε] > $1 - \varepsilon - (1-d)^r$, and Proposition 4.1 follows.

Choose n large and even (how large is to be determined later). Let $c = e^{-100^n}$. Let $B^0(s) = W(s,1)$. For i = 1, 2, . . . n/2, define $B^i(s) = W(s, 1+e^{-100^{n-i}})$. Define the stopping times $T^i = T^{B^i}(1+c)$. Clearly (by Lemma 4.1, our choice of M and elementary considerations) for n sufficiently large, P[for some $0 \le i \le n/2$, $T^i > M$] + P[for some i ≤ n/2, C(1,1) is non-empty but $(1, 1+e^{-100^{n-i}})$ is not in C(1,1)] < ε/4. Consider the stopping times $V^j = T^1 \;\; T^2 \dots \;\; T^j$. By Lemma 4.2 (a) and the observation above, outside of the event $\bigcup_{i=0}^{n/2} A^{B^i}(1+c, e^{-100^{n-i}/2})^c$, we have $P[T^j > T^{j-1} + (e^{-100^{n-j}4/2}, e^{-100^{n-j}4/2}) \mid F_{V^{j-1}}] > 1/3$. By Lemma 4.2 (b), for large n, $P[\bigcup_{i=0}^{n/2} A^{B^i}(1+c, e^{-100^{n-i}/2})^c, \cup \bigcup_{i=0}^{n/2} \{T^i(1+c) > M\}] < \varepsilon/4$. Also outside of the events

$$\bigcup_{i=0}^{n/2} \{ \mid T^{i-1}(c) - T^i(1+c) \mid \ge e^{-100^{n-i}/8}, T^{i-1}(1+c) < M \},$$

it is the case that $\{T^j > T^{j-1} + (e^{-100^{n-j}4/2}, e^{-100^{n-j}4/2})\}$ implies $\{T^j > V^{j-1} + (2c, 2c)\}$. Also, by Lemma 4.1, we have for large n,

$$P\left[\bigcup_{i=0}^{n/2} \{ \mid T^{i-1}(1+c) - T^i(1+c) \mid \ge e^{-100^{n-i}/8}, T^i(1+c) < M \}\right]$$

$$< C(M) \sum_{i=0}^{n/2} e^{-100^{n-i}/8} < \varepsilon/8$$

Collecting these bounds together, we conclude that if N = #{j ≤ n/2: $T^j > V^{j-1} + (2c, 2c)$}, then for n large, P[N < r] < ε/4 + ε/4 + ε/8 + ε/8 + P[B(n/2, 1/3) < r], where B(n/2, 1/3) is a binomial random variable with parameters n/2 and 1/3. If n is sufficiently large our result follows.

□

The thoughts for this proof were suggested to the authour during a conversation with Robert Dalang, Steve Evans and Davar Khoshnevisan.

Section Five

In this section we establish that the Hausdorff dimension of boundary points is strictly less than 3/2. We will show that for any v greater than zero, the dimension of the set B = {$\underline{t} \in [1,2]^2$: $W(\underline{t})$ is a boundary point of a component of diameter greater than v}
has dimension bounded below 3/2 - c for some c > 0 not depending on v. The proof and elementary scaling ideas will convince the reader that this is enough.

The j-ring around the point $\underline{t}$ is the set $R(\underline{t}, j) = \{\underline{s} : |\underline{s} - \underline{t}|_\infty = 2^{-j}\}$. A j-ring is *good* if $\sup_{\underline{s} \in R(\underline{t}, j)} W(\underline{s}) - W(\underline{t})\} < -2^{-j/2}$.

Given δ ∈ (0, 1/2), we say a point $(\frac{j}{2^n}, \frac{k}{2^n}) \in [1,2]^2$, is n-bad if

i $\quad |W(\frac{j}{2^n}, \frac{k}{2^n}) - 1| \le n^2 2^{-n/2}$

ii $\quad$ There does not exist a good j-ring for $j \in [(1-\delta)n, \sqrt{n}\,]$

Orey and Pruitt (1973) prove that a.s. for n sufficiently large, $\underline{s} \in [1,2]^2$, $|\underline{s} - \underline{t}|_\infty < 2^{-n+1}$ implies that $|W(\underline{s}) - \underline{W(t)}| < n^2 2^{-n/2}$. Therefore for n large

$$B \subset \bigcup_{\substack{j,k\in[1,2^n] \\ (\frac{j}{2^n},\frac{k}{2^n})\, n-bad}} [1+\frac{j}{2^n}, 1+\frac{j+1}{2^n}]\times[1+\frac{k}{2^n}, 1+\frac{k+1}{2^n}].$$

We proceed to estimate the probability of a time point being n-bad. For notational simplicity we will work with the time point (1,1) but all conclusions found will be valid for an arbitrary time point $\underline{t}$ in $[1,2]^2$. Define the processes

$B_1(t) = W(t+1,1) - W(1,1), t\in [0,1]$

$B_2(t) = W(1,t+1) - W(1,1), t\in [0,1]$

$B_3(t) = \frac{W(1,1-t)}{1-t} - W(1,1)\ t \in [0,1),$

$B_4(t) = \frac{W(1-t,1)}{1-t} - W(1,1)\ t \in [0,1).$ These four processes are independent of each other and of W(1,1). The first two processes are Brownian motions, the last two are such that $B_i(\frac{s}{s+1})$ are Brownian motions and so in a neighbourhood of 0, the properties of all B_j will be Brownian. For i ∈ {1, 2, 3, 4} and j ∈ $(\sqrt{n}, (1-\delta)n]$, define the event

$$A_j^{M,i} = \sup_{s\le 2^{-j}} B_i(s) \le M 2^{-j/2}$$

Lemma 5.1

There exists strictly positive c such that for i ∈ {1, 2, 3, 4} and all n large enough

$$P\left[\sum_{j=\sqrt{n}}^{(1-\delta)n} I_{A_j^{M,i}} \ge (1-\delta)n\frac{15}{16}\right] \ge 1-e^{-cn}$$

Proof

We give the proof for i = 1, the prove for the cases i = 3 or 4 is essentially the same, that for i = 2 is of course exactly the same.

For j ∈ $(\sqrt{n}, (1-\delta)n]$, define

$$T_j = \inf\{s>0: B_1(s) \ge M 2^{-j/2}$$

The T_j are stopping times with respect to the natural filtration of B_1 and the corresponding σ-fields F_{T_j} form a reverse filtration. The quantity $P[A_j^{M,1} | F_{T_{j+1}}]$ is equal to 0 if $T_{j+1} \ge 2^{-j}$; if $T_{j+1} < 2^{-j}$ then it is equal to

$$P\left[\sup_{0\le s\le 2^{-j}-T_{j+1}} B(s) \ge M^{-j/2} - M^{-(j+1)/2}\right]$$

where B is a Brownian motion. This term is less than or equal to

$$P\left[\sup_{0\le s\le 2^{-j}} B(s) \ge M^{-j/2}(1-\frac{1}{\sqrt{2}})\right] = 2P[B(1) \ge M(1-\frac{1}{\sqrt{2}})] = f(M).$$

Since clearly $P[A_{(1-\delta)n}^{M,1}] \le f(M)$, we conclude that $\sum_{j=\sqrt{n}}^{(1-\delta)n} I_{A_j^{M,i}}$ is stochastically greater than a Binomial random variable with parameters $(1-\delta)n-\sqrt{n}$ and 1-f(M). Since f(M) tends to zero as M tends to infinity, the result follows from standard large deviations estimates for binomial random variables. See for example Bollabas (1985).

□

Define the two dimensional processes:

$$W^1(s,t) = W(1+s,1+t) - W(1,1+t) - W(1+s,1) + W(1,1)$$
$$W^2(s,t) = W(1-s,1+t) - W(1,1+t) - W(1-s,1) + W(1,1)$$
$$W^3(s,t) = W(1+s,1-t) - W(1,1-t) - W(1+s,1) + W(1,1)$$
$$W^4(s,t) = W(1-s,1-t) - W(1,1-t) - W(1-s,1) + W(1,1)$$

The estimates below follow directly from estimates in Orey and Pruitt (1973).

Lemma 5.2

Let V_j^i, i ∈ {1, 2, 3, 4}, j ∈ $(\sqrt{n}, (1-\delta)n)$ be the event { there exists s ∈ $[0, 2^{-j}]$ s.t. $W^i(s,2^{-j}) \geq 2^{-j}/4$ or $W^i(2^{-j},s) \geq 2^{-j}/4$}. There exist strictly positive K and c so that

$$P\left[\bigcup_{j=\sqrt{n}}^{(1-\delta)n} \bigcup_{i=1}^{4} V_j^i\right] \leq Ke^{-c2^{\sqrt{n}}}$$

Lemma 5.3

Let C_j be the event { for every i ∈ {1,2,3,4} $B_i(2^{-j+1}) \leq -(M+2)2^{-j}$, and $\sup_{s \leq 2^{-j+1}} B_i(s) \leq (M+\frac{1}{2})2^{-j}$ }. Then there exists strictly positive c so that

$$P\left[\bigcup_{j=\sqrt{n}}^{(1-\delta)n} C_j\right] > 1 - e^{-cn}$$

for all n large enough.

Proof

Let $F'_j = \sigma\text{-}\{B_s^i: i=1,2,3,4,\ s \leq 2^{-j}\}$. There exists k > 0 so that $P[C_j \mid F_j'] > k$ on $\bigcap_i A_j^{M,i}$. Therefore

$$P\left[(\bigcup_{j=\sqrt{n}}^{(1-\delta)n} C_j)^c\right] \leq P\left[\sum_{j=\sqrt{n}}^{(1-\delta)n} I_{\bigcap_i A_j^{M,i}} \geq (1-\delta)n\frac{12}{16}\right] + (1-k)^{(1-\delta)3n/4}$$

The result now follows from Lemma 5.1

□

Proposition 5.1

For every $(\frac{j}{2^n},\frac{k}{2^n}) \in [1,2]^2$ and some finite K, $P[(\frac{j}{2^n},\frac{k}{2^n})]$ is n-bad $\leq Kn^2 2^{-n(\frac{1}{2}+c)}$.

Proof

We prove this just for $j = k = 2^n$.

If (for j ∈ $(\sqrt{n}, (1-\delta)n)$) C_j occurs, then the j+1-ring will be good unless for some i ∈ {1, 2, 3, 4}

$$\sup_{s \le 2^{-j+1}} W^i(s, 2^{-j+1}) > 2^{-j}/2$$

or

$$\sup_{s \le 2^{-j+1}} W^i(2^{-j+1}, s) > 2^{-j}/2.$$

By Lemma 5.2 this can only happen on a set of measure bounded by $Ke^{-c2^{\sqrt{n}}}$. Given this bound Lemma 5.3 implies that the probability that there does not exist a good j-ring for j $\in [\sqrt{n}, (1-\delta)n]$ is less than or equal to 2^{-nc} for some c strictly positive. This event is measureable with respect to $G([\frac{1}{2},\frac{3}{2}]\times[0,\frac{3}{2}]\cup[0,\frac{3}{2}]\times[\frac{1}{2},\frac{3}{2}]]$. W(1,1) is equal to W(1/2, 1/2) plus a random variable measureable with respect to $G([\frac{1}{2},\frac{3}{2}]\times[0,\frac{3}{2}]\cup[0,\frac{3}{2}]\times[\frac{1}{2},\frac{3}{2}]]$. Since W(1/2, 1/2) is independent of the latter sigma-field we deduce

$$P[\,|W(1,1)| < n^2 2^{-n/2} \mid G([\frac{1}{2},\frac{3}{2}]\times[0,\frac{3}{2}]\cup[0,\frac{3}{2}]\times[\frac{1}{2},\frac{3}{2}]] < Kn^2 2^{-n/2}$$

and so P[(1,1) is n-bad] $\le Kn^2 2^{-n(\frac{1}{2}+c)}$.

□

We can now complete the proof of Theorem One with the following proposition.

Proposition 5.2

The Hausdorff dimension of the set of time points that are in the boundary of some component of $\{\underline{s}: W(\underline{s}) > 0\}$ is less than or equal to 3/2 - c. Here c is the positive constant of Lemma 5.3 and Proposition 5.1.

Proof

As was mentioned in the introductory paragraph of this section, a.s. for all n large

$$\bigcup_{\substack{j,k\in[1,2^n] \\ (\frac{j}{2^n},\frac{k}{2^n})\, n-bad}} [1+\frac{j}{2^n}, 1+\frac{j+1}{2^n}]\times[1+\frac{k}{2^n}, 1+\frac{k+1}{2^n}].$$

is a covering of B. For $\alpha > 3/2 - c$, by Proposition 4.1,

$$E\left[\sum_{\substack{j,k\in[1,2^n] \\ (\frac{j}{2^n},\frac{k}{2^n})\, n-bad}} 2^{-n\alpha}\right] < 2^{2n}Kn^2 2^{-n(1/2+c)} 2^{-n\alpha}$$

which tends to zero as n tends to infinity. The result now follows by Fatou's lemma.

□

Section Six

In this section we use the ideas of Section Two to establish Theorem Two, stated in the introduction.

To prove this result we follow a path close to that of Section Two.

We reason along the following lines: if there are such points then with positive probability there ought to be points in a given rectangle which are on the boundary both of components of {W > 1} of diameter > 1, and of components of {W < 1} of diameter > 1. We consider time points in the square $[L, L+1]^2$, where L is a large constant to be fully specified later. Suppose we can "pick out" some finite number of components of {W > 1} (hereafter *positive* components) of diameter > 1 and some finite number of components of {W < 1} (hereafter *negative* components) of diameter > 1. The boundaries of these components intersected with $[L,L+1]^2$ are compact sets disjoint boundaies, then these boundaries should be separated by a strictly positive distance. Accordingly, if we can show that , with probability bounded away from zero, for each n there exists a point in $[L,L+1]^2$ which is within 2^{-n} of a positive component and a negative component, then we will have shown that with positive probability there exist points which are boundary points of both positive and negative components. A routine application of a 0-1 law of Orey and Pruitt (1973) will complete the proof.

We now introduce, recall or redefine some notation.
$D_n^L = \{ (\frac{i}{2^n},\frac{j}{2^n}) : (\frac{i}{2^n},\frac{j}{2^n}) \in [L,L+1]^2\}$. Let M and d be chosen (and fixed) so that $x(d,M)$, the constant introduced in Section Two, is strictly less than 1. Let L be a large number much larger than $12M^2$ We define, for $\underline{t} \in D_n^L$, the stopping points $U^1(\underline{t}), U^2(\underline{t}), \ldots, U^N(\underline{t})$ as before (with our fixed M). We need a new definition for $V(\underline{t})$ however:
If $U^N(\underline{t}) < \infty$, we define $V(\underline{t})$ to equal $(L+2, (U^1)_2)$ if for each s in $[(U^N)_1, L+2]$, $W(s,(U^N)_2) > 3/2$. Otherwise V is equal to infinity.
We also require some random points in the quadrant below $\underline{t}$, $U^{j\,\prime\prime}$. We define the Brownian sheet

$$W_{\underline{t}}'(s_1,s_2) = \frac{s_1}{t_1}\frac{s_2}{t_2}W(\frac{t_1^2}{s_1},\frac{t_2^2}{s_2})$$

and the let the stopping points $U^{1\prime}(\underline{t}), U^{2\prime}(\underline{t}), \ldots, U^{N\prime}(\underline{t})$ be defined for the sheet W' above.

Finally define the random points $U^{1\prime\prime}, U^{2\prime\prime}, \ldots, V''$ by,

$$\underline{U^{j\prime\prime}} = [(U^{j\prime\prime})_1, (U^{j\prime\prime})_2] = [\frac{t_1^2}{(U^{j\prime})_1}, \frac{t_2^2}{(U^{j\prime})_2}],$$

if $U^{j\prime} < \infty$; $= \infty$ otherwise. We now define a random subset of D_n^L analogous to B_n. Let V_n^L consist of those elements $\underline{t}$ of D_n^L such that

a $|W(\underline{t})-1| < 2^{-n/2}$,

b $W(\underline{t}+(2^{-n},0)) \in (1+2^{-n/2}, 1+2\,2^{-n/2})$,

c $W(\underline{t}-(2^{-n},0)) \in (1-2\,2^{-n/2}, 1-2^{-n/2})$,

d $V(\underline{t}) < \infty$

e $V''(\underline{t}) < \infty$. The following lemmas follow in the same way as their Section Two and Three counterparts:

Lemma 6.1

For some strictly positive K not depending on n, $E[\,|V_n^L|\,] \geq K\,2^{3n/2}2^{-x(d,M)n}$.

Lemma 6.2

There exists a finite K' not depending on n so that for $\underline{t}$ and $\underline{s}$, elements of D_n^L with $|\underline{t}-\underline{s}|_{\max} = \max\{\,|t_1-s_1|, |t_2-s_2|\} \in [2^{-i}, 2^{-i+1})$ and $|\underline{t}-\underline{s}|_{\min} = \min\{\,|t_1-s_1|, |t_2-s_2|\} \in [2^{-j}, 2^{-j+1})$,

$$P[t \text{ and } s \in V_n^L] \le K' 2^{-n/2} 2^{-x(d,M)n} 2^{-(n-i)/2} 2^{-(n-j)x(d,M)}$$

We are now in a position to prove Theorem Two.

Proof of Theorem Two

We first estimate $E[|V_n^L|^2]$.

This quantity is equal to

$$\sum_{s,t \in D_n^L} \sum_{i=0}^{n} \sum_{j=i}^{n} \sum_{\substack{|t-s|_{\min} \in [2^{-j}, 2^{-j+1}) \\ |t-s|_{\max} \in [2^{-i}, 2^{-i+1})}} P[s, t \in V_n^L]$$

By Lemma 6.2, the above is bounded by

$$H 2^{2n} \sum_{i=0}^{n} 2^{(n-i)} \sum_{j=i}^{n} 2^{(n-j)} 2^{-n/2} 2^{-x(d,M)n} 2^{-(n-i)/2} 2^{-x(d,M)(n-j)},$$

which equals

$$H 2^{+3n} 2^{-2x(d,M)n} \sum_{i=0}^{n} 2^{-i} 2^{i/2} \sum_{j=i}^{n} 2^{-j} 2^{x(d,M)j}. < H' 2^{3n} 2^{-nx(d,M)}.$$

We conclude that $E[|V_n^L|^2] < K(E[|V_n^L|])^2$ for some finite constant K. Therefore, as before, it follows that there is a constant c, not depending on n so that $P[|V_n^L| > 0] >$ c for all n. Therefore, with probability at least c, the set V_n^L must be non-empty for infinitely many n. If $t \in V_n^L$, then $t + (2^{-n},0)$ must belong to a component of $\{W > 1\}$ which intersects the line segment $\{L+2\} \times [L, L+2]$ at points where W is greater than 1+1/2, similarly $t - (2^{-n},0)$ must belong to a component of $\{W < 1\}$ which intersects the line segment $\{\frac{L^2}{L+2}\} \times [\frac{L^2}{L+2}, L]$ at points where W is less than 1/2. It follows that (with positive probability) the boundaries of these two sets of finite components are not disjoint. Hence with positive probability there exist in $[L, L+1]^2$ points which are on the boundary of both positive nad negative components.

□

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