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KENNETH DAVID ELWORTHY
MARC YOR

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CONDITIONAL EXPECTATIONS FOR DERIVATIVES OF CERTAIN STOCHASTIC FLOWS

K.D. Elworthy[(1)] *and M. Yor*[(2)]

(1) *Mathematics Institute - University of Warwick - Coventry CV4 7 AL UK*

(2) *Laboratoire de Probabilités - Université Paris VI - 4, place Jussieu - Tour 56 - 3ème Etage - 75252 PARIS CEDEX 05*

1. Introduction.

Consider a stochastic differential equation

$$dx_t = X(x_t)\circ dB_t + A(x_t)dt \tag{1}$$

on an n-dimensional C^∞ manifold M. Here $\{B_t : t \geq 0\}$ is a Brownian motion on some Euclidean space $\mathbb{R}^m$ and for each x in M, $X(x) : \mathbb{R}^m \longrightarrow T_xM$ is a linear map into the tangent space at x to M. Both X and the vector field A are assumed C^∞.

Given a solution $\{x_t : t \geq 0\}$ to (1), assumed to exist for all time, and $v_o \in T_{x_o}M$, there is a derivative process $\{v_t : t \geq 0\}$ with $v_t \in T_{x_t}M$. This can be obtained by differentiating the solutions of (1), with respect to their initial point, in the direction v_o. It can be given as the solution of a certain S.D.E. on the tangent bundle TM ([E1] or [E3]) or more concisely by the covariant equation

$$Dv_t = \nabla X(v_t)\circ dB_t + \nabla A(v_t)dt \tag{2}$$

using an affine connection on M, where

$$\nabla A \in L(TM\ ;\ TM) \quad \text{and} \quad \nabla X \in \mathbb{L}(TM\ ;\ \mathbb{L}(\underline{\mathbb{R}}^m\ ;\ TM))$$

are the covariant derivatives, with $\underline{\mathbb{R}}^m$ the trivial bundle $M \times \mathbb{R}^m$ over M. Recall that (2) is to be interpreted as the corresponding Stratonovich equation for $T_{x_o}M$-valued processes obtained by parallel translation back along

$\{x_t, t \geq 0\}$; compare *(3)* below.

This derivative process $\{v_t, t \geq 0\}$ plays a fundamental role in the ergodic theory of solution flows of stochastic differential equations, in particular in the definition of Lyapunov exponents and so in related questions of stability e.g. see [E3]. It also contains geometrical and topological information : see [K] and [E2]. Here we consider the conditional expectation

$$\mathbb{E}\{v_t | \mathcal{F}_t^{x^-}\}$$

of v_t with respect to the σ-algebra $\mathcal{F}_t^{x^-} := \sigma\{x_s : 0 \leq s \leq t\}$.

By definition, this will be another process over $\{x_t : t \geq 0\}$ i.e.

$$\mathbb{E}\{v_t | \mathcal{F}_t^{x^-}\} \in T_{x_t} M$$

given by

$$\mathbb{E}\{v_t | \mathcal{F}_t^{x^-}\} = //_t \, \mathbb{E}\{//_t^{-1} v_t | \mathcal{F}_t^{x^-}\} \tag{3}$$

where $//_t : T_{x_o} M \longrightarrow T_{x_t} M$ denotes parallel translation along $\{x_t : t \geq 0\}$.

<u>Note</u> : As Michel Emery pointed out, the definition (3) does not depend on the choice of the connection. Indeed, the "difference" $//_t \, \backslash\backslash_t^{-1}$ between two parallel transports is a linear operation from $T_{x_t} M$ into itself, which is measurable with respect to $\mathcal{F}_t^{x^-}$; hence, it commutes with conditional expectations. □

We will also consider analogously defined conditional expectations for certain processes of vectors. Our main result is that for gradient Brownian systems with drift, this conditional expectation gives the Hessian flow, see [E2], or the Weitzenböck flow for q-vectors. In a corollary, the results in [E2] on topological obstructions to moment stability for gradient systems are considerably strengthened. We identify the conditional distribution for 1-dimensional Brownian flows. We also give more limited results for more general Brownian systems on Ricci flat, constant curvature, and other Riemannian manifolds.

2. Preliminaries.

A - Suppose M is Riemannian and compact for simplicity. We will use its Levi-Civita connection. Suppose that the differential generator for (1) is $\frac{1}{2}\Delta + A^X$ where the vector field A^X will be given by

$$A^X(x) = \frac{1}{2} \text{ trace } \nabla X(X(x)(-))(-) + A(x) \qquad (x \in M) \tag{4}$$

This holds if and only if

$$X(x)\, X(x)^* v = v \qquad v \in T_x M \tag{5}$$

from which follows

$$\nabla X(w)\, X(x)^*(v) + X(x)\, \nabla X(w)^* v = 0 \tag{6}$$

for all v,w in $T_x M$.

Let $\pi : OM \longrightarrow M$ be the orthonormal frame bundle of M, so if $u \in OM$ with $\pi(u) = x$, then $u : \mathbb{R}^n \longrightarrow T_x M$ is an isometry. Given $u_o \in \pi^{-1}(x_o)$, a solution $\{x_t : t \geq 0\}$ to (1) has a horizontal lift $\{u_t : t \geq 0\}$ starting at u_o. Then, $\pi(u_t) = x_t$ and $u_t \in \mathcal{F}_t^{\bar{x}}$ so that, for $t \geq 0$,

$$\mathcal{F}_t^{\bar{u}} = \mathcal{F}_t^{\bar{x}} . \tag{7}$$

Define a Brownian motion on $\mathbb{R}^n$ by $\tilde{B}_t = \int_0^t u_s^{-1}\, X(x_s) dB_s$ (8)

The following is fairly well known

Lemma : For $t \geq 0$, $\qquad \mathcal{F}_t^{\bar{x}} = \mathcal{F}_t^{\bar{\tilde{B}}} = \mathcal{F}_t^{\bar{u}}$

Proof : Clearly $\mathcal{F}_t^{\bar{\tilde{B}}} \subset \mathcal{F}_t^{\bar{u}}$. On the other hand, let

$$H_u : T_x M \longrightarrow T_u OM \qquad u \in \pi^{-1}(x)$$

be the horizontal lift map for the Levi-Civita connection. Then

$$du_t = H_{u_t}(X(x_t) \circ dB_t + A^X(x_t) dt)$$

and so

$$du_t = H_{u_t}(u_t \circ d\tilde{B}_t) + \tilde{A}^X(u_t) dt \tag{9}$$

(the canonical equation on OM) with $\tilde{A}^X$ the horizontal lift of A^X.

Thus $\mathcal{F}_t^{u} \subset \mathcal{F}_t^{\tilde{B}}$. □

B - For an orthonormal base $e_1, \dots, e_m$ of $\mathbb{R}^m$, let X^i be the vector field $X(\cdot)e_i$ and let $S_r^i : M \longrightarrow M$, $r \in \mathbb{R}$ be its solution flow. This has derivative flow $TS_r^i : TM \longrightarrow TM$ which induces $\Lambda^q TS_r^i : \Lambda^q TM \longrightarrow \Lambda^q TM$, linear on fibres and determined by

$$\Lambda^q TS_r^i(v^1 \wedge \dots \wedge v^q) = TS_r^i(v^1) \wedge \dots \wedge TS_r^i(v^q)$$

for a q-vector $v^1 \wedge \dots v^q$ in $\Lambda^q T_x M$.

Define $$Q_x^q : \Lambda^q T_x M \longrightarrow \Lambda^q T_x M \qquad (x \in M,\ q = 0 \text{ to } n)$$

by $$Q_x^q(V) = \sum_{i=1}^{m} \frac{D^2}{\partial r^2} \Lambda^q TS_r^i(V)\Big|_{r=0} \qquad (10)$$

When (1) is a gradient Brownian system with drift, Q^q depends only on the curvature of M. Such systems are defined by an isometric immersion

$$f : M \longrightarrow \mathbb{R}^m$$

(for example, the standard inclusion of the sphere S^n into $\mathbb{R}^{n+1}$). The diffusion coefficient $X(x)$ is defined to be the orthogonal projection of $\mathbb{R}^m$ onto the tangent space at x to M (considered as a subset of $\mathbb{R}^m$ by using $T_x f$ as an identification). Thus if $f^i(x) = \langle f(x), e_i \rangle_{\mathbb{R}^m}$, then

$$X^i = \nabla f^i .$$

In this case, (1) has generator $\frac{1}{2}\Delta + A$.

As an example, for the standard inclusion of S^1 in $\mathbb{R}^2$, the corresponding equation (1) can be written

$$dx_t = (\sin x_t)dB_t^1 + (\cos x_t)dB_t^2$$

for $B_t = (B_t^1, B_t^2)$ Brownian motion on $\mathbb{R}^2$, parametrizing S^1 by angle as usual. Then (2) is

$$dv_t = (\cos x_t)v_t\, dB_t^1 - (\sin x_t)v_t\, dB_t^2 .$$

It makes no difference whether these are considered as Itô or Stratonovich equations.

Proposition [E2] : Let $\mathcal{R}^q_x : \Lambda^q T^*_x M \longrightarrow \Lambda^q T^*_x M$, $x \in M$ be the Weitzenböck curvature tensor for $q = 0,\ldots,n$. Then for a gradient Brownian system with drift

$$Q^q_x = -(\mathcal{R}^q_x)^*.$$

In particular $\langle Q^1_x(u),v\rangle_x = -\mathrm{Ric}(u,v) \qquad u,v \in T_x M$

for $\mathrm{Ric}(-,-)$ the Ricci tensor. For general Brownian systems,

$$\langle Q^1_x(u),v\rangle_x = -\mathrm{Ric}(u,v) + \langle\sum_i \nabla(\nabla X^i(X^i))(u),v\rangle_x .$$

The Weitzenböck curvature arises in the Weitzenböck formula :

$$\Delta^q\varphi = \mathrm{trace}\ \nabla^2\varphi - \mathcal{R}^q(\varphi) \tag{11}$$

where $\Delta^q = -(dd^* + d^*d)$ is the Hodge Laplacian (with probabilist's sign convention) and $\mathcal{R}^q$ is the zero order operator on q-forms :

$$(\mathcal{R}^q(\varphi))_x = \mathcal{R}^q_x(\varphi_x), \qquad \text{see [E3], [G].}$$

C - Let $\{F_t : t \geq 0\}$ be a solution flow for (1). It can be chosen to consist of C^∞ diffeomorphisms of M and, in particular, has derivative flow $TF_t : TM \longrightarrow TM$ with $v_t = TF_t(v_o)$ satisfying (2) for $v_o \in T_{x_o}M$. As for the deterministic flows S^i_t , there are induced processes $\Lambda^q(TF_t)(V_o) \in \Lambda^q T_{x_t}M$ for $V_o \in \Lambda^q T_{x_o}M$.

Set $V_t = \Lambda^q(TF_t)(V_o)$ and for $\Psi \in \Lambda^q\mathbb{R}^m$, set $\Psi_o = \Lambda^q(u_o)(\Psi)$ and $\Psi_t = \Lambda^q(u_t)(\Psi) \in \Lambda^q T_{x_t}M$, where $\{u_t : t \geq 0\}$ is as in § 2A. There are then the covariant equations along $\{x_t : t \geq 0\}$:

$$D\Psi_t = 0 \tag{12}$$

and

$$DV_t = (d\Lambda)^q\,(\nabla X(\cdot)\circ dB_t)V_t + (d\Lambda)^q\,(\nabla A^X(\cdot))V_t\,dt \tag{13}$$

where, for any linear $S : E \longrightarrow E$ of a vector space, $(d\Lambda)^q(S) : \Lambda^q E \longrightarrow \Lambda^q(E)$ is the linear map determined by

$$(d\Lambda)^q(S)\ (v^1 \wedge \dots \wedge v^q) = \sum_{j=1}^{q} v^1 \wedge \dots v^{j-1} \wedge Sv^j \wedge v^{j+1} \wedge \dots \wedge v^q.$$

By Itô's formula, e.g. [E3], Prop. I.3A,

$$<\Psi_t, V_t>_{x_t} = <\Psi_o, V_o>_{x_o} + \int_0^t <\psi_s, (d\Lambda)^q(\nabla X(\cdot)dB_s)V_s>_{x_s}$$

$$+ \int_0^t <\Psi_s, \frac{1}{2} Q^q_{x_s}(V_s) + (d\Lambda)^q\ (\nabla A(\cdot))V_s>_{x_s} ds \tag{14}$$

by (10) and (13).

D - To calculate conditional expectations, take $\Phi \in L^2(\Omega, \mathcal{F}^{\tilde{B}}_t, \mathbb{P}\ ;\ \mathbb{R})$, for $(\Omega, \mathcal{F}, \mathbb{P})$ our underlying probability space. There is then an $\mathcal{F}^{\tilde{B}}_{\cdot}$-predictable

$$\varphi : [0,t] \times \Omega \longrightarrow \mathbb{R}^n$$

with $\varphi_s := \varphi(s,-)$ in L^2 for each s, and

$$\Phi = \mathbb{E}(\Phi) + \int_0^t <\varphi_s, d\tilde{B}_s>_{\mathbb{R}^n}. \tag{15}$$

From (14),

$$\mathbb{E}\left\{\Phi<V_t,\Psi_t>_{x_t}\right\} = \mathbb{E}\left\{\Phi\left(<V_o,\Psi_o>_{x_o} + \int_0^t <\Psi_s, \frac{1}{2} Q^q_{x_s}(V_s) + (d\Lambda)^q(\nabla A(\cdot)V_s)>ds\right)\right\} + \Lambda_t \tag{16}$$

where $\Lambda_t = \mathbb{E}\left\{\Phi \int_0^t <\Psi_s, (d\Lambda)^q(\nabla X(\cdot)dB_s)V_s>_{x_s}\right\}$

$$= \mathbb{E}\left\{\int_0^t <\varphi_s, d\tilde{B}_s> \int_0^t <\Psi_s, (d\Lambda)^q\ (\nabla X(\cdot)dB_s)V_s>_{x_s}\right\}$$

$$= \mathbb{E}\left\{\int_0^t <u_s\varphi_s, X(x_s)dB_s> \int_0^t <\Psi_s, (d\Lambda)^q\ (\nabla X(\cdot)dB_s)V_s>_{x_s}\right\}$$

$$= \sum_{i=1}^{m} \mathbb{E}\left\{\int_0^t <u_s\varphi_s, X^i(x_s)><\Psi_s, (d\Lambda)^q(\nabla X^i(\cdot))V_s>_{x_s} ds\right\} \tag{17}$$

using (8).

§ 3. Main results :

A - Theorem A : For a gradient Brownian system with drift A on a compact Riemannian manifold M if $V_o \in \Lambda^q T_{x_o} M$ with $q = 0,\dots,n$,

$$E\{\Lambda^q TF_t(V_o) | \mathcal{F}_t^{x}\} = W_t^q(V_o)$$

where $\{W_t^q(V_o) : t \geq 0\}$ satisfies the equation along $\{x_t : t \geq 0\}$

$$\frac{D}{\partial t} W_t^q(V_o) = -\frac{1}{2} (\mathcal{R}^q_{x_t})^* W_t^q(V_o) + (d\Lambda^q)(\nabla A)(W_t^q(V_o)) \tag{18}$$

$$W_o^q(V_o) = 0$$

for $\mathcal{R}^q$ the Weitzenböck curvature.

Proof : For a gradient system at each point x of M, an orthonormal basis for $\mathbb{R}^m$ can be chosen so that either $X^i(x) = 0$ or $\nabla X^i(x) = 0$, see [E3]. From (17), this implies that $\Lambda_t = 0$ so that (16) together with the Proposition yields

$$E\left\{\Phi\langle V_t,\Psi_t\rangle_{x_t}\right\} = E\left\{\Phi\left(\langle V_o,\Psi_o\rangle_{x_o} + \int_0^t \langle\Psi_s,\frac{D}{\partial s} W_s^q(V_o)\rangle_{x_s} ds\right)\right\}$$

$$= E\left\{\Phi(\langle V_o,\Psi_o\rangle + \langle\Psi_t,W_t^q(V_o)\rangle_{x_t})\right\}.$$

The Theorem follows since $\{\Psi_t : t \geq 0\}$ was an arbitrary parallel field of q-vectors along $\{x_t : t \geq 0\}$. □

B - Theorem : Suppose M is compact and (1) is a Brownian system with drift satisfying $\nabla A^X = 0$. Assume there is a constant σ with

$$Ric(u,v) = \sigma\langle u,v\rangle_x \qquad u,v \in T_x M,\ x \in M$$

(i.e. M is an Einstein manifold, and so has constant curvature if $\dim M \leq 3$). Then, for $V_o \in T_{x_o} M$ and $\Psi_t = u_t\Psi$ as above with $q = 1$

$$E\{\langle\Psi_t,V_t\rangle_{x_t} \mid \langle\Psi,\tilde{B}_s\rangle : 0 \leq s \leq t\} = e^{-1/2\,\sigma t} \langle\Psi_o,V_o\rangle_{x_o}.$$

Proof : Apply the same proof as for Theorem A but now with Φ of the special form

$$\Phi = \mathbb{E}\,\Phi + \int_0^t \varphi_s^o <\Psi, d\tilde{B}_s>$$

for some predictable $\varphi_s^o : [0,t] \times \Omega \longrightarrow \mathbb{R}$. Then, by (17)

$$\Lambda_t = \mathbb{E}\left(\int_0^t \varphi_s^o <\Psi_s\,,\ \nabla X(V_s)X(x_s)^* \ \Psi_s>ds\right) = 0$$

by the skew symmetry (6). By (16),

$$\mathbb{E}(\Phi<V_t,\Psi_t>_{x_t}) = \mathbb{E}\left\{\Phi\left(<V_o,\Psi_o>_{x_o} - \int_0^t \frac{1}{2}\,\sigma<\Psi_s,V_s>ds\right)\right\}$$

Corollary : For M and $\{V_t : t \geq 0\}$ as above

$$\mathbb{E}\ //_t^{-1}\ V_t = e^{-1/2\ \sigma t}\ V_o$$

Remark : The Corollary can also be easily seen from the Itô form of (2) for an arbitrary Brownian motion system with drift :

$$Dv_t = \nabla X(v_t)dB_t + \nabla A^X(v_t)dt - \frac{1}{2}\,\mathrm{Ric}(v_t,-)^{\#} \qquad (19)$$

Here, $\mathrm{Ric}(v_t,-)^{\#} : T_{x_t}M \longrightarrow T_{x_t}M$ corresponds to the Ricci tensor and, as usual, the equation refers to the Itô equation obtained after parallel translation back to x_o. It comes from the Proposition in paragraph 2B.

C - For $M = \mathbb{R}$ or S^1, it is possible to identify the conditional distribution of v_t given $\{x_s : s \geq 0\}$ when $\{x_s : s \geq 0\}$ is a Brownian motion. In this case, for $M = \mathbb{R}$, equation (2) reduces to

$$dv_t = X'(x_t)dB_t$$

giving

$$v_t = \exp\left(\int_0^t X'(x_s)dB_s - \frac{1}{2}\int_0^t |X'(x_s)|^2 ds\right)v_o.$$

Using (6), there is therefore a 1-dimensional Brownian motion $\{\xi_s : s \geq 0\}$ independent of $\{x_s : s \geq 0\}$ with

$$v_t = \exp\left(\int_0^t |X'(x_s)|d\xi_s - \frac{1}{2}\int_0^t |X'(x_s)|^2 ds\right) v_o$$

$$= \exp(\gamma_u - \frac{u}{2})v_o, \quad \text{at} \quad u = \int_0^t |X'(x_s)|^2 ds,$$

for a Brownian motion $\{\gamma_u : u \geq 0\}$ on $\mathbb{R}$ independent of $\{x_s : s \geq 0\}$. Thus, conditionally on $\{x_s : s \geq 0\}$, the derivative $F_t'(x_o)$ has the distribution of $\exp(\gamma_u - \frac{1}{2} u)v_o$ with $u = \int_0^t |X'(x_s)|^2 ds.$

4 - Topological and geometric obstructions to moment stability.

For M compact Riemannian, define the moment exponents $\mu_{x_o}(p)$, $x_o \in M$, $p \in \mathbb{R}$, by

$$\mu_{x_o}(p) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}(\| T_{x_o} F_t \|^p)$$

and write

$$\mu_{x_o}^q(1) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}(\| \Lambda^q T_{x_o} F_t \|) \qquad q = 1,\dots,m.$$

Then, from [A] or [E3], μ_{x_o} is convex and $p \longrightarrow \frac{1}{p}\mu_{x_o}(p)$ is increasing, with $\mu_{x_o}(0) = 0$. Clearly (with suitable choice of norms) :

$$\mu_{x_o}^q(1) \leq \mu_{x_o}(q) \qquad q = 1 \text{ to } m.$$

On the other hand, following [ERI] and [ERII], define $\mathcal{R}^q(x_o)'$ to be the lowest eigenvalue of the Weitzenböck tensor $\mathcal{R}^q$ at x_o. (Thus, $\mathcal{R}^1(x_o)'$ is a lower bound for the Ricci tensor at x_o). Set

$$\nu^q(x_o) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\left[\exp\left(-\int_0^t \mathcal{R}^q(x_s)' ds\right)\right]$$

for $\{x_s : 0 \leq s \leq t)$ a Brownian motion on M starting from x_o.

In [ERI], [ERII], there were shown to be strong topological consequences of having $\nu^q < 0$.

In fact, these were consequences of $\mu^R_{x_o}(q) < 0$ for each x_o in M where

$$\mu^R_{x_o}(q) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\|W^q_t\|_{x_o}$$

for $\{W^q_t : t \ge 0\}$ given by (18) with $A = 0$. Using Theorem A, for a gradient Brownian system on compact M

$$\mu^R_{x_o}(q) \le \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\{\mathbb{E}\{\|\Lambda^q T_{x_o} F_t\| \,|\, \mathcal{F}^x_t\}\} = \mu^q_{x_o}(1) \le \mu_{x_o}(q). \qquad (20)$$

Thus, the results of [ERI], [ERII] are implied by stability conditions such as $\mu_{x_o}(q) < 0$ or $\mu^q_{x_o}(1) < 0$. For example :

<u>Theorem</u> : <u>For a gradient Brownian flow on a compact manifold</u> M

(i) <u>if</u> $\mu_{x_o}(1) < 0$ <u>for each</u> $x_o \in M$ ("<u>moment stability</u>"), <u>then</u> $H^1(M,Z) = 0$. <u>If also</u> dim M = 3, <u>then</u> $\pi_2 M = 0$.

(ii) <u>if</u> $\mu^2_{x_o}(1) < 0$, <u>for each</u> $x_o \in M$, <u>then</u> $\pi_2 M$ <u>is a torsion group and the orders of the elements of</u> $\pi_2 M$ <u>are bounded</u>. <u>If</u> dim M = 4 <u>and</u> $\pi_1 M = 0$, <u>then</u> $\mu^2_{x_o}(1) < 0$ <u>implies that</u> M <u>is diffeomorphic to the sphere</u> S^4.

Proof : Part (i) comes from (20) and the proof of Corollary 5A of [ERI] and (ii) comes from (20), and the proof of Corollary 3.23 of [ERII]. □

The first part of (i) is proved in [E2] for more general systems. It should also be noted that the main results of [ERII] are concerned with the universal cover of M when $\pi_1 M$ is infinite. However, as pointed out in [E4], if

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \sup_{x_o \in M} \log \mathbb{E}|T_{x_o} F_t| < 0$$

("strong moment stability") then $\pi_1 M = 0$, for any stochastic flow on a compact M, (for the non-compact case, see [L]).

For more general systems, we can use (19) to obtain

Theorem : Suppose M is a compact Riemannian manifold, and (1) is a Brownian system with drift satisfying $\nabla A^X = 0$. If the Ricci curvature is non positive, the flow is not moment stable, and if the curvature is negative, then $\mu_{x_o}(1) > 0$.

Appendix : Consider a general non-degenerate stochastic differential equation of the form (1) and give M the Riemannian metric and associated connection so that the generator is $\frac{1}{2}\Delta + \underline{A}^X$ as in § 2 above.

For $x \in M$, the adjoint of $X(x)$ gives an isometric inclusion

$$X(x)^* : T_xM \longrightarrow \mathbb{R}^m ;$$

write $T_xM^\perp$ for the orthogonal complement of its image (i.e : the kernel of $X(x)$) and let $Y(x)$ be the projection of $\mathbb{R}^m$ onto $T_xM^\perp$, so that :

$$Y(x)e = e - X(x)^* X(x)e.$$

Let $TM^\perp$ be the subbundle of $\underline{\mathbb{R}}^m$ with fibres $T_xM^\perp$, and give it the Riemannian metric induced from the standard, trivial, metric of $\mathbb{R}^m$. Take any metric connection on $TM^\perp$. We will use $//_t$ to denote parallel translation of the normal space $T_{x_o}M^\perp$ along $\{x_s ; 0 \le s \le t\}$ to $T_{x_t}M^\perp$, as well as parallel translation of tangent vectors. Identifying $T_{x_t}M$ with the corresponding subspace of $\mathbb{R}^m$, we obtain $//_t : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $t \ge 0$.

For $\tilde{B}_t$ defined by (8), set $\hat{B}_t = X(x_0)^* u_0 \tilde{B}_t$. Consider the $T_{x_o}M^\perp$ valued process $\{\beta_t ; t \ge 0\}$ defined by :

$$\beta_t = \int_0^t //_s^{-1} Y(x_s)dB_s \tag{21}$$

and set $\bar{B}_t = \hat{B}_t + \beta_t \qquad (t \ge 0)$.

The following generalizes a result of Price and Williams [PW] on S^2.

Proposition : *The process* $\{\bar{B}_t\ ;\ t \geq 0\}$ *is Brownian motion on* $\mathbb{R}^m$*, with :*

$$B_t = \int_0^t //_s d\bar{B}_s \tag{22}$$

In particular, $\{\beta_t\ ;\ t \geq 0\}$ *is a Brownian motion independent of* $\{x_t\ ;\ t \geq 0\}$.

Proof : That (22) holds is clear by definition of β_t, and (8), using the fact that $u_s = //_s u_0$ and $//_t^{-1} X(x_t)^* //_t = X(x_0)^*$. However, (22) gives :

$$\bar{B}_t = \int_0^t //_s^{-1}\, dB_s\ ,$$

showing that $\bar{B}_.$ is a BM ($\mathbb{R}^m$), since each $//_s$ is orthogonal.

The final result follows since $\sigma\{x_t\ ;\ t \geq 0\} = \sigma\{\tilde{B}_t\ ;\ t \geq 0\}$. □

For the standard embedding of the sphere S^n in $\mathbb{R}^{n+1}$ with corresponding gradient Brownian system, we can now identify the conditional distribution of v_t given $\{x_s, s \geq 0\}$. Indeed, in this case, (19) reduces to

$$Dv_t = -v_t\, d\beta_t^1 - \frac{1}{2}(n-1)v_t dt, \text{ where : } \beta_t^1 = \int_0^t \langle x_s, dB_s\rangle$$

giving :
$$v_t = e^{-\beta_t^1 - \frac{1}{2}nt}\, //_t v_o\ .$$

This is because ∇X is essentially the shape operator for the submanifold and so, for S^n in $\mathbb{R}^{n+1}$:

$$\nabla X(v)e = -\langle x,e\rangle v, \qquad v \in T_xM,\ e \in \mathbb{R}^m$$

e.g. see [E3].

Now, $\{\beta_t^1\ ;\ t \geq 0\}$ is a 1-dimensional Brownian motion and, as for β_t above, we see that it is independent of $\mathcal{F}_\infty^{x}$. Thus, $//_t^{-1} v_t$ is independent of $\mathcal{F}_\infty^{x}$. □

Comment : So, this turns out to be rather uninteresting. However, the more general case for a gradient Brownian system :

$$Dv_t = A_{x_t}(v_t, //_t \, d\beta_t) - \frac{1}{2} \mathrm{Ric}^{\#}_{x_t}(v_t)dt$$

where $A_x : T_xM \times (T_xM)^{\perp} \longrightarrow T_xM$ looks difficult to treat.

Here, A is the shape operator :

$$\langle A_x(v_1,z),v_2\rangle = \langle \alpha(v_1,v_2),z\rangle$$

for α the second fundamental form, and

$$\nabla X(v)(e) = A(v,Y(\bar{x})e) \qquad v \in T_xM,$$

so :

$$\nabla X(v_s)dB_s = A(v_s,//_s \, d\beta_s)$$

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