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# AN

# OPERATOR THEORETIC APPROACH

# TO

# STOCHASTIC FLOWS ON MANIFOLDS

by

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**Abstract**

Operator theoretic methods are used to construct stochastic flows of diffeomorphisms on smooth manifolds as solutions of stochastic differential equations driven by a single Brownian motion or a Poisson process. Our only assumption is that the infinitesimal motion of the flow is described by a complete smooth vector field.

## 1. Introduction

In this paper we will be concerned with the construction of stochastic flows of diffeomorphisms on manifolds by means of solving stochastic differential equations. Although a great deal of work has been done in this area (see e.g. [IkWa], [Kun 1, 2] and references therein), we feel that the theory has two major deficiencies,

(a) Insufficient global analytic insight into its structure,

(b) An overemphasis on path–continuous flows which are driven by Brownian motion.

In this paper, we aim to take some first steps towards remedying both of these.

As regards (a) we observe that in the case of deterministic flows there are three perspectives from which the flow can be investigated which, for want of better words, we will call topological, analytic and algebraic (respectively). Topologically, the flow is given as a two–parameter family $\Phi = \{\Phi_{s,t}\, ;\, s, t \in \mathbb{R}, t \geq s\}$ of diffeomorphisms of the manifold M.

Analytically, we realise the flow as unitary operators $U = \{U_{s,t}\, ;\, s, t \in \mathbb{R}, t \geq s\}$ on the intrinsic Hilbert space $h_0$ of the manifold by the prescription

$$U_{s,t}\,\psi = \psi \circ \Phi_{s,t} \qquad \text{... (1.1)}$$

for $\psi \in h_0$. Algebraically we obtain automorphisms $J = \{J_{s,t}\, ;\, s, t \in \mathbb{R}, t \geq s\}$ of the algebra $C^{\infty}(M)$ of smooth functions on the manifold by

$$J_{s,t}\,f = f \circ \Phi_{s,t} \qquad \text{... (1.2)}$$

for $f \in C^{\infty}(M)$ (for details see [AMR]). The analytic and algebraic perspectives are further related by the formula (for bounded f),

$$J_{s,t}(f) = U_{s,t}\, f\, U^{-1}_{s,t} \qquad \text{... (1.3)}$$

If the flow consists of the family of integral curves obtained by solving a differential equation on M, we obtain also differential equations for U and J (see §2 below).

When we come to look at the stochastic case, it turns out that exactly the same scenario holds true as is described above only now all three of $\Phi$, U and J satisfy appropriate stochastic differential equations. In fact the essence of our approach is to perturb the above procedure by constructing U first and obtaining J and then $\Phi$ from it. In particular we are then able to show that in the simple case where the flow is driven by a single Brownian motion without any drift that $\Phi$ consists of diffeomorphisms under the sole (global) assumption that the vector field which describes its infinitesimal behaviour is complete.

We do not here discuss the more general case of multidimensional noise with drift where we expect the analysis to be more complicated due to the possibility of explosions.

We note that although our work is entirely classical, many of the formulae we obtain will come as no surprise to devotees of quantum probability where the analytic and algebraic perspectives reign supreme. In particular the infinitesimal expression for U will be familiar to readers of [HuPa] and that of J to readers of [Hud] and [App 1]. Furthermore we observe that J is essentially a quantum stochastic process in the sense of [AFL]. Before leaving the subject of quantum probability we remark that in §3 we give an algebraic characterisation of a stochastic flow of diffeomorphisms which when appropriately generalised may be of use in the study of stochastic flows on "non–commutative manifolds".

Turning our attention now to (b), we find that the form of U is such that it is natural to replace the incidence of Brownian motion therein by a Poisson process (c.f. again [HuPa]). When we do this, the procedure outlined above leads us to flows of diffeomorphisms of M which are no longer path continuous and which arise as the solutions of stochastic differential equations driven by a Poisson process. A specific example is described in local co–ordinates at the end of §5.

This present work has arisen out of a series of papers by the author on a quantum probabilistic generalisation of the concept of stochastic parallel transport (see [App 1] and references therein). The transition from the viewpoint of [App 1] to that employed here (i.e. the use of vector fields rather than covariant derivatives) is inspired by the discussion given in [Mey 2].

**Notation**

All Hilbert spaces will be complex however the algebras $C^\infty(M)$ (and $C^\infty_K(M)$) always comprise real smooth functions (of compact support) on the manifold M.

If $\mathscr{D}_i$ are dense subspaces of Hilbert spaces $h_i (i = 1, 2)$, we denote their algebraic tensor product by $\mathscr{D}_1 \underline{\otimes} \mathscr{D}_2$ and note that it is dense in $h_1 \otimes h_2$. A densely defined, linear, closeable operator T on a Hilbert space $h$ has domain $\mathscr{D}(T)$ and closure $\overline{T}$.
We will use the convention that inner products are conjugate–linear on the left.

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## 2. Deterministic Flows on Manifolds

Let M be a smooth finite–dimensional manifold. A <u>flow</u> on M is a family $\{\xi_{s,t} \,;\, s, t \in \mathbb{R}, t \geq s\}$ of diffeomorphisms of M which satisfy

| | | |
|---|---|---|
| F (i) | $\xi_{s,s}(x) = x$ | for all $s \in \mathbb{R}$, $x \in M$ |
| F (ii) | $\xi_{s,t} \circ \xi_{r,s} = \xi_{r,t}$ | for all $r < s < t$. |

A flow is said to be <u>autonomous</u> if F(iii) $\xi_{s,t}$ depends only on t–s.
Writing $\xi_t = \xi_{0,t}$, we see that for such a flow $\{\xi_t, t \in \mathbb{R}\}$ is a one–parameter group of diffeomorphisms of M with $\xi_t^{-1} = \xi_{-t}$ for all $t \in \mathbb{R}$.

Given a flow on M, we define for all $s,t \in \mathbb{R}$, with $t \geq s$ a family of automorphisms of $C^\infty(M)$ by the prescription

$$j_{s,t}(f) = f \circ \xi_{s,t} \qquad \dots (2.1)$$

for $f \in C^\infty(M)$. We note that each $j_{s,t}$ leaves the subalgebra $C^\infty_K(M)$ invariant. (For completeness, we prove this result in Appendix 1).

From F(i) and F(ii), we deduce

F′ (i) $\quad j_{s,s}(f) = f \quad$ for all $s \in \mathbb{R}$, $f \in C^\infty(M)$

F′(ii) $\quad j_{r,s} \circ j_{s,t} = j_{r,t} \quad$ for all $r < s < t$

We call $\{j_{s,t} : s,t \in \mathbb{R}, t \geq s\}$ a flow of automorphisms of $C^\infty(M)$

A flow of automorphisms is said to be autonomous if

F′(iii) $\quad j_{s,t}$ depends only on $t-s$.

We write $j_{0,t} = j_t$ for all $t \in \mathbb{R}$.

Autonomous flows of diffeomorphisms of M induce autonomous flows of automorphisms of $C^\infty(M)$ through the formula

$$j_t(f) = f \circ \xi_t \qquad \text{... (2.2)}$$

for all $t \in \mathbb{R}$, $f \in C^\infty(M)$.

In [AMR] pp. 230–3 it is shown that the correspondence between flows of diffeomorphisms of M and flows of automorphisms of $C^\infty(M)$ given by (2.2) is in fact one–to–one.

We recall the standard construction of autonomous flows of diffeomorphisms of M from solutions of differential equations. A smooth vector field Y on M is said to be complete if the initial value problem

$$\left.\begin{aligned} c'(t) &= Y(c(t)) \\ c(0) &= x \end{aligned}\right\} \qquad \text{... (2.3)}$$

has a unique solution for all $x \in M$ and all $t \in \mathbb{R}$. For each $x \in M$, let $\{c_x(t), t \in \mathbb{R}\}$ be the integral curve obtained by solving (2.3), then our flow is given by

$$\xi_t(x) = c_x(t) \qquad \text{... (2.4)}$$

We will now seek to obtain some functional analytic insight into the ideas we have described. To this end, we assume that M is oriented and let $\mu_v$ be a volume form on M. We denote by $\mu$ the unique Borel measure on M induced by $\mu_v$. Every half–density on M can be written in the form $h\,\mu_v^{\frac{1}{2}}$ where $h \in L^2(M, \mu)$ and

$$\mu_v^{\frac{1}{2}}(x)\,(Y_1(x), \ldots, Y_d(x)) = |\mu_v(x)\,(Y_1(x), \ldots, Y_d(x))|^{\frac{1}{2}} \qquad \text{... (2.5)}$$

for each $x \in M$ and $Y_j(x) \in T_x(M)$ $(1 \leq j \leq d)$ where d is the dimension of M. The half–densities on M form a Hilbert space, which we denote as $h_0$ with inner product

$$\langle h_1 \mu_v^{\frac{1}{2}}, h_2 \mu_v^{\frac{1}{2}} \rangle = \int \bar{h}_1 h_2 \, d\mu \qquad \text{... (2.6)}$$

for $h_1, h_2 \in L^2(M, \mu)$. The mapping $h \longrightarrow h\mu_v^{\frac{1}{2}}$ is thus a canonical isomorphism between $L^2(M, \mu)$. and $h_0$.

The divergence of a smooth vector field Y with respect to $\mu$ is the unique map $\mathrm{div}_\mu(Y) \in C^\infty(M)$ such that

$$L_Y(\mu_v) = \mathrm{div}_\mu(Y)\,\mu_v \qquad \dots (2.7)$$

where $L_Y$ denotes the Lie derivative.

In local co–ordinates, if $X(x) = a^j(x)\dfrac{\partial}{\partial x^j}$ and $\mu_v(x) = v(x)\,dx^1 \wedge \dots \wedge dx^d$ where $v(x) \neq 0$ for all $x \in M$ then

$$\mathrm{div}_\mu(Y) = v(x)^{-1}\sum_{j=1}^{d}\frac{\partial}{\partial x^j}(a^j(x)\,v(x)) \quad \dots (2.8)$$

Now let $\{\xi_t,\ t \in \mathbb{R}\}$ be the autonomous flow of diffeomorphisms of M defined by (2.4) and (2.3). We obtain a strongly continuous one–parameter group of unitary operators $\{U(t),\ t \in \mathbb{R}\}$ on $h_0$ by the prescription

$$U(t)\,(f\,\mu_v^{\frac{1}{2}})(x) = f(\xi_t(x))(\xi_t^*(\mu_v)(x))^{\frac{1}{2}} \quad \dots \quad (2.9)$$

for $x \in M$, where $\xi_t^*(\mu_v)$ is the pull back of $\mu_v$.

Let $\mathscr{D}_0$ be the dense subspace of $h_0$ comprising $\{h\,\mu_v^{\frac{1}{2}};\ h \in C_K^\infty(M)\}$, then it is shown in ([AMR] pp 435–40) that the infinitesimal generator of $\{U(t),\ t \in \mathbb{R}\}$ is the closure of the essentially self adjoint operator $-\,i\,T_Y$ on $\mathscr{D}_0$ where

$$T_Y = Y + \frac{1}{2}\,\mathrm{div}_\mu(Y) \qquad \dots (2.10)$$

so we may write

$$U(t) = e^{t\overline{T}_Y} \qquad \dots (2.11)$$

for all $t \in \mathbb{R}$.

Now recall the flow $\{j_t,\ t \in \mathbb{R}\}$ of automorphisms of $C^\infty(M)$ constructed in (2.2).

**Proposition 1**

For all $f \in C_K^\infty(M)$ and $t \in \mathbb{R}$ we have

$$j_t(f) = U(t)f\,U(t)^* \qquad \dots (2.12)$$

**Proof** Since U(t) is unitary for each $t \in \mathbb{R}$, any element of $h_0$ can be written in the form $U(t)\,h\,\mu_v^{\frac{1}{2}}$ for $h \in L^2(M, \mu)$ thus for $f \in C_K^\infty(M)$,

$$U(t)f\,U(t)^*\,U(t)h\,\mu_v^{\frac{1}{2}}(x) \quad = \quad U(t)f\,h\,\mu_v^{\frac{1}{2}}(x)$$

$$= f(\xi_t(x))h(\xi_t(x))(\xi_t^*(\mu_V)(x))^{\frac{1}{2}} \text{ by (2.9)}$$
$$= j_t(f)\ U(t)\ h\ \mu_V^{\frac{1}{2}}(x) \qquad \square$$

Using (2.11) and (2.12) we can write, for all $t \in \mathbb{R}$ and $f \in C_K^\infty(M)$

$$\begin{aligned} j_t(f) &= e^{t\overline{T}_Y} f e^{-t\overline{T}_Y} \\ &= e^{t[\overline{T}_Y, \cdot]} f \\ &= e^{tY} f \qquad \text{...(2.13)} \end{aligned}$$

since $[T_Y, f] = [Y, f] = Yf$.

We note that $\{U(t), t \in \mathbb{R}\}$ and $\{j_t, t \in \mathbb{R}\}$ satisfy the following analogues of the Schrödinger and Heisenberg equations of non–relativistic quantum theory

$$\frac{dU(t)}{dt} \psi = U(t)\ T_Y\ \psi \qquad \text{... (2.14)}$$

for all $\psi \in \mathscr{D}_0$

$$\frac{dj_t}{dt}(f) = j_t(Yf) \qquad \text{... (2.15)}$$

for all $f \in C_K^\infty(M)$.

These results have a partial converse in the Povzner–Nelson theorem which states that if the symmetric operator $T_Y$ is essentially skew adjoint on $\mathscr{D}_0$ then the flow $\{\xi_t, t \in \mathbb{R}\}$ given by (2.4) and (2.3) is defined for all but a possible set of initial conditions in M of $\mu$–measure zero.

A proof may be found in [AMR] pp 435–40.

**Stochastic Flows**

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with $\Omega$ also a Polish space. By a random variable on $\Omega$, taking values in M, we will mean an equivalence class X of measureable functions from $\Omega$ to M which agree almost everywhere with respect to P.

Let J be a unital *–homomorphism from $L^\infty(M, \mu)$ into $L^\infty(\Omega, \mathscr{F}, P)$ which is normal in the sense that given any net $(f_\alpha)\alpha \in I$ of positive, increasing elements of $L^\infty(M, \mu)$ for which $\sup_{\alpha \in I} f_\alpha \in L^\infty(M, \mu)$ then

$$J\left[\sup_{\alpha \in I} f_\alpha\right] = \sup_{\alpha \in I} J(f_\alpha) \qquad \text{... (3.1)}$$

Any such J will be called a <u>random morphism</u> associated to $(M, \Omega)$. It is shown in [Acc] (see also [Var]) that there is a one–to–one correspondence between random variables on $\Omega$ taking values in M and random morphisms associated to $(M, \Omega)$ given by the prescription

$$J(f) = f \circ x \qquad \dots (3.2)$$

for each $f \in L^\infty(M, \mu)$, where x is any representative of X.

By a stochastic process of morphisms associated to $(M, \Omega)$ we will mean a family $(J(t), t \in \mathbb{R}^+)$ of random morphisms. It follows from (3.2) that there is a one–to–one correspondence between such objects and stochastic processes $(X(t), t \in \mathbb{R}^+)$ of random variables on $\Omega$ taking values in M.

Now for each $s,t \in \mathbb{R}^+$ $t \geq s$, let $\phi_{s,t}$ be a measurable function from $M \times \Omega$ into M. We will call $\{\phi_{s,t} ; t \geq s\}$ a stochastic pre–flow. We will use the notation $\Phi_{s,t}$ to denote equivalence classes of such pre–flows which agree almost everywhere with respect to $\mu \times P$. By (3.2) again, we can associate to each $\{\Phi_{s,t} ; t \geq s\}$ a family $\{J_{s,t} ; t \geq s\}$ of unital, normal *–homomorphisms from $L^\infty(M, \mu)$ into $L^\infty(M \times \Omega, \mu \times P)$ for which

$$J_{s,t}(f) = f \circ \phi_{s,t} \qquad \dots (3.3)$$

for each $f \in L^\infty(M, \mu)$.

We will say that a stochastic pre–flow has the fixed time covering property if for each $s,t \in \mathbb{R}$ with $t \geq s$ there exists $A_{s,t} \subset M$ with $\mu(A_{s,t}) = 0$ such that

$$\{\phi_{s,t}(x,\omega); x \in M, \omega \in \Omega\} = M - A_{s,t} \quad \dots (3.4)$$

**Theorem 2** A stochastic pre–flow $\{\Phi_{s,t} ; t \geq s\}$ has the fixed time covering property if and only if $J_{s,t}$ is an isometric embedding of $L^\infty(M, \mu)$ into $L^\infty(M \times \Omega, \mu \times P)$ for each $s,t \in \mathbb{R}^+$, with $t \geq s$

**Proof.** (To avoid introducing further notational complexity, we have here taken the liberty of identifying functions with their equivalence classes).

(Necessity). Suppose that $\phi$ has the fixed time covering property then for each $s,t \in \mathbb{R}^+$, $t \geq s$, $f \in L^\infty(M, \mu)$

$$\begin{aligned} \|J_{s,t}(f)\| &= \inf\{K ; |J_{s,t}(f)(x, \omega)| \leq K \text{ for almost all } (x, \omega) \in M \times \Omega\} \\ &= \inf\{K ; |f(\phi_{s,t}(x, \omega))| \leq K \text{ for almost all } (x, \omega) \in M \times \Omega\} \\ &= \inf\{K ; |f(x)| \leq K \text{ for almost all } x \in M\} \\ &= \|f\| \end{aligned}$$

(Sufficiency). Suppose that for all $s,t \in \mathbb{R}^+$ with $t \geq s$ and for all $f \in L^\infty(M, \mu)$, $\|J_{s,t}(f)\| = \|f\|$ but that $\phi$ does not satisfy the fixed time covering property. Hence there exists $N_{s,t} \subset M$ with $\mu(N_{s,t}) > 0$ such that $\{\Phi_{s,t}(x, \omega) ; x \in M, \omega \in \Omega\} = M - N_{s,t}$

then we have $$J_{s,t}\left[\chi_{N_{s,t}}\right] = \chi_{N_{s,t}}(\phi_{s,t}) = 0$$

and we have obtained our desired contradiction. □

Now let $\phi$ be a stochastic preflow on M and define a map $\phi^{\omega}_{s,t} : M \longrightarrow M$ by

$$\phi^{\omega}_{s,t}(x) = \phi_{s,t}(x, \omega) \qquad ...(3.5)$$

for each $s,t \in \mathbb{R}^{+}$ $t \geq s$, $\omega \in \Omega$, $x \in M$.

We say that $\phi$ is a stochastic flow of diffeomorphisms of M (c.f. [Kun 1, 2]) if, for almost all $\omega \in \Omega$, each of the mappings $\phi^{\omega}_{s,t}$ is a diffeomorphism of M and the axioms F(i) and F(ii) of §1 are satisfied.

We will again use the notation $\Phi$ for equivalence classes of stochastic flows of diffeomorphisms of M which agree almost everywhere on $\Omega$ with respect to P.

By (2.1), for almost all $\omega \in \Omega$, we obtain a flow of automorphisms of $C^{\infty}(M)$, $\{j^{\omega}_{s,t} ; t \geq s\}$ by

$$j^{\omega}_{s,t}(f) = f \circ \phi^{\omega}_{s,t} \qquad ... (3.6)$$

We introduce the normal unital *–homomorphisms $\{E^{\omega}, \omega \in \Omega\}$ from $\mathscr{L}^{\infty}(M \times \Omega, \mu \times P)$ into $\mathscr{L}^{\infty}(M, \mu)$ by the prescription

$$(E^{\omega} f)(x) = f(x, \omega) \qquad ... (3.7)$$

for $x \in M$, $\omega \in \Omega$, $f \in \mathscr{L}^{\infty}(M \times \Omega, \mu \times P)$

We can now give an algebraic analogue of a stochastic flow of diffeomorphisms. A stochastic flow of automorphisms associated to M will mean a family $J = (J_{s,t} ; t \geq s)$ of normal unital *–homomorphisms from $L^{\infty}(M, \mu)$ into $L^{\infty}(M \times \Omega, \mu \times P)$ which is such that for almost all $\omega \in \Omega$, the family $J^{\omega} = (J^{\omega}_{s,t} ; t \geq s)$ of operators on $C^{\infty}_{K}(M)$ extends to a flow of automorphisms of $C^{\infty}(M)$ where

$$J^{\omega}_{s,t} = E^{\omega} \circ J_{s,t} \qquad ... (3.8)$$

for each $t \geq s$, $\omega \in \Omega$.

It is not difficult to verify (using (2.1)) that if such automorphisms exist then they are unique.

**Proposition 3.** There is a one–to–one correspondence between equivalence classes of stochastic flows of diffeomorphisms of M and stochastic flows of automorphisms associated to M.

**Proof** If $\phi$ is a stochastic flow of diffeomorphisms on M we define J by (3.3).

We then obtain for each $s,t \in \mathbb{R}^{+}$, $t \geq s$, $\omega \in \Omega$, $f \in C_{K}(M)$

$$J^{\omega}_{s,t}(f) = E^{\omega} \circ J_{s,t}(f)$$

$$= E^{\omega} \circ f \circ \Phi_{s,t} = f \circ \phi^{\omega}_{s,t}$$

which extends to the required flow of automorphisms of $C^{\infty}(M)$.

Conversely, given $J = (J_{s,t} : t \geq s)$, for each $x \in M$, $t \geq s$, $J^{x}_{s,t}$ is a normal unital *–homomorphism from $\mathscr{L}^{\infty}(M, \mu)$ into $\mathscr{L}^{\infty}(\Omega, \mathscr{F}, P)$ where

$$(J^{x}_{s,t} f)(\omega) = (J_{s,t} f)(x, \omega)$$

for $\omega \in \Omega$, $f \in \mathscr{L}^{\infty}(M, \mu)$.

Hence by (3.2), there exists a measurable function $\phi^{x}_{s,t} : \Omega \longrightarrow M$ such that

$$(J^{x}_{s,t} f) = f \circ \phi^{x}_{s,t}$$

Now by (2.1), for $f \in C^{\infty}(M)$ we have

$$J^{\omega}_{s,t} (f) = (J_{s,t} f)(x, \omega) = f \circ \phi^{\omega}_{s,t}$$

where $(\phi^{\omega}_{s,t} ; t \geq s)$ is a flow of diffeomorphisms of M for almost all $\omega \in \Omega$.

Now combining the above results we see that for each $x \in M$, $\omega \in \Omega$ we can find $\phi^{x}_{s,t}: \Omega \longrightarrow M$, $\phi^{\omega}_{s,t} : M \longrightarrow M$ such that

$$f(\phi^{x}_{s,t}(\omega)) = f(\phi^{\omega}_{s,t} (x))$$

for all $f \in C^{\infty}_{K}(M)$. By taking f to be a suitable bump function as in [AMR] p. 215, we see that

$$\phi^{x}_{s,t}(\omega) = \phi^{\omega}_{s,t} (x)$$

thus we obtain our required pre–flow $(\phi_{s,t}; t \geq s)$ by defining

$$\phi_{s,t}(x, \omega) = \phi^{x}_{s,t} (\omega) = \phi^{\omega}_{s,t} (x)$$

for each $x \in M$, $\omega \in \Omega$ □

## 4. Brownian Flows

Let $(\Omega, \mathscr{F}, P)$ be the canonical space for n–dimensional Brownian motion $(B_1, \ldots, B_n)$ and let $Y_0, \ldots, Y_n$ be smooth vector fields on M. We denote by $(\eta^{s}_{x}(t), t \geq s)$ the unique solution of the stochastic differential equation (SDE), written in Stratonovitch form,

$$\begin{aligned} d\eta(t) &= \sum_{j=1}^{n} Y_j(\eta(t)) \circ dB^j(t) + Y_0(\eta(t))dt \\ \eta(s) &= x \qquad \text{a.e.} \end{aligned} \qquad \text{... (4.1)}$$

for $x \in M$.

If $Y_0, \ldots, Y_n$ are complete and generate a finite dimensional Lie algebra, it is shown in [Kun 2], p.194 that the prescription, for $\omega \in \Omega$,

$$\lambda_{s,t}(x,\omega) = \eta_x^s(t)\omega \qquad \text{... (4.2)}$$

yields a stochastic flow of diffeomorphisms ($\lambda_{s,t}$ ; $t \geq s$). Flows of this type will be called Brownian (c.f.[Kun 1, 2]). Note that for $f \in C_K^\infty(M)$, Ito's formula yields

$$df(\eta(t)) = \sum_{j=1}^{n}(Y_jf)(\eta(t))\, dB^j(t) + \left[(Y_0f)(\eta(t)) + \frac{1}{2}\sum_{j=1}^{n}(Y_j^2f)(\eta(t))\right] dt \quad \text{... (4.3)}$$

We will now consider Brownian flows from the algebraic and analytical perspectives. For simplicity we will here only consider the case of flows driven by a single Brownian motion $B = (B(t), t \in \mathbb{R}^+)$. Using the canonical realisation on paths

$$B(t)\omega = \omega(t) \qquad \text{... (4.4.)}$$

for $\omega \in \Omega$, $t \in \mathbb{R}^+$, we consider each $B(t)$ as a multiplication operator acting in $L^2(\Omega, \mathcal{F}, P)$. We note that each $B(t)$ is self adjoint with domain

$$\mathcal{D}(B(t)) = \left\{y \in L^2(\Omega, \mathcal{F}, P);\ \int_\Omega \omega(t)^2 \mid y(\omega) \mid^2 dP(\omega) < \infty\right\}$$

A common core for the $B(t)$'s is the dense domain $\mathcal{E}$ comprising finite linear combinations of exponential martingales (see e.g. [Mey 1]

Now let Y be a complete smooth vector field on M so that the linear operator $T_Y = Y + \frac{1}{2}\operatorname{div}_\mu(Y)$ acting in $h_0$ is essentially skew–adjoint on $\mathcal{D}_0$.

We will work in the complex separable Hilbert space $h = L^2(\Omega, \mathcal{F}, P; h_0)$ which is canonically isomorphic to $h_0 \otimes L^2(\Omega, \mathcal{F}, P)$. For convenience we will identify these two spaces. Note that $L^\infty(M \times \Omega, \mu \times P) \subset B(h)$. For each $t \in \mathbb{R}^+$, the linear operator $T_Y \otimes B(t)$ is essentially skew–adjoint on the dense domain $\mathcal{D}_0 \underline{\otimes} \mathcal{E}$ in $h$. We denote its closure by $A(t)$ and observe that for $\psi \in \mathcal{D}(A(t))$ we have

$$(A(t)\psi)(\omega) = \omega(t)\, \overline{T}_Y\, \psi(\omega) \qquad \text{... (4.5)}$$

We now form a family $U = (U(t), t \in \mathbb{R}^+)$ of unitary operators in $h$ by the prescription

$$U(t) = e^{A(t)} \quad \dots (4.6)$$

For each $t \in \mathbb{R}^+$, $\mathscr{D}(A(t))$ contains a dense set of analytic vectors for $A(t)$ which we denote by $\mathscr{D}^a(t)$.

By (4.5) and (4.6) we have, for $\psi \in \mathscr{D}^a(t)$, in the strong topology on $h$,

$$(U(t)\,\psi)(\omega) = \sum_{m=0}^{\infty} \left[\frac{A(t)^m}{m!}\,\psi\right](\omega)$$

$$= \sum_{m=0}^{\infty} \frac{\omega(t)^m\,\overline{T}_Y^m}{m!}\,\psi(\omega) \quad \dots (4.7)$$

Let $Y(\omega(t))$ denote the complete smooth vector field $\omega(t)Y$, then it is not difficult to verify that

$$T_{Y(\omega(t))} = \omega(t)\,T_Y \quad \dots (4.8)$$

and we see from (4.7) that for each $\psi \in \mathscr{D}^a(t)$, $\omega \in \Omega$, $\psi(\omega)$ is analytic for $T_{Y(\omega(t))}$

For each $t \in \mathbb{R}^+$, define

$$U^{\omega}(t) = e^{\overline{T_{Y(\omega(t))}}} \quad \dots (4.9)$$

then each $U^{\omega}(t)$ is a unitary operator on $h_0$ and we have

$$(U(t)\psi)(\omega) = U^{\omega}(t)\,\psi(\omega) \quad \dots (4.10)$$

for $\psi \in h$, $\omega \in \Omega$.

Now for each $t \in \mathbb{R}^+$, note that

$$U(t)^{-1} = U(t)^* = e^{-A(t)}$$

and define, for $t \geq s$,

$$U(s, t) = U(s)^{-1}\,U(t) = e^{A(t) - A(s)}$$

$$= e^{\overline{T_Y \otimes B(s,t)}} \quad \dots (4.11)$$

where $B(s, t) = B(t) - B(s)$

We obtain a family of automorphisms of $B(h)$, $J = (J_{s,t}\,;\, t \geq s)$ by defining

$$J_{s,t}(X) = U(s, t)\,X\,U(s, t)^{-1} \quad \dots (4.12)$$

In the sequel we will only be concerned with the restriction of these automorphisms to $L^\infty(M, \mu)$ which we regard as a *–subalgebra of $L^\infty(M \times \Omega, \mu \times P)$

**Theorem 4.** J is a stochastic flow of automorphisms on M.

**Proof** For each $\omega \in \Omega$, we define $J^\omega = (J^\omega_{s,t}\,; t \geq s)$ by

$$J^\omega_{s,t}(f) = U^\omega(s, t)\, f\, U^\omega(s, t)^{-1}$$

on $C^\infty_K(M)$, then by (2.13) we see that each $J^\omega_{s,t}$ is the restriction to $C^\infty_K(M)$ of the automorphism $e^{(\omega(t) - \omega(s))Y}$ of $C^\infty(M)$. It follows from (4.13) that $J^\omega$ satisfies $F'$(i) and $F'$(ii) and thus is a flow of automorphisms of $C^\infty(M)$. Hence by (2.1), there exists a flow of diffeomorphisms of M, $\{\phi^\omega_{s,t}; t \geq s\}$ for which

$$J^\omega_{s,t}(f) = f \circ \phi^\omega_{s,t}$$

for each $f \in C^\infty(M)$.

Now for $f \in L^\infty(M, \mu)$ we have $(J_{s,t}(f)\, \psi)(\omega) = (U^\omega(s,t) f\, U^\omega(s,t)^{-1})\psi(\omega)$ for each $\psi \in h$, $\omega \in \Omega$, hence each $J_{s,t}(f) \in L^\infty(M \times \Omega, \mu \times P)$. (3.8) is now easily verified □

By proposition 3, we may now associate to J a stochastic flow $\phi = (\phi_{s,t}; t \geq s)$ of diffeomorphisms of M. It follows from the contruction of theorem 4, that for each $t \geq s$, $\omega \in \Omega$. $x \in M$

$$\phi_{s,t}(x, \omega) = \xi_{\omega(t) - \omega(s)}(x)$$

where $\xi$ is the deterministic autonomous flow of §2 obtained from the integral curves of Y.

Note that by continuity of the Brownian paths, the stochastic flow $\phi$ is itself path continuous so that the equivalence class of flows given by proposition 3 has only one member. Our aim in the rest of this section is to investigate the relationship between the flows $\lambda$ and $\phi$, where $\lambda$ is given by (4.2). In order to do this, we must first investigate the differential structure of $\phi$.

Now by [Re Si II; p. 205] we may write $\mathscr{D}^a(t)$, for each $t \in \mathbb{R}^+$, as the linear span of a set of product vectors of the form $v \otimes \psi$ wherein v ($\psi$) is analytic for $\overline{T}_Y$ (B(t)), thus for $u \in h_0$, $\chi \in L^2(\Omega, \mathscr{F}, P)$ we may write

$$<u \otimes \chi, U(t)(v \otimes \psi)> = \sum_{m=0}^{\infty} \frac{1}{m!} < u \otimes \chi, \overline{T}_Y^m\, v \otimes B(t)^m\, \psi > = E(\overline{\chi}\, G_t(u, v)\, \psi)$$

where

$$G_t(u, v) = \sum_{m=0}^{\infty} \frac{<u, \overline{T}_Y^m\, v>_{h_0}}{m!} B(t)^m \qquad \ldots (4.13)$$

and the limit is in the weak topology in $h$. We will prove elsewhere([App 2]) that each $G_t(u, v)$ is, in fact, a bona–fide random variable on $(\Omega, \mathcal{F}, P)$ and that the process $t \longrightarrow G_t(u, v)$ is thus a smooth function of Brownian motion.

**Lemma 5.** For each $t \in \mathbb{R}^+$, we have

$$dG_t(u, v) = G_t(u, \overline{T}_Y v)\, dB(t) + \frac{1}{2} G_t(u, \overline{T}_Y^2 v)\, dt \quad \text{... (4.14)}$$

**Proof** Applying Itô's formula to (4.13) we find

$$dG_t(u, v) = \sum_{m=1}^{\infty} \frac{<u, \overline{T}_Y^m v>}{(m-1)!} B(t)^{m-1} dB(t) + \frac{1}{2} \sum_{m=2}^{\infty} \frac{<u, \overline{T}_Y^m v>}{(m-2)!} B(t)^{m-2} dt$$

and the result follows □

By analogy with the notation of [HuPa], we write (4.14) as

$$\begin{aligned} dU &= U(\overline{T}_Y\, dB(t) + \tfrac{1}{2} \overline{T}_Y^2\, dt) \qquad \text{... (4.15)} \\ &= (\overline{T}_Y\, dB(t) + \tfrac{1}{2} \overline{T}_Y^2\, dt)\, U \end{aligned}$$

with initial condition $U(0) = I$.

It is easily verified that the process $U_s = (U(s,t); t \geq s)$ also satisfies (4.15) with the initial condition $U_s(s) = I$.

Now taking adjoints in (4.11) yields

$$U(s, t)^* = e^{-\overline{T_Y \otimes B(s,t)}}$$

so that given $v \otimes \psi \in \mathcal{D}^a(t)$ for $t \geq s$, we have

$$U(s, t)^* (v \otimes \psi) = \sum_{m=0}^{\infty} \frac{(-\overline{T}_Y)^m v \otimes B(s, t)^m \psi}{m!} \quad \text{...(4.16)}$$

A similar calculation to lemma 5 with analytic vectors yields the familiar result that for each $s,t \in \mathbb{R}^+$ with $t \geq s$ and for each $f \in C_K^\infty(M)$, we have

$$df(\phi_{s,t}) = (Yf)(\phi_{s,t})dB(t) + \frac{1}{2} (Y^2 f)(\phi_{s,t})\, dt \quad \text{... (4.17)}$$

Comparing (4.3) (with $n = 1$, $Y_0 = 0$) and (4.17) we see that these are identical in form so that each flow of stochastic diffeomorphisms $\lambda$ and $\phi$ yields a solution of (4.17).

Now let $\pi = (\pi_{s,t}; t \geq s)$ be any stochastic flow of diffeomorphisms of M which is a solution of

(4.17) and define a family of linear operators on $h$, $V = (V_{s,t}\,;\, t \geq s)$ by

$$((V_{s,t}\,\psi)(\omega))(x) = \psi(\omega)(\pi_{s,t}(x, \omega)) \quad \ldots (4.18)$$

for $\psi \in h$, $x \in M$, $\omega \in \Omega$.

In the sequel we will write (4.18) in the simplified form

$$V_{s,t}\,\psi = \psi \circ \pi_{s,t} \qquad \ldots (4.19)$$

**Lemma 6** $V_{s,t}$ is unitary for each $t \geq s$.

**Proof.** Let $\psi \in h$, $\omega \in \Omega$ be such that

$$\psi(\omega) = f\,\mu_V^{\frac{1}{2}}$$

where $f \in L^2(M, \mu)$ and define $V^{\omega}_{s,t}$ by

$$V^{\omega}_{s,t}\,\psi(\omega) = (V_{s,t}\psi)\,(\omega) = (\psi \circ \pi_{s,t})(\omega) = f \circ \pi^{\omega}_{s,t}\,((\pi^{\omega}_{s,t})^{*}\mu_V)^{\frac{1}{2}}$$

Hence $V^{\omega}_{s,t}$ is unitary by (2.9) and we have

$$\left[V_{s,t}^{*}\;V_{s,t}\,\psi\right](\omega) = \left[V^{\omega}_{s,t}\right]^{*} V^{\omega}_{s,t}\,\psi(\omega) = \psi(\omega)$$

A similar calculation for $V_{s,t}\,V_{s,t}^{*}$ confirms the unitarity of $V_{s,t}$ □

Now let $\mathscr{D}_1$ denote the dense domain $\mathscr{D}_0 \underline{\otimes}\, \mathscr{E}$ in $h$.

**Lemma 7** $V = (V_{s,t}\,;\, t \geq s)$ is a solution of equation (4.15) on $\mathscr{D}_1$.

**Proof.** Let $\psi \in \mathscr{D}_1$ be such that $\psi(\omega) = f\,\mu^{\frac{1}{2}}$ for $\omega \in \Omega$, $f\,\mu^{\frac{1}{2}} \in \mathscr{D}_0$, then

$$V_{s,t}\,\psi = f(\pi_{s,t})(\pi_{s,t})^{*}\mu_V^{\frac{1}{2}}$$

Now by Ito's product formula, we obtain

$$\begin{aligned}
dV_{s,t}\,\psi &= df(\pi_{s,t})\;\pi_{s,t}^{*}(\mu_V)^{\frac{1}{2}} + (f\pi_{s,t})\;d\,\pi_{s,t}^{*}(\mu_V)^{\frac{1}{2}} + df(\pi_{s,t})\;d\,\pi_{s,t}^{*}(\mu_V)^{\frac{1}{2}} \\
&= \left[(Yf)(\pi_{s,t})\,dB(t) + \tfrac{1}{2}\,(Y^2f)(\pi_{,t})\,dt\right]\pi_{s,t}^{*}(\mu_V)^{\frac{1}{2}} + f(\pi_{s,t})(\tfrac{1}{2}\,\mathrm{div}_{\mu}(Y)(\pi_{s,t})dB(t) \\
&\quad + \left[\tfrac{1}{4}(Y(\mathrm{div}_{\mu}(Y))(\pi_{s,t}))dt + \tfrac{1}{8}\,(\mathrm{div}_{\mu}(Y)(\pi_{s,t}))^2\,dt\right]\pi_{s,t}^{*}(\mu_V)^{\frac{1}{2}} \\
&\quad + \tfrac{1}{2}\,(Y(f)\,\mathrm{div}_{\mu}(Y)(\pi_{s,t})\;\pi_{s,t}^{*}(\mu_V)^{\frac{1}{2}}\,dt \\
&= (\overline{T}_Y\,\psi)(\pi_{s,t})dB(t) + \tfrac{1}{2}(\overline{T}_Y^{\,2}\,\psi)(\pi_{s,t})\,dt \\
&= V_{s,t}(\overline{T}_Y dB(t) + \tfrac{1}{2}\,\overline{T}_Y^{\,2}\,dt)\psi
\end{aligned}$$

□

Now by the appendix, we see that the solution to (4.15) is unique on $\mathscr{D}_1$. Hence we deduce that

$$\psi \circ \lambda_{s,t} = \psi \circ \Phi_{s,t}$$

for all $t \geq s$, $\psi \in h_0$, so that

$$f \circ \lambda_{s,t} = f \circ \Phi_{s,t}$$

for all $f \in C_K^\infty(M)$ from which we deduce that $\lambda_{s,t} = \Phi_{s,t}$ as required.

We also note that

$$U(s, t) = V_{s,t} \text{ for all } t \geq s.$$

The extension of these results to the more general case wherein the noise is a multidimensional Brownian motion with a drift is far more problematic. However, we observe that in the simple case, where each $Y_j$ is divergence–free and $[Y_j, Y_k] = 0$ for $0 \leq j, k \leq n$ we may write

$$U(t) = \exp\left[\sum_{j=1}^{n} Y_j \otimes B^j(t) + Y_0\, t\right]^{-} = e^{Y_0 t} \prod_{j=1}^{n} \overline{\exp(Y_j \otimes B^j(t))}$$

which is the unique solution of the SDE

$$dU = U\left[\sum_{j=1}^{n} Y_j\, dB^j + (Y_0 + \frac{1}{2}\sum_{j=1}^{n} Y_j^2)\, dt\right]$$

with $U(0) = I$. In this particular case we will still obtain a stochastic flow of diffeomorphisms however in the general multidimensional case, we cannot expect all the symmetric operators $(A(t), t \in \mathbb{R}^+)$ where $A(t) = T_{Y_0}\, t + \sum_{j=1}^{n} T_{Y_j} \otimes B_j(t)$ to be essentially skew adjoint on $\mathcal{D}_0 \underline{\otimes} \mathcal{E}$.

## 5. Poisson Flows

In this section we will consider a class of flows which unlike the Brownian ones discussed in the previous section, no longer possess continuous sample paths.

To this end, we here take $(\Omega, \mathcal{F}, P)$ to be the sample space for a Poisson process $(N_\lambda = (N_\lambda(t), t \in \mathbb{R}^+)$ of intensity $\lambda > 0$, and consider each $N_\lambda(t)$ as a self–adjoint multiplication operator on $L^2(\Omega, \mathcal{F}, P)$. Now, for each $t \in \mathbb{R}^+$, let $C(t)$ denote the closure of the essentially skew–adjoint operator $T_Y \otimes N_\lambda(t)$ on the domain $\mathcal{D}(\overline{T}_Y) \underline{\otimes} \mathcal{D}(N_\lambda(t))$ in $h$ and let $U = (U(t), t \in \mathbb{R}^+)$ be the unitary operator valued process defined by

$$U(t) = e^{C(t)} \qquad \ldots (5.1)$$

We shall also have need of the unitary operator $e^{T_Y}$ on $h_0$ which we denote by $W_Y$.

**Theorem 8** $U = (U(t), t \in \mathbb{R}^+)$ is a solution of the operator valued SDE

$$dU = U(W_Y - I)\, dN_\lambda \qquad \text{... (5.2)}$$

with $U(0) = I$

*Proof.* We use the same technique as in theorem 3 and expand $U(t)$ as a series on analytic vectors of the form $v \otimes \psi$.

We write for $u \in h_0$, $\chi \in L^2(\Omega, \mathscr{F}, P)$, $v$, $\psi$ as above:

$$< u \otimes \chi, U(t)\,(v \otimes \psi)> = E(\overline{\chi}\, G_t(u, v)\, \psi) \text{ where}$$

$$G_t(u, v) = \sum_{n=0}^{\infty} \frac{<u, \overline{T}_Y^n\, v>}{n!} N_\lambda(t)^n \qquad \text{... (5.3)}$$

The required result will follow if we can show that

$$dG_t(u, v) = G_t(u, (W_Y - I)v)\, dN_\lambda(t)$$

By Itô's formula in (5.3), we obtain

$$dG_t(u, v) = \sum_{n=1}^{\infty} \frac{<u, \overline{T}_Y^n\, v>}{n!} [N_\lambda(t) + 1)^n - N_\lambda(t)^n]\, dN_\lambda(t)$$

$$= \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n!} \begin{bmatrix} n \\ r \end{bmatrix} <u, \overline{T}_Y^n\, v> N_\lambda(t)^r\, dN_\lambda(t)$$

However, since v is analytic for $\overline{T}_Y$, we obtain

$$G_t(u, (W_Y - I)v) = \sum_{n=0}^{\infty} \frac{1}{n!} <u, \overline{T}_Y(W_Y - I)\, v> N_\lambda(t)^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n!\, m!} <u, \overline{T}_Y^{n+m}\, v> N_\lambda(t)^n$$

$$= \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{r!(n-r)!} <u, \overline{T}_Y^n\, v> N_\lambda(t)^r$$

$$= \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n!} \begin{bmatrix} n \\ r \end{bmatrix} <u, \overline{T}_Y^n\, v> N_\lambda(t)^r$$

as required □

Now as in §4, for $t \geq s$, write

$$U(s,t) = U(s)^{-1}\, U(t) \qquad \text{... (5.4)}$$

and

$$J_{s,t}(X) = U(s,t)\, X\, U(s,t)^{-1} \qquad \text{... (5.5)}$$

for $X \in B(h)$. Again we can define for each $\omega \in \Omega$, the unitary operators

$$U^{\omega}(s,t) = e^{(\omega(t)-\omega(s))\overline{T}_Y} \qquad \dots (5.6)$$

but note that in this case each $\omega(s)$, $\omega(t)$ is a natural number.

We can now imitate the argument of theorem 4 to assert that there exists an equivalence class of stochastic flows of diffeomorphisms of M, $\Phi = (\Phi_{s,t};\ t \geq s)$ for which

$$J_{s,t}(f) = f \circ \Phi_{s,t} \qquad \dots (5.7)$$

for all $f \in L^{\infty}(M, \mu)$. However it follows from (5.6) that the map $\omega \longrightarrow \phi_{s,t}(x, \omega)$ (for $x \in M$, $t \geq s$, $\omega \in \Omega$) will not be continuous. We will call $\Phi$ a Poisson flow. Similar algebraic manipulations to those of theorem 8 show that for each $f \in C_K^{\infty}(M)$, $t \geq s$ we have

$$df(\phi_{s,t}) = (e^{Y}f(\phi_{s,t}) - f(\phi_{s,t}))\, dN_{\lambda}(s,t) \qquad \dots (5.8)$$

By (2.2) and (2.13), we can write

$$e^{Y}f = f \circ \xi_1 \qquad \dots (5.9)$$

where $\xi = (\xi_t,\ t \in \mathbb{R})$ is an autonomous (deterministic) flow on M. Since Y is complete, we can use (2.4) to obtain $\xi$ from a family of integral curves $\{c_x,\ x \in M\}$.

Writing $Y(x) = a^j(x)\frac{\partial}{\partial x_j}$ in local co–ordinates we have, for $1 \leq j \leq d$, $t \in \mathbb{R}^+$

$$c_x^j(t) = x + \int_{-\infty}^{t} a_j(c(\tau))\, d\tau = x + h^j(t)$$

Let $h^j = h^j(1)$, then we can write (5.8) in local co–ordinates as

$$f(\phi_{s,t}) = (f(\phi_{s,t} + h) - f(\phi_{s,t}))dN_{\lambda}(t) \qquad \dots (5.10)$$

where $h = (h^1, \dots, h^d)$.

Using Ito's formula for Poisson processes, we see that a solution to (5.10) is given by

$$\phi_{s,t}(x) = \overset{s}{\eta}_x(t) \qquad \dots (5.11)$$

where, in local co–ordinates, for each $x \in M$, $1 \leq j \leq d$, we have

$$\overset{s}{\eta}_x(t)^j = x^j + h^j\ N_{\lambda}(t) \qquad \dots (5.12)$$

$$\text{with } \overset{s}{\eta}_x(s)^j = x^j$$

i.e. each $\eta^j$ is a point process with jumps of size $h^j$.

Similar techniques to those used in §4 establish that (5.12) is the unique solution to (5.10), in the sense of agreement almost everywhere with respect to P.

## Appendix 1

We show here that the process $U = (U(t), t \geq 0)$ of §4 is the unique solution of (4.5). To do this, it is more convenient to work in Fock space rather than Wiener space and we will assume here some familiarity with the decomposition therein of Brownian motion into a sum of annihilation and creation processes and the realisation of exponential martingales as exponential vectors (see [HuPa], [Mey] for details). In fact we will prove a slightly stronger result.

Let $h_0$ be a complex separable Hilbert space and $L_0$ be a skew adjoint operator on $h_0$ with invariant domain $\mathscr{D}_0$.
Consider the equation

$$dU = U(L_0\, dB + \tfrac{1}{2} L_0^2\, dt) \qquad \ldots (A1)$$
$$U(0) = I \qquad \text{on } \mathscr{D}_0 \underline{\otimes}\ \mathscr{E}$$

**Theorem** If a solution $U = (U(t)\ t \geq 0)$ exists to (A1) then it is unique.

**Proof** Let $V = (V(t), t \geq 0)$ be another solution to (A1) and write

$$W(t) = U(t) - V(t) \text{for each } t \in \mathbb{R}^+$$

so that $W = (W(t), t \geq 0)$ solves the SDE

$$dW = W(L_0\, dB + \tfrac{1}{2} L_0^2\, dt)$$
$$W(0) = 0$$

Now let $u \in \mathscr{D}_0$, $\psi(f) \in \mathscr{E}$ with f a locally bounded function, then by the Itô product formula of [HuPa], we obtain

$$\|W(t)\, u \otimes \psi(f)\,\|^2$$
$$= \int_0^t \{2\mathrm{Re}(f(s)).\ \ 2\mathrm{Re}\langle W(s)\, L\, u \otimes \psi(f), W(s)\, u \otimes \psi(f)\rangle + \mathrm{Re}\langle W(s)\, L^2\, u \otimes \psi(f),$$
$$W(s)\, u \otimes \psi(f)\rangle + \langle W(s)\, L\, u \otimes \psi(f), W(s)\, L\, u \otimes \psi(f)\rangle\}\, ds$$
$$\leq \int_0^t \Big[4|f(s)|(\|W(s)L\, u \otimes \psi(f)\,\|^2 + \|W(s)\, u \otimes \psi(f)\,\|^2) + \|W(s)L^2\, u \otimes \psi(f)\,\|^2$$
$$+ \|W(s)\, u \otimes \psi(f)\,\|^2 + \|W(s)L\, u \otimes \psi(f)\,\|^2\Big] ds \quad \ldots (A.2)$$

c.f. [HuPa], corollary 1 p.310.

Let $\alpha_k(t) = \|W(s)\, L^k\, u \otimes \psi(f)\,\|^2$ for $k = 0, 1, 2$, then (A.2) may be written

$$\alpha_0'(t) \leq \sum_{k=0}^{2} c_k(t)\, \alpha_k(t)$$

where $c_0(t) = c_1(t) = 4\,|f(t)| + 1$ and $c_2(t) = 1$.

By a slight extension of Gronwall's inequality we obtain

$$\alpha_0(t) \le \alpha_0(0) \int_0^t \exp\left[\int_s^t c_0(\tau)d\tau\right]\left[c_1(s)\alpha_1(s) + c_2(s)\ \alpha_2(s)\right] ds$$

But $\alpha_0(0) = 0$ and the result follows. □

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